Chapter 1

Introduction

1.1. Overview

In this chapter, we first present a brief introduction to the topics concerning the present investigations followed by detailed literature on the mechanics of laminated composites and smart composite plate structures. The chapter starts with the introduction of the laminated composites and sandwich structures in which the various types of composite materials, sandwich materials, and their significance concerning practical applications are discussed. Then an introduction to the topic of smart structures is presented in which the various classifications of smart structures, types of smart materials, and their applications in various industries are discussed. In this study, the piezoelectric materials are mainly considered as the smart material for the structural analysis of smart composite plates, therefore, discussions on the piezoelectric materials are also presented. Literature on the modeling of multi-layered laminated composites and smart composite structures is presented in detail followed by the various solution methods adopted in the literature to solve the governing equations of the problems. The motivation behind this research and the literature gap is then presented followed by the objective and scope of the present work. A chapter-wise organization of the thesis is presented in the end that summarizes the contents of all the chapters in the thesis.

1.2. Laminated composites and Sandwich structures

A composite is a structural material that consists of two or more constituents combined at a macroscopic level and not soluble in each other. The constituent materials in composites have dissimilar properties and are combined to create a new material with advanced properties, unlike the individual constituents. There are two phases in composites mainly, the reinforcing phase and the matrix. The reinforcing phase is embedded in the matrix. The reinforcing phase is generally in the form of discrete fibers and the matrix phase is continuous. Typical fibers include graphite, boron, cellulose and, glass and some of the matrix materials are epoxies, polyimides, titanium, and aluminium, etc. The typical engineered composite materials are

- Composite wood such as plywood
- Reinforced Concrete
- Reinforced Plastics such as fiber-reinforced polymers (FRPs)
- Ceramic matrix composites
- Metal matrix composites

Composite materials are less expensive, stronger, and lighter in comparison to common materials. They are widely used in bridges, buildings, ship hulls, automobile bodies, storage tanks, aircraft wings, helicopter gliders, turbine disks, and many more. Composite materials are mainly formed in three different types:

- Fibrous composites
- Particulate composites
- Laminated composites

The fibrous composite consists of lightweight and high modulus fibers of a material embedded in the matrix of a different material. The properties of the composites can be varied by changing the direction of the fibers. Particulate composites are made of macro-sized particles of a material suspended in a matrix of a different material. Laminated composites are made up of several layers of different materials generally stacked in the thickness direction and can have the composites of the first two types (Reddy, 2004). Fibrous composites like the FRPs are very popular in the mechanical, aerospace, and naval industries, because of the high stiffness and strength. This is attributed to the strong and stiff fibers embedded in a matrix material. Laminated composites consisting of fibrous composites stacked in the thickness direction are widely used in the above-mentioned industries because such type of composite construction can provide the required engineering properties like the in-plane and bending stiffness and coefficient of thermal expansion. The fibers in the lamina can be continuous or discontinuous, unidirectional, bi-directional, and woven. The difference between the continuous and discontinuous fibers is that continuous fibers have a long aspect ratio and generally have a preferred orientation while the discontinuous fibers have a short aspect ratio and have random orientations. Continuous fibers exhibit the highest strength when orientated unidirectionally, however, the composites exhibit very low strength in the direction perpendicular to the fiber direction. Boron, carbon, alumina, and silicon carbide are the most researched continuous fiber-reinforcements in composites (Kapranos *et al*., 2014). Halpin and Karoos (1978) determined the strength of short fiber composites having random orientations. Unidirectional fiber-reinforced laminae are generally stacked in the thickness direction with orientation in the same or different directions as shown in Figure 1.1. Dong and Davies (2015) determined the flexural strength of bi-directional composites. Experimental investigation for understanding the mechanical properties of bi-directional composites is presented by Pamar *et al*. (2015). Research on woven fiber composites is presented by Ratim *et al*. (2012), Ye and Daghyani (1997), Rath and Sahu (2012), and Panda *et al*. (2013).

Laminates made of fiber-reinforced composites also have distinct disadvantages. The mismatch of the material properties in between the layers produces significant shear stresses which cause delamination. Similarly, the mismatch of material properties between fiber and matrix creates fiber debonding. This problem is circumvented by using a new material known as the Functionally Graded Material (FGM). FGMs are novel materials in which the properties change gradually with dimensions.

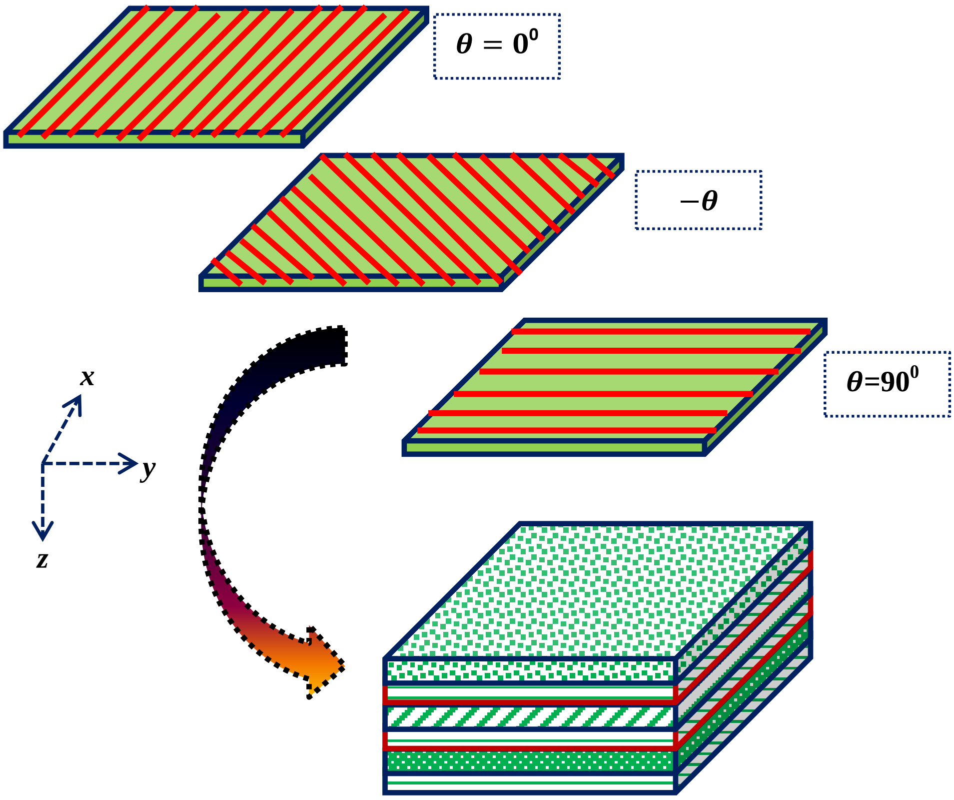

Figure 1.1. Laminae with different fiber orientations to form a laminate

The concept of FGM was first introduced in Japan in the year 1984 during a spaceplane project. A combination of materials was used which would serve the objective of a thermal barrier that is capable of withstanding a surface temperature of 2000 K with a temperature gradient of 1000 K across a section of 10 mm (Ruys and Sun, 2002). The aerospace and computer circuit industry is keen to use materials that can withstand high thermal gradients. This is achieved with a ceramic layer in conjunction with a metallic layer. FGMs are also used in the power plant boiler shells. Significant research works (Swaminathan and Naveenkumar, 2014; Singh and Harsha, 2019 and Bouguenina *et al.*

2015) on FGMs have been carried out in the recent past in which FGMs are sandwiched between metal and ceramic layers in various engineering applications.

A sandwich structure is also a composite material that is fabricated by combining two stiff and thin skins with a lightweight, and thick core. The core material is generally a low strength material yet its thickness provides higher bending stiffness with overall low density. The materials used in the core are structured foams like polyvinylchloride, polystyrene, polyethylene, polyurethane, polyethersulfone, balsa woods and syntactic foams, etc. The laminates of carbon or glass fiber-reinforced composites are commonly used as skin material. Sandwich construction is extensively used in both commercial and aerospace industries as it is extremely lightweight and at the same time exhibits a high bending stiffness and high strength-to-weight ratio. The face sheets or the skins carry the bending loads and the core carries the shear loads. The honeycomb sandwich structures have minimal material cost and minimal weight as the construction of these structures is based on the minimization of the used materials to reach the criteria of low weight and cost. An example of a sandwich structure is shown in Figure 1.2. The common geometrical feature of these structures is an array of hollow cells in between thin vertical walls. The out-of-plane compression and shear properties are relatively high in honeycomb construction. A plate-like assembly is made by layering a honeycomb structure in between two thin and stiff layers. Applications of honeycomb sandwich structures are found in space applications, aircraft, and rockets as they are lightweight and can also be used as curved surfaces. It is mentioned in Campbell (2004) that when the thickness of the core is doubled, then the stiffness increases over 7× with only 3 % weight gain, while quadrupling the core thickness increases the stiffness over 37× with a 6 % weight gain. Therefore, sandwich constructions are very efficient and widely used in various disciplines wherever possible. More information on the area of sandwich structures is available in the review articles by Mackerle (2002), and Birman and Kardomateas (2018).

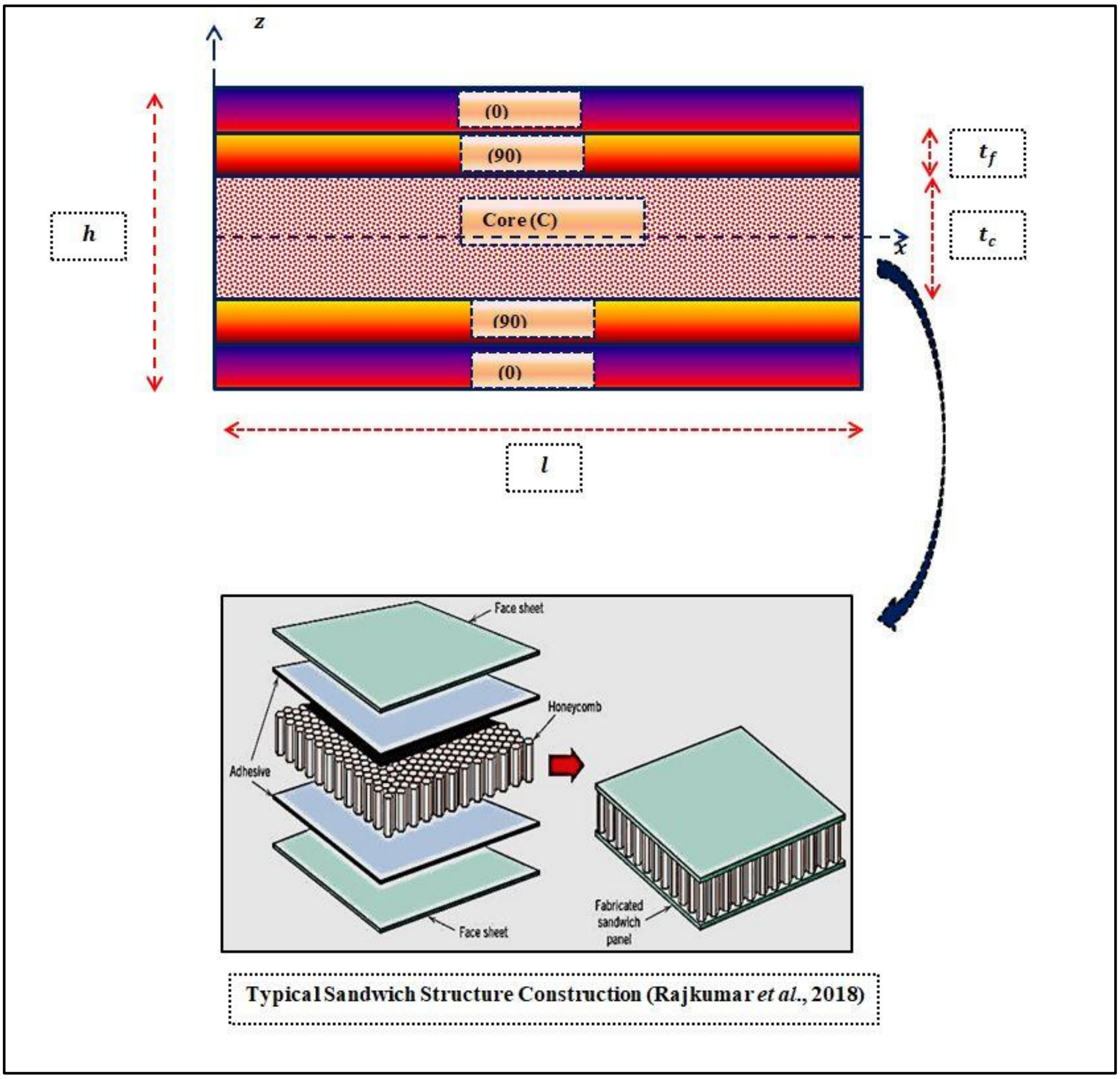

Figure. 1.2. Sandwich Structure

1.3. Smart Structures

A smart structure is a structure that can sense external stimuli such as a change in velocity, temperature, density, or pressure and respond in a controlled manner in real-time. Therefore, it is also known as an adaptive structure or an intelligent structure. A smart structure has distributed actuators, sensors, and microprocessors for analyzing the response from the sensors and with the help of some control theory gives a command to the actuators to counteract strains or displacements to alter the system response. It can react to a changing external environment (change in shape) and internal environment (damage). It has actuators that can alter the system characteristics like stiffness and damping and also the system response like the stresses/strains in a controlled manner. In general, smart structures have five important components: actuators, sensors, power and signal conditioning electronics, control strategies, and a computer (Chopra, 2002).

Typical examples of actuators and sensors are piezoelectric materials, electrostrictive materials, shape memory alloys, magnetostrictive materials, electro, and magneto-rheological fluids. These materials can be integrated or embedded in a load-carrying structure without affecting the structural stiffness and mass. The smart structures technology is applied to control the vibration and noise, damping of vibration, shape change, aeroelastic stability, and stress distribution. It is used in various applications like rotary-wing aircraft, space systems to fixed-wing aircraft, civil structures, automotive, machine tools, and medical systems.

1.3.1. Classification of smart structures

- **Adaptive Structures**

  Adaptive structures have distributed actuators to change the characteristics in a prescribed manner. Actuators are used to control the system response like stresses/strains, displacements. Sensors may not be present in an Adaptive structure.

- **Sensory Structures**

  Sensory structures have distributed sensors to monitor the characteristics of a structure like structural health monitoring. Sensors are used to detect the strains, acceleration, displacement, temperature, and also the extent of the damage.

- **Controlled Structures**

Controlled structures are adaptive as well as sensory structures. They constitute sensors, actuators, and a control system to actively control the characteristics of the structures.

- **Active Structures**

  Active structures are a type of controlled structures. The actuators and sensors also have the load-carrying capacity in the active structures.

- **Intelligent Structures**

  Intelligent structures are a type of active structures. They have a highly integrated control system and power electronics.

1.3.2. Piezoelectric materials

The word piezoelectric means electricity generated from pressure. French physicists Jacques Currie and Pierre Curie discovered piezoelectricity in the year 1880 (Manbachi and Cobbold, 2011). The piezoelectric effect is a result of the linear electromechanical interaction between electrical and mechanical states in crystalline materials with no inversion symmetry (Gautschi, 2002). The effect is a reversible process, *i.e*, materials producing electricity when mechanical strains are applied (Piezoelectric effect) also exhibit the reverse effect of generating mechanical strains when an electric field is applied. The piezoelectric effect is also known as the direct effect while the reverse piezoelectric effect is known as the converse effect. It is given in Krautkrämer and Krautkrämer (2013) that lead zirconate crystal (PZT) which is a piezoelectric material generates electricity when the crystal is deformed by 0.1 % of the original dimension and the same crystal change about 0.1 % of the original dimension when an external electric field is applied. Therefore, the piezoelectric materials are used as actuators and sensors.

Piezoceramics such as lead zirconate titanate (PZT) is a widely used piezoelectric material that is available in thin sheets and can be readily embedded or attached to laminated composite beams, plates, and shell structures. The maximum strain that can be actuated with PZT is one thousand microstrains. Polyvinylidene fluoride (PVDF) is a piezoelectric polymer, which is also mentioned in the literature, is used as a distributed actuator and sensor integrated or embedded with laminated composites. The maximum actuation strain noted in PVDF is seven hundred microstrains. Significant research (Bailey and Hubbard, 1985; Baz and Poh, 1988; and Hanagud *et al*., 1992) has been carried out in the past to achieve active control of smart structures using piezoelectric materials acting as distributed actuators and sensors. Additionally, piezoelectric fiber-reinforced (PFRC) composites have also been effectively employed for underwater transducers and medical imaging applications (Bennett and Hayward, 1997 and Sigmund *et al*. 1998). The PFRCs show improvement in the mechanical performance and electromechanical coupling coefficients in comparison to the conventional piezoelectric materials. An example of a smart composite with a PFRC layer is shown in Figure 1.3. Research on the development of piezoelectric composite structures is directed towards the estimation of mechanical and electromechanical properties using micromechanical analysis. Aboudi (1998) predicted the piezoelectric coefficients of piezoelectric fiber-reinforced composites (PFRC) with a micromechanical analysis by considering different electric fields in the matrix and fiber. However, the piezoelectric constant which is responsible for actuation along the fiber direction did not show any improvement over the conventional piezoelectric materials. Smith and Auld (1991) predicted marginal improvement of the piezoelectric constants of the PFRC. In their work, rods of piezoelectric materials were placed vertically aligned with the thickness of the composite material and the electric field was applied along the length of the piezoelectric fibers.

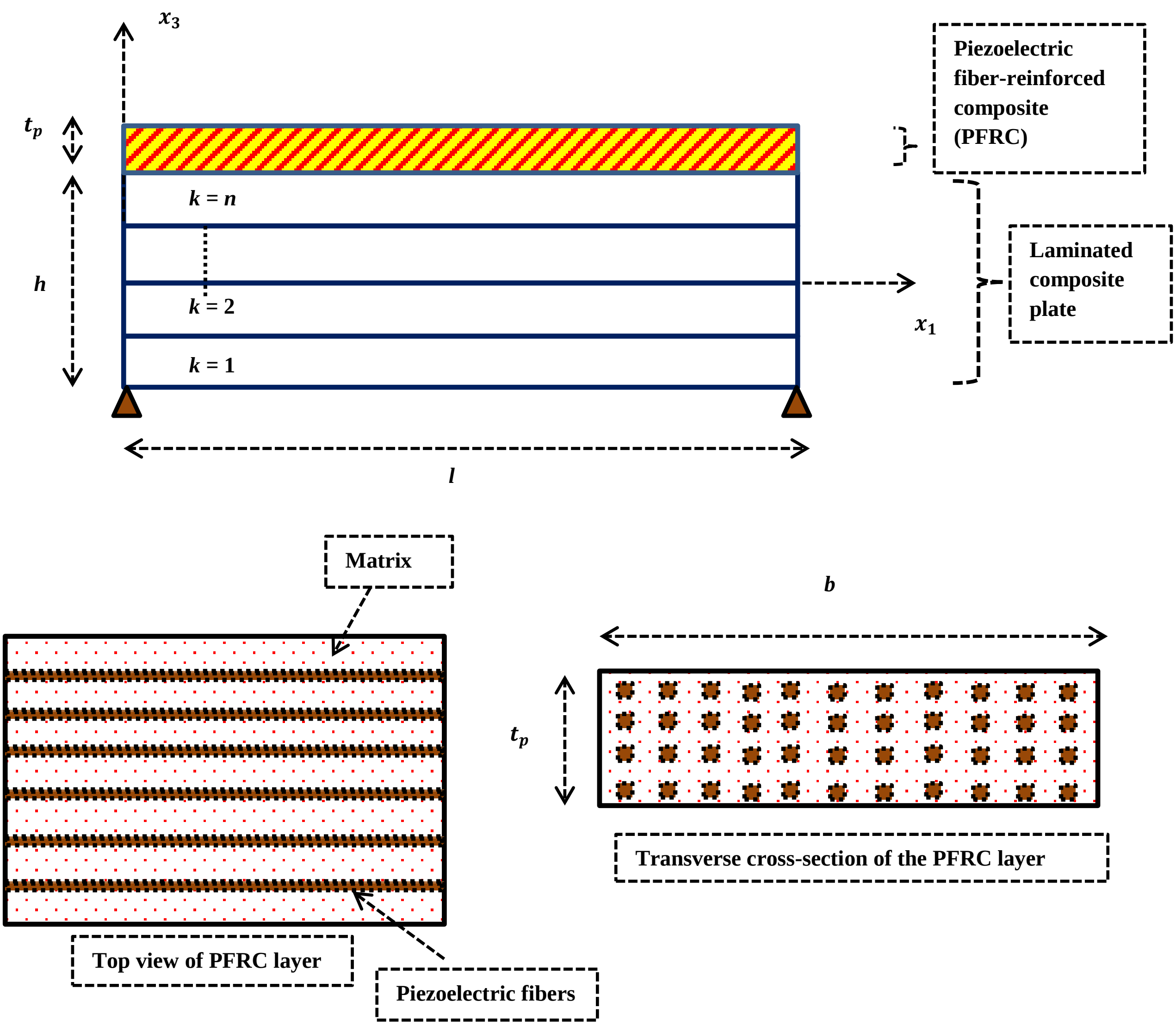

Figure 1.3. Smart composite plate with a piezoelectric fiber-reinforced composite (PFRC) layer.

Also, the electric field was assumed to be constant in both matrix and piezoelectric fibers. To use the PFRC as distributed actuators for flexural vibration control, the piezoelectric fibers should be placed in the longitudinal direction to render bending mode of actuation (Mallik and Ray, 2003). It may not be practically possible to apply a constant electric field in the longitudinal direction, *i.e*, along the length of the fibers when they are very long. A practical option is to apply the electric field in the thickness direction of the composites, *i.e*, transverse to the fiber direction. Since the thickness of the composite layer is sufficiently low in comparison to the in-plane dimensions, therefore, it will not be much difficult to maintain constant electricity in the thickness direction of the composites. Mallik and Ray (2003) predicted the effective coefficients of PFRC by carrying out a micromechanical analysis and a constant electric field was applied in the transverse direction of the fiber direction. Piezoelectric materials can be used in numerous ways like high voltage and power sources, actuators, sensors, piezoelectric motors, reduction of vibration and noise and surgery, etc.

1.4. Literature Review

In this section, the literature review on the mechanics of traditional laminated composites and smart composite plate structures is presented. The various approaches of modeling the structural responses of laminated composites and smart composite structures are discussed in detail followed by the literature on the various solution methods for solving the governing equations.

1.4.1. Introduction

Structural analysis of laminated composite beams, plates, and shells is very crucial for an efficient design. Deriving the governing equations in terms of partial differential equations (PDEs) or ordinary differential equations (ODEs) describing the physical phenomenon like bending, vibration, and buckling, etc. is the primary step in any analysis. The formulations can be exact or approximate. In the exact formulations, there are no assumptions involved and the responses are free from any numerical error. The approximate formulations are derived based on some approximations made in the variation of the displacements and stresses of the structures. Though the solutions from the approximate formulation are not exact, yet, the formulations are often useful because of the mathematical complexities involved in deriving the exact governing equations and also solving them to get the responses. The second step is to find the solutions to the governing equations. The solutions can be closed-form analytical solutions or numerical solutions. Closed-form analytical solutions do not invoke any additional error in the results as the solutions are valid for the entire domain and satisfy the boundary conditions, and the governing equations exactly at all the points in the domain. The numerical solutions are found at some specific points in the form of discrete values of the field variables. The values are further interpolated with some interpolation functions to obtain the magnitude of the field variables in between the selected points. The solutions from the numerical methods are approximate as some error gets initiated while selecting the interpolation functions, discretization of the domain, integration schemes, approximation in the derivatives, etc. However, numerical solutions are often useful as assuming closed-form analytical solutions for every problem is a difficult task. An extensive literature is available on different aspects of traditional laminated composites and smart composite beams, plates, and shell structures. It is extremely difficult to compile all the references in one common platform. The objective of this chapter is to present the significant research carried out in the past which are relevant to the present study. In an attempt to present the available literature of the present research, the literature review is divided into the following categories:

- Elasticity Solutions
- Modeling of plates using Plate Theories
    - Development of the plate theories
        - Classical Plate theory
        - First Order Shear Deformation theory
        - Higher-Order Shear Deformation theories

- Extension of the plate theories for the modeling of multi-layered structures
    - Equivalent Single Layer (ESL) Approach
    - Layer Wise (LW) Approach
    - Zigzag (ZZ) Approach
- Carrera Unified Formulation (CUF)

❖ Solution Techniques

1.4.2. Elasticity Solutions (3 D)

The most efficient way of modeling the deformation responses of beams, plates, and shell structures to date is to start the formulation with the 3 D equilibrium equations (EE) of elasticity. However, these equations alone are not very useful as six unknown stresses are associated with three equations. Therefore the equations are converted in terms of strains and the resulting equations in terms of the displacements (*U*, *V*, and *W*) using the 3 D constitutive equations and the strain displacement relations, respectively. In this way, 15 unknowns are associated with 15 equations which now make the problem determinate. The resulting partial differential equations are then solved for the unknown displacements. Of course, we require the boundary conditions of the problem to assume mathematical solutions for the unknown variables. Significant contributions in this direction of deriving exact elasticity solutions of bending and vibration analysis of laminated composites and sandwich plates are presented by Pagano (1970), Pagano and Hatfield (1972), Srinivas and Rao (1970), Noor (1973) and Bhaskar *et al*. (1996). Additionally, the 3 D solutions of multifield problems mainly, the coupled electromechanical formulations of piezoelectricity are presented by Heyliger (1994), Ray *et al*. (1993, 1998), Mallik and Ray (2004), and Ray and Sachade (2006). The foremost reason for employing the elasticity formulations is that there are no approximations involved in any step which ensures that the responses obtained are free from any sort of numerical error. Numerical error creeps into the solutions from the approximate governing equations as well as approximate solution methods. In the above-mentioned works, the formulations are exact and the solutions are closed-form analytical solutions that exactly satisfy the boundary conditions and the PDEs at every point in the domain. It is important to note that the exact solutions are restricted to a particular geometry, lamination sequence, and boundary conditions. In all the above-mentioned references, trigonometric functions in the spatial domain (x, y) are assumed for the primary variables that exactly satisfy the diaphragm-supported boundary conditions. The governing PDEs are converted to ODEs in the thickness domain (z) with the assumed mathematical functions for the primary variables. However, assuming the initial functions for other types of boundary conditions is a difficult task. Another simplicity that is noticed in the above-mentioned references is that the trigonometric solutions assumed for the diaphragm-supported boundary conditions transform the PDEs to ODEs in the thickness domain with constant coefficients. This is achieved by collecting the coefficients of the trigonometric functions and equating the coefficients of the right-hand side and left-hand side of the PDEs. This might not be the case in the problems with complicated boundary conditions even if someone manages to assume solutions for the primary variables by satisfying the boundary conditions. After substituting the assumed solutions in the PDEs, there are possibilities that we may not be in a situation to equate the coefficients of the trigonometric functions at both sides of the PDEs and exclusively get the ODEs in the z-direction. Therefore, it is evident that the methodology used in the aforementioned references has limitations at least in the context of boundary conditions. Significant progress in this regard is due to Kapuria and Kumari (2012, 2013) and Kumari and Behera (2017) in which a different methodology is presented and the 3 D solutions for static and dynamic responses of traditional laminated composite plates and smart composite plates are derived for various boundary conditions using the Extended Kantorovich method (EKM). One set of a boundary is assumed to be diaphragm-supported while the other boundary set can have any conditions. The solution methodology is powerful in the sense that EKM converts the PDEs into two ODEs and then a closed-form solution or a numerical solution of the ODEs is feasible. Initially mathematical functions are assumed for the primary variables in an in-plane direction (x or y) with the simply-supported boundary conditions. The solutions for the other in-plane direction and the thickness-direction are then derived. This approach yields 3 D solutions at the cost of high computations. Due to the advancements in the theory of material science, many advanced materials are getting developed which have a rather complicated constitutive model than those of the traditional orthotropic materials. While deriving solutions in the framework of 3 D elasticity formulations for any general problem with arbitrary boundaries, geometries, constitutive model, lamination sequences, and loading conditions results in PDEs whose solution methods are prohibitively difficult and require heavy computations. Nevertheless, the exact solutions for the simpler cases can be considered as benchmark solutions for comparing the responses from approximate formulations and new solution methodologies. Apart from the elasticity solutions, there are semi-analytical or pseudo 3 D solutions for the bending responses of simply-supported traditional laminated composites and also for the multifield problems of thermoelasticity and piezoelectricity. Contributions in this direction are made by Kant *et al*. (2007, 2008), Pendhari *et al*. (2012, 2015), Sawarkar *et al*. (2016, 2020) and LomtePatil *et al*. (2018). Similar to the exact elasticity formulations, these formulations also start with the 3 D equilibrium equations of elasticity and the charge equilibrium equations depending on the type of problem considered. Using the constitutive relations and the strain-displacement equations, the PDEs are converted to a system of six first-order ODEs in the thickness direction corresponding to the primary variables. The three displacements and the three transverse stresses are considered as the primary variables in the works. Then the ODEs are solved numerically for the unknowns. The accuracy of the solutions presented in the aforementioned references is equally good as that of the elasticity solutions. The name, 'Pseudo 3 D' is because the solutions of the ODEs are obtained numerically in the final step in contrary to the closed-form solutions in the exact elasticity formulations. The mathematical difficulties encountered while carrying out these works are less than those of the 3 D elasticity formulations, yet the solution methodology is not very straightforward.

Relatively simpler approach for modeling the multi-layered laminated plate structures is by exploiting the geometry of the structures and somehow expresses the 3 D displacement variables in terms of mathematical functions in the thickness domain and unknown deformation modes defined in the spatial domain. Such an approach reduces the 3 D displacements to two-dimensional (2 D) deformation modes, subsequently a 3 D formulation to a 2 D formulation in the case of plate structures. Any structural system bounded by two planes and separated by a distance that is very small in comparison to the in-plane dimensions is called a plate. The fact of relatively small thickness in comparison to the in-plane dimensions helps in converting the 3 D structure to a 2 D midplane based on some approximations. Thus, it is now only required to model the deformation behavior of the midplane and obtain the solutions of the deformation modes. The known mathematical functions of the thickness coordinate should be carefully chosen so that the deformations are consistent with the actual responses of the system. In a nutshell, a theory is developed for deriving the deformation responses of the plates where the 3 D displacements of the plate are reduced to 2 D deformation modes at the midplane by expressing the displacements as a linear combination of the mathematical functions of the thickness domain and 2 D deformation modes. After solving for the deformation modes, we can extrapolate the displacements, stresses, strains at any point in the plate with the assumed mathematical functions. A detailed description of the various plate theories and their underlying assumptions is presented in the next section.

1.4.3. Modeling of plates using Plate Theories

The motivation behind the development of the various plate theories in the literature is the complicated theoretical formulations and solution methods associated with the 3 D approaches. In general, when the modeling of plates is accomplished using a plate theory, the formulation is such that the deformation modes defined at the midplane are the primary unknowns. The displacement components (U, V and W) are written in terms of known mathematical functions of the thickness coordinate and unknown deformation modes defined at the midplane. Once, the deformation modes are obtained, the stresses and strains at any point in the thickness domain can be derived. A flow-chart on the modeling of a plate structure in the framework of a plate theory is shown in Figure 1.4.

1.4.3.1. Development of the plate theories

The foremost hypothesis in the early development of the theory of plates is that the thickness of the plate is very small in comparison to the in-plane dimensions. For very thick systems, the 3 D elasticity formulations are essential for modeling. Based on the consideration of a thin plate, the following assumptions will prove to be useful for deriving a theory for the plate.

- The inplane 3 D displacements (U, V) vary linearly through the thickness of the plate. This assumption allows expressing the 3 D displacements of any arbitrary point '*P*' at a depth z from the midplane in terms of a constant deformation mode and a rotational mode defined at the midplane associated with a linear function (z) of the thickness coordinate.

- The transverse displacement (W) is constant through the thickness of the plate. This simply means that the transverse normal will not undergo any change in length in the thickness direction.

Under these assumptions, we can express the 3 D displacement components in the following manner.

$$U(x, y, z) \approx u_o(x, y) + z\theta_x(x, y)$$
$$V(x, y, z) \approx v_o(x, y) + z\theta_y(x, y)$$
$$W(x, y, z) \approx w_o(x, y) \qquad 1.1$$

where u_o, v_o and w_o are the constant deformation modes in the x, y and z-direction and θ_x, θ_y are the rotational deformation about the y and x-axis, respectively. The above deformation profile is also shown in Figure 1.5 for physical realization. Equation 1.1 is the displacement field of the first-order plate theory about which is described in detail in the subsequent sections. Based on this background, we are now in a position to introduce and discuss the first plate theory in the literature known as the Kirchhoff plate theory or the so-called Classical Plate theory.

1.4.3.2. Classical Plate theory

The Classical plate theory is a 2 D theory that is used to predict the deformation responses of thin plates only under the action of forces. It is known to be an extension of Euler-Bernoulli beam theory and the deformation of any point in the plate is based on the *Kirchhoff hypothesis*. The *Kirchhoff hypothesis* is stated as follows:

- Straight lines perpendicular to the mid-surface (transverse normal) shall remain straight under the deformation of the plate.

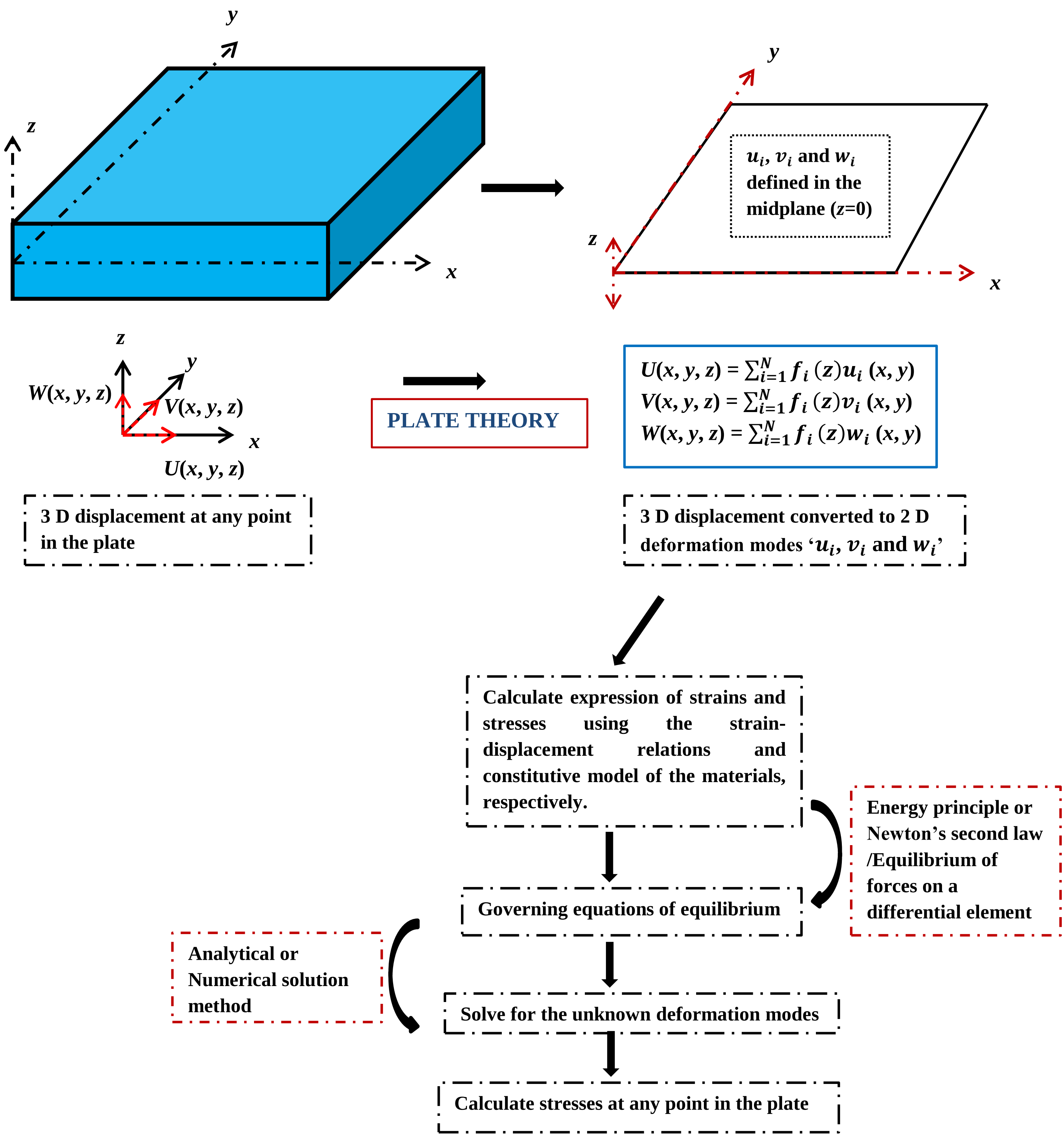

Figure 1.4. Flowchart on the modeling of a plate structure using any Plate theory

- The transverse normal does not undergo any change in length under the deformation of the plate.
- The transverse normal rotate in a fashion such that it shall remain perpendicular to the midplane under the deformation of the plate.

The first two assumptions have already been introduced earlier by which the displacement field in Eq. 1.1 is created. The first assumption clearly states that the in-plane displacements (U, V) have a linear variation across the thickness of the plate structures. It also means that the transverse cross-section to the midplane shall not warp. The second assumption states that the transverse deflection at any point along the thickness-direction will remain constant, *i.e*, no elongation or contraction of transverse normal takes place. If we look into the displacement field in Eq. 1.1, we see that the transverse deformation (w_o) and the rotations (θ_x,θ_y) are independent of each other. The third assumption is useful to establish a relationship between the transverse deformation and the rotational modes. The mechanistic meaning of the third assumption is that the transverse shear strains are zero as the angle in between the transverse normal and the midplane does not change during the deformation, also shown in Figure 1.6.

The expressions of the transverse shear strains based on Eq. 1.1 are shown below.

$$\gamma_{xz} = \left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right) = \theta_x + \frac{\partial w_o}{\partial x} \text{ and } \gamma_{yz} = \left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right) = \theta_y + \frac{\partial w_o}{\partial y} \qquad 1.2a$$

According to the third assumption, we can write that the transverse shears strains, γ_{xz}= γ_{yz} = 0. Thus Eq. 1.2a can be modified and written as follows:

$$\theta_x = -\frac{\partial w_o}{\partial x} \text{ and } \theta_y = -\frac{\partial w_o}{\partial y} \qquad 1.2b$$

We have now established a relationship between the rotational and the transverse degrees of freedom and have also reduced the total number of field variables from five (see Eq. 1.1) to three. Using Eq. 1.2b, the modified form of Eq. 1.1 is presented below.

$$U\,(x, y, z) \approx u_o(x, y) - z\frac{\partial w_o(x,y)}{\partial x}$$

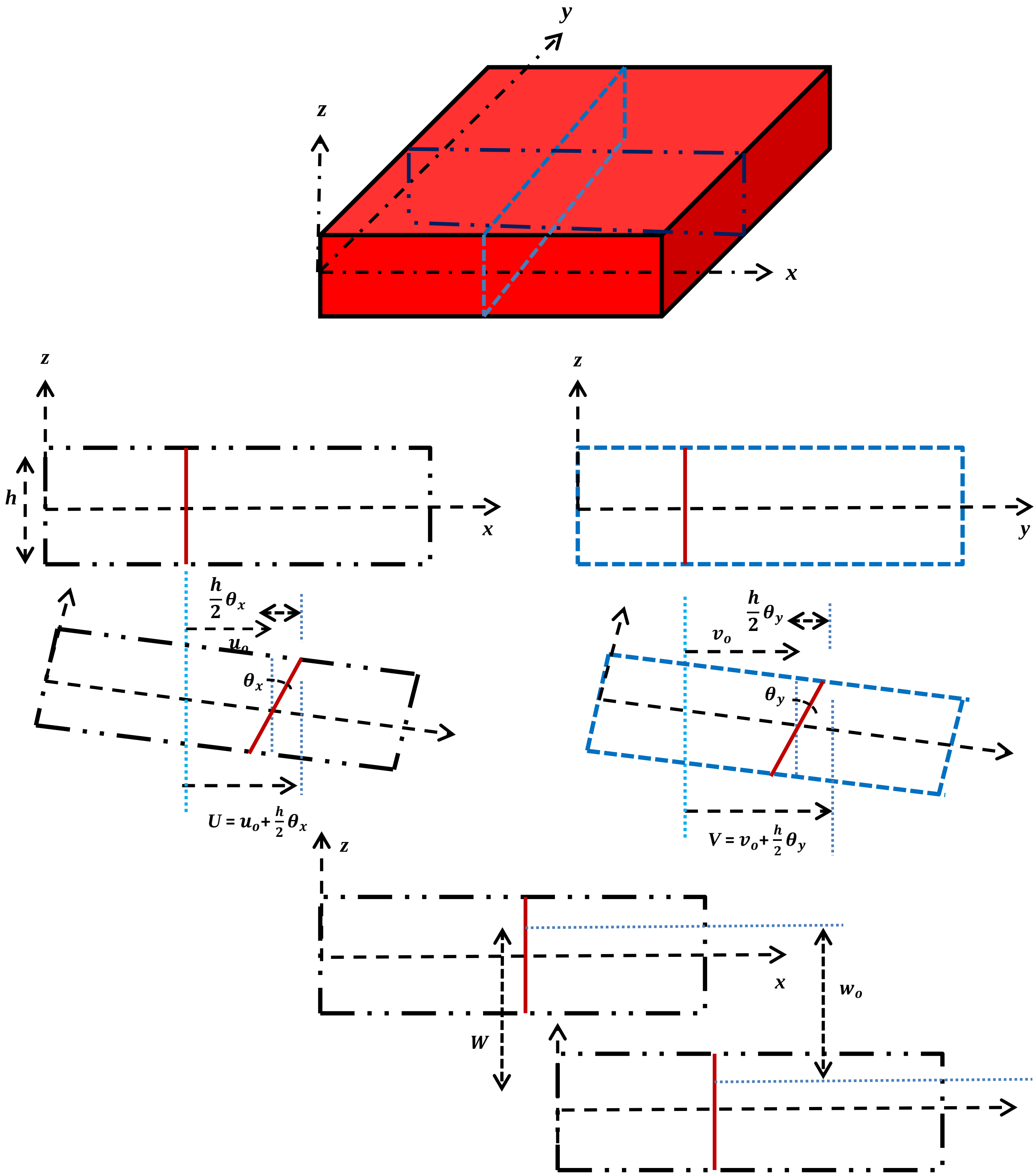

Figure 1.5. 3 D displacements (U, V, W) written in terms of the 2 D deformation modes (u_0, v_0, w_0, θ_x, θ_y) and linear mathematical function of the thickness coordinate

$V(x, y, z) \approx v_o(x, y) - z\frac{\partial w_o(x,y)}{\partial y}$

$W(x, y, z) \approx w_o(x, y)$ 1.3

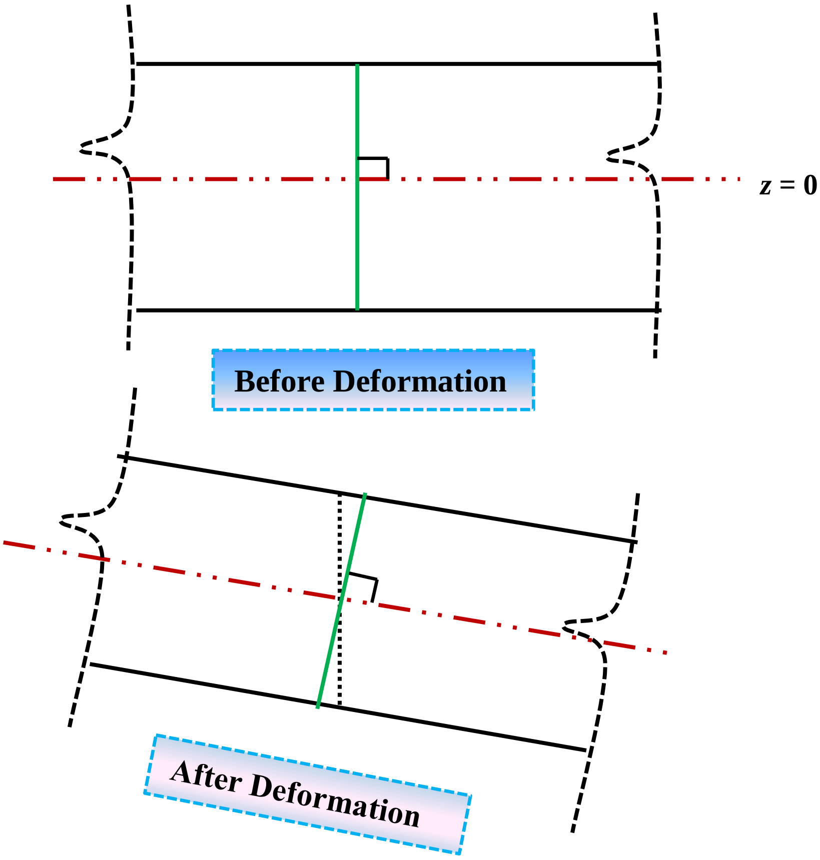

Figure 1.6. Transverse normal remaining perpendicular to the midplane during the deformation

The above equation, Eq. 1.3 is the displacement field of the so-called Classical Plate Theory (CPT). Ashton and Whitney (1970) used the CPT for modeling the deformation responses of laminated composite plates. The classical laminated plate theory (CLPT) is the simplest plate theory which is an extension of the CPT to laminated composite plates (Reissner and Stavsky, 1961; Stavsky, 1961; Dong *et al*., 1962 and Yang *et al*., 1966). As mentioned at the beginning of this section, CPT is used to predict the behavior of thin plates only. Therefore, under the condition of a thin plate (*Kirchhoff* *hypothesis*), the mathematical developments presented above will be meaningful. In the case of thick isotropic plate structures, the transverse shear strains are significant, and discarding the effects of the transverse strains will lead to erroneous responses. Also, in the traditional laminated composites and sandwich plates, the ratio of Young's modulus to the transverse shear modulus is very high. Therefore, it becomes very important to include the transverse shear strains for understanding the real behavior of composite plate structures. The refinement of the CPT is the first-order shear deformation theory (FSDT) which takes into account the effects of transverse shear strains without many theoretical complexities (Whitney and Pagano, 1970; Librescu and Reddy, 1987 and Reddy, 2004).

1.4.3.3. First Order Shear Deformation theory

All the assumptions that are taken in CPT hold in the FSDT except the third assumption which constraints the transverse normal to rotate in a fashion so that it remains perpendicular to the midplane during the deformation of the plate. The rotation of the transverse normal to the mid-plane according to FSDT is shown in Figure 1.7. We have already introduced earlier at the beginning of section 1.4.3.1 the displacement field of FSDT. Comparing Eq. 1.1 and 1.3, we now see that CPT is a special case of the FSDT when the total rotation of the transverse normal 'θ_x,θ_y' is equal to the negative slope of transverse displacement $\left(-\frac{\partial w_o}{\partial x}, -\frac{\partial w_o}{\partial y}\right)$ in the *x* and *y*-directions, respectively. Reissner (1944, 1945) has first provided a theory to incorporate the effect of shear deformation. The basic assumption made by Reissner (1944, 1945) gives a consistent representation of stress distribution across the thickness, which results in a through-thickness linear variation of in-plane displacements. Mindlin (1951) derived a theory using a displacement based approach at the same level of approximation as the one utilized by Reissner (1944, 1945). As per the Mindlin's theory, the transverse shear stresses are assumed to be constant in the thickness domain. The constant transverse shear strains can be observed in the expressions given in Eq. 1.2a. It is observed in the equation that the transverse shear strains are independent of the thickness coordinate, *i.e*, having a constant variation in the thickness domain. Consequently, the transverse shear stresses also become constant through the thickness of the plate structures. In reality, the stresses have a parabolic through-thickness variation in the thickness domain. The linear through-thickness variation of the in-plane displacements in Eq. 1.1 is the reason for constant shear deformations. The responses of FSDT can be improved using an indirect post-processing approach which is based on the multiplication of a shear-correction factor (SCF) with the transverse shear stiffness coefficients. The magnitudes of the transverse shear coefficients in the stiffness matrix can be adjusted to the results of the 3 D elasticity formulations by multiplying the coefficients with the SCF. However, the value of the SCF is unknown at the beginning of the formulation and can only be decided once the exact responses from the 3 D formulations are obtained. Therefore, the indirect post-processing approach has severe limitations for a more general problem with different material properties, loading conditions, boundary conditions and lamination sequences, etc. whose elasticity solutions are not available. In this regard, research articles are presented by Whitney (1973), Noor and Burton (1989), Pai (1995) and Meunier and Shenoi (1999) in which attempts are made to determine the appropriate values of the SCF using different methods and also establish the dependencies of the SCFs with the various conditions of the problem like the loading, boundaries and material properties, etc.

To have a more realistic model that can accommodate the non-linear nature of the transverse shear stresses without application of any indirect post-processing techniques, then the displacement field equation in Eq. 1.1 requires some modifications. The modification in the displacement field equation is brought by using Taylor's series expansion. The higher-order shear deformation theories (HSDTs) utilize the higher-order terms of Taylor's series expansion to accommodate higher-order modes in the displacement field of CPT and FSDT which are essential to refine the overall responses of the plate structures.

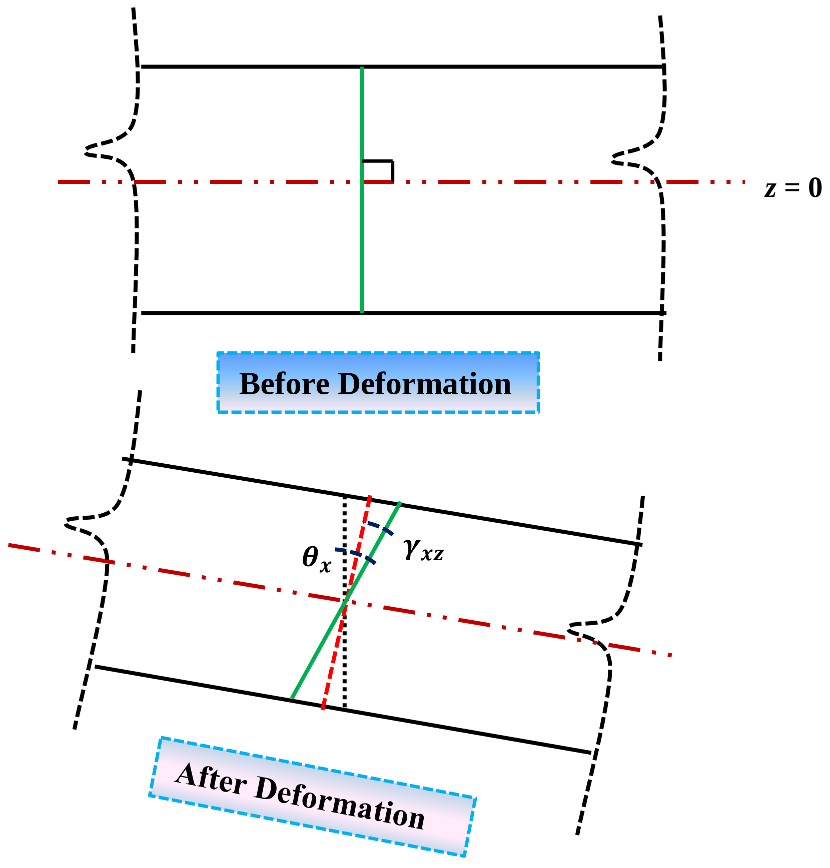

Figure 1.7. Rotation of the transverse normal to the mid-plane in FSDT

1.4.3.4. Higher-Order Shear Deformation theories

Koiter (1960) made a recommendation, popularly known as the Koiter's recommendation (KR) in his article of 2 D modeling of traditional isotropic elastic shells, which is stated as "*a refinement of Love's first approximation theory is indeed* *meaningless, in general, unless the effects of the transverse shear and normal strains (stresses) are taken into account at the same time*". There are numerous HSDTs to date which account for the effects of transverse shear strains only or both transverse shear and normal strains in the kinematic model that partially or completely follow the KR, respectively. The HSDTs are refinements of the CPT and FSDT which try to bridge the gap between the mathematical complexities in obtaining the exact responses from the 3 D formulations and the unsatisfactory performances of the CPT and FSDT for isotropic and traditional composite plate structures. In the literature, the HSDTs are available in the form of polynomial-based higher-order theories (PHSDTs) and non-polynomial-based higher-order theories (NHSDTs). The foremost difference among the two classes of theories is the use of polynomial and non-polynomial mathematical functions for expressing the displacement components in terms of the deformation modes. The idea of constructing the kinematic field of the PHSDTs came from the higher-order terms in the Taylor series expansion. Using the Taylor series, we can express any function '$f(x)$' in the following manner:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^{n} \quad 1.4$$

The pictorial presentation of expressing any function using the Taylor series is shown in Figure 1.8. It is observed in the figure as well as in Eq. 1.4 that the function '$f(x)$' can be expressed as a sum of linear and non-linear terms of *x*, *i.e*, higher-order terms. The higher-order terms shown in Figure 1.8 can be thought of as the higher-order deformation modes of the in-plane and transverse displacement components in HSDT. Theoretically, an infinite number of terms should be taken to express *U*, *V* and *W*. Practically, the series is truncated after taking a specific number of terms which are decided based on the accuracy of the responses. Using the same technique as shown in Eq. 1.4 and Figure 1.8, the displacement components *U*, *V* and *W* are written as

$$U(x,y,z) = U(x,y,0) + z\,U^1(x,y,0) + \frac{z^2}{2!}U^2(x,y,0) + \frac{z^3}{3!}U^3(x,y,0) + \ldots\ldots\infty$$
$$V(x,y,z) = V(x,y,0) + z\,V^1(x,y,0) + \frac{z^2}{2!}V^2(x,y,0) + \frac{z^3}{3!}V^3(x,y,0) + \ldots\ldots\infty$$
$$W(x,y,z) = W(x,y,0) + z\,W^1(x,y,0) + \frac{z^2}{2!}W^2(x,y,0) + \frac{z^3}{3!}W^3(x,y,0) + \ldots\ldots\infty$$

1.5

Eq. 1.5 is similar to Eq. 1.4, except that Eq. 1.5 is expanded in the *z*-direction and the value of 'a' in Eq. 1.4 is considered to be 0 in Eq. 1.5. U^1, U^2...V^1, V^2...W^1,W^2.... are the derivatives of the respective displacement functions with respect to *z*. Now, we would like to extract some physical interpretations and significance of the various terms in Eq. 1.5. $U(x,y,0)$, $V(x,y,0)$ and $W(x,y,0)$ are the constant variations of the 3 D functions, *U*, *V* and *W* at *z* = 0 in the *x*, *y* and *z*-direction, respectively. These modes simply represent the rigid body motion, *i.e*, gives a measure of the amount of constant axial deformation of any point in the plate in the *x*, *y* and *z*-direction. The constant deformation modes are important to include because when a homogeneous plate is subjected to direct uniform traction, then the constant deformation modes give a measure of the amount of stretching experienced by the plate. It is important to note that these deformation modes are not constant in the spatial domain and constant only in the thickness domain. While studying the bending behavior of non-homogeneous plate structures, the constant modes also contribute in the bending deformations as non-homogenous structures experiences coupled stretching-bending deformation under the action of transverse loads and vice versa. Therefore, these modes cannot be neglected even when no axial force is acting. $U^1(x,y,0)$, $V^1(x,y,0)$ are the derivatives of the in-plane 3 D displacement functions with respect to *z*, and simply gives a measure of the amount of rotation of the transverse normal about the *y* and *x*-axes at *z* = 0. The higher-order terms, '$U^2(x,y,0)$, $V^2(x,y,0)$' are the rate of change in slopes (curvature) of the in-plane 3 D displacement functions in the *x* and *y*-direction at *z* = 0, respectively.

Similarly, $W^1(x, y, 0)$ and $W^2(x, y, 0)$ are the slope and curvature of the 3 D transverse displacement function, *W* (*x*, *y*, *z*) in the *z*-direction. Further terms in Eq. 1.5 do not represent any physical quantity as such, however, required to refine the overall response of the system. The coefficient of the rotational deformation modes '$U^1(x, y, 0)$, $V^1(x, y, 0)$' is '*z*' and has a linear anti-symmetric variation about the *z* = 0 midplane. Such a deformation shape truly represents the linear bending of the system. It is mentioned above that the constant deformation modes represent the stretching phenomenon in the *x* and *y*-directions. The modes have a constant coefficient associated with them, which also confirms to the above physical realization. It can be argued that the two deformation modes (constant and rotation) are the basic deformations and the higher-order modes are simply included for refinement of the stretching and rotational modes.

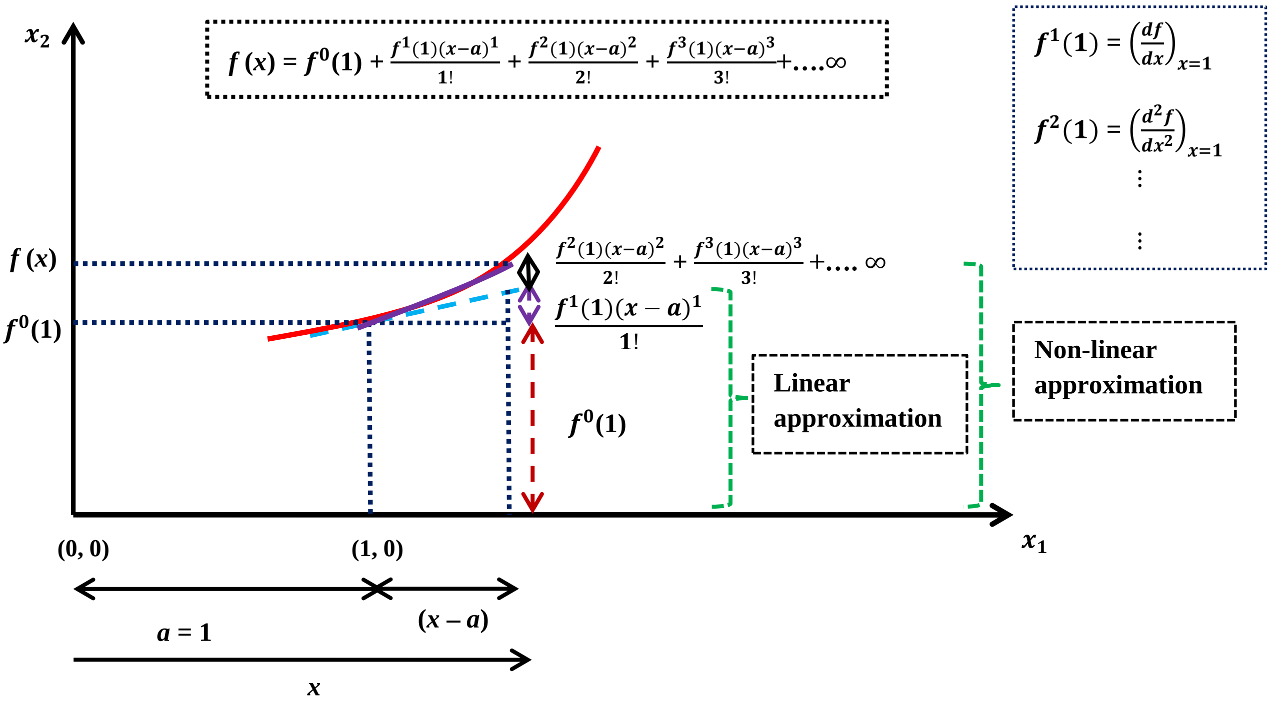

Figure 1.8. Expressing a function '*f* (*x*)' with the Taylor Series expansion centered at *x* = 1.

If we closely look at the coefficients of the higher-order modes, $U^2(x, y, 0)$ and $U^3(x, y, 0)$, it is observed that a quadratic and a cubic function of z is associated with them, respectively. Therefore the higher-mode '$U^2(x, y, 0)$' is refining the constant deformation mode '$U(x, y, 0)$' and consequently the overall stretching response of the system. The inclusion of this mode will now make the mathematical model capable of modeling a non-uniform stretch experienced by the plate. The $U^3(x, y, 0)$ mode is refining the simple linear rotational mode, '$U^1(x, y, 0)$' and consequently, the bending response of the system. The inclusion of this mode will make the mathematical model capable of modeling the non-linear bending profile of a plate structure, *i.e*, warping of the transverse cross-section. Therefore, it is now evident from the preceding discussions that the odd and even powered terms of z in conjunction with the higher-order modes in the in-plane displacement components are used for refining the bending and stretching of the system, respectively. While, the exact opposite phenomenon can be found in the expansion of the 3 D transverse displacement function, 'W' in which the odd and even powered terms of z contribute to the stretching and bending of the system, respectively. The rotation of the transverse normal in HSDTs is shown in Figure 1.9.

Hildebrand *et al*. (1949) were the first to present improved higher-order theories of plates/shells by expanding the displacement components with the help of Taylor Series expansion. Some more contributions are made by Nelson and Lorch (1974), Lo *et al*. (1977), Levinson (1980), Kant (1982), and Reddy (1984) by developing various HSDTs for determining the deformation responses of isotropic and orthotropic multilayered plate structures. In all the HSDTs, unknowns are introduced in the midplane with an additional increase in the power of the thickness coordinate. Another interesting way of reducing the 3 D displacement components to 2 D deformation modes is by employing a non-polynomial mathematical function of the thickness-coordinate. An example is shown below in which the 3 D displacement component, U (x, y, z) is expressed in terms of the deformation modes with the help of a polynomial or a non-polynomial function.

$$U(x, y, z) = u_o(x, y) - z\frac{\partial w_o(x,y)}{dx} + f(z)\, u_2(x, y)$$
$$W(x, y, z) = w_o(x, y) \qquad 1.6$$

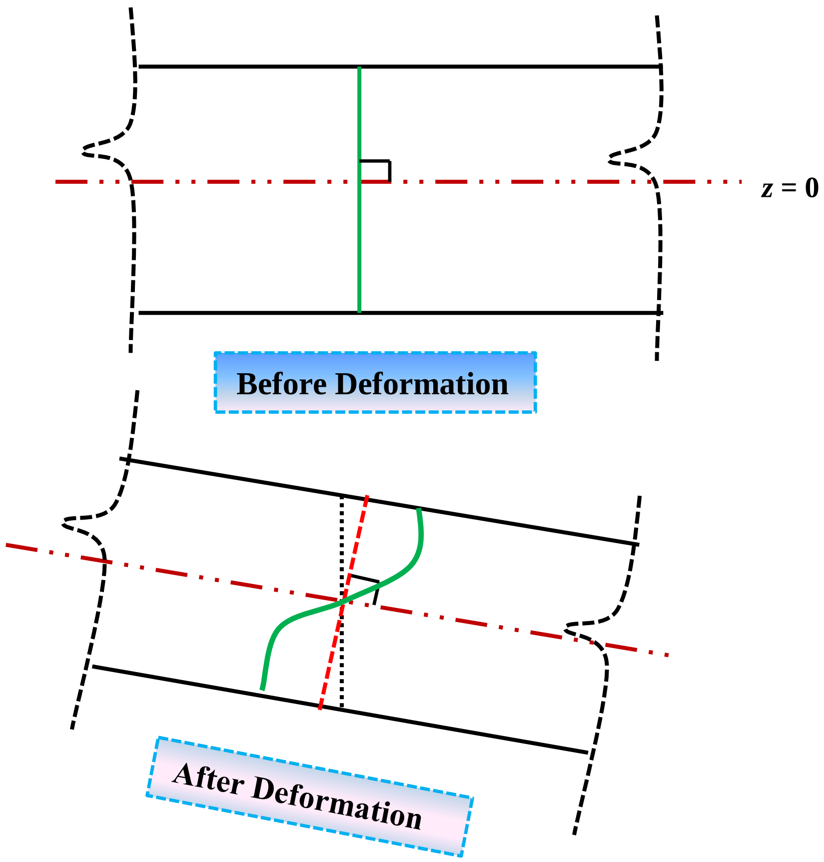

Figure 1.8. Rotation of the transverse normal to the mid-plane in HSDTs

We can see in Eq. 1.6, that the displacement field is a refinement of the CLPT, and the refinement is done with the help of the mathematical function '$f(z)$'. The choice of $f(z)$ can be a polynomial or a non-polynomial. If a polynomial function is used then the choices could be 'z^3' and $\left(z - \frac{4z^3}{3h^2}\right)$ so that the parabolic profile of the transverse shear strains can be attained. The first function is assumed with the help of Taylor-Series and the second function is given by Reddy (1984). Both the functions are cubic powered and can generate the quadratic variation of the transverse shear strains. There is an additional advantage of using Reddy's function as it automatically satisfies the traction-free boundary conditions of the transverse shear stresses/strains at the extreme surfaces of the plates. The through-thickness variations of the Reddy's function and its derivative with respect to z are shown in Figure 1.10. It is observed in the figure that $f(z)$ is zero at the mid-plane and maximum at the extreme surfaces and the derivative is maximum at the mid-plane and zero at the extreme surfaces of the plate. The variations are exactly similar to the through-thickness bending deformations and transverse shear strain variations in the plate structures. These variations can also be used as a basis for developing various non-polynomial mathematical functions of the thickness coordinate in HSDTs. Various non-polynomial mathematical functions are proposed to date by Touratier (1991), Soldatos (1992), Aydogdu (2009), Mieche *et al*. (2011), Mantari *et al*. (2012a), and Mahi *et al*. (2015) to name a few and implemented in the HSDTs for modeling the deformation responses of plate structures. The mathematical function, $f(z)$ is also known as shear-strain function in the literature. The various shear-strain functions adopted in the above-mentioned research works are collected in Table 1.1. Substituting for f (z) in Eq. 1.6 with the shear-strain functions in the table will create various non-polynomial higher-order shear deformation theories (NHSDTs).

Eq. 1.6 is just presented as an example and every NHSDTs would not exactly be in that form. It's just that we have shown the refinement over the CPT to create various NHSDTs and similarly, the refinements can be done over FSDT using the same shear-strain functions. The following conditions can be used as a basis for developing new shear-strain functions for the HSDTs.

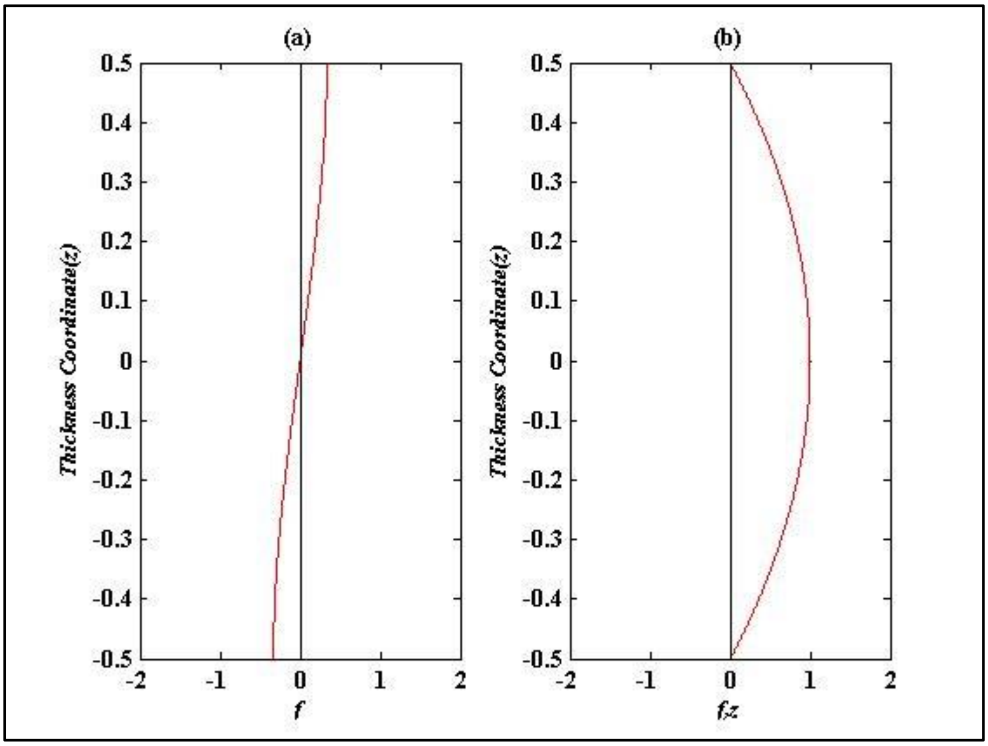

Figure 1.10. (a). Through-thickness variation of Reddy's shear strain function. (b) Through-thickness variation of the derivative of Reddy's shear strain function with *z*

Table 1.1. Various shear strain functions used in the kinematic models

Reference	Shear Strain function	
	$f\,(\mathrm{z})$	Ω
Touratier (1991)	$\frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)$	0
Soldatos (1992)	$h\sinh\left(\frac{z}{h}\right)$	$\cosh\left(\frac{1}{2}\right)$
Aydogdu (2009)	$z\alpha^{-2\left(\frac{z}{h}\right)2/ln(\alpha)}$	0
Mieche *et al*. (2011)	$0.6626\frac{h}{\pi}\sinh\left(\frac{\pi z}{h}\right)$	-0.6626
Mantari *et al*. (2012a)	$\tan(mz)$	$-m\sec^2\left(\frac{mh}{2}\right)$
Mahi *et al*. (2015)	$\frac{h}{2}\tanh\left(2\frac{z}{h}\right)-\frac{4}{3\cosh^2(1)}\left(\frac{z^3}{h^2}\right)$	0

- $f\,(0) = 0$ 1.7

The value of the shear-strain function at the mid-plane (z = 0) is zero. The shear-strain functions proposed by Touratier (1991), Soldatos (1992), Aydogdu (2009), Mieche *et al*. (2011), Mantari *et al*. (2012), and Mahi *et al*. (2015) satisfies this condition. It is shown earlier in Figure 1.10a that the through-thickness variation of the function '$f\,(z)$' represents the non-linear bending profile of the plate structure, *i.e*, useful for refinement of the overall bending response of the structure. From the theory of bending, it is known that when a homogeneous structure experiences bending under the action of transverse loads, the midplane does not undergo any change in length. Thus the value of the function '$f\,(z)$' is zero at the mid-plane. The same phenomenon may not happen in non-homogeneous plate structures and the plate can also experience axial deformation in addition to the bending deformation under the action of transverse loads. In that case, the constant deformation modes and the higher-order refinements can be used to model the axial deformations.

- $f^{/}(0)$ = 1 1.8a
- $f^{/}(\pm h/2)$ = 0 1.8b

The value of the derivative of the function at the mid-plane (z = 0) is 1 and at the extreme surfaces $(\pm h/2)$ is 0. The derivative of the shear-strain function is required while calculating the transverse shear strains. By satisfying equation 1.8a, we ensure that the transverse shear strains are maximum at the midplane, and zero at the extreme surfaces of the plate structures by satisfying equation 1.8b. In a more general sense, we wish to obtain the maximum value of the transverse shear strains/stresses at the mid-plane using Eq. 1.8a. All the shear-strain functions in the above-mentioned literature satisfy Eq. 1.8b. The functions in Soldatos (1992) and Mantari *et al*. (2012) do not satisfy Eq. 1.8a however, the results obtained are efficient. It is also important to note that the maximum value of the transverse shear stresses is not always at the midplane especially, in the two-layered antisymmetric laminated plates. Eq. 1.8b is very useful as it states that the extreme surfaces of the plate are free from transverse shear strains.

- $\int_{-\frac{h}{2}}^{\frac{h}{2}} f^{/}(z)\mathrm{dz} = 0$ 1.9

By satisfying Eq. 1.9, we ensure that the chosen shear-strain function is odd function. Eq. 1.7 constrains the shear-strain function to have a zero value at the mid-plane and Eq. 1.9 constrains the function to have an antisymmetric variation about the midplane. This is exactly similar to the bending profile of a structure. This completes the discussion on the HSDTs of both polynomial and non-polynomial nature. It can be said that there are only three different kinematic models available in the literature namely, CPT, FSDT and HSDT. The models were initially used for modeling the single-layered isotropic beams, plates and shell structures. Due to the advancements in the research on materials, advanced materials like composites, functionally graded materials, carbon-nanotubes, smart materials, etc. are developed and successfully applied in various industrial applications. It is found that these materials are often used as discrete layers stacked in the thickness-direction (laminated composites) or sandwiched in between some other materials (functionally graded sandwich structures, CNT reinforced sandwich plates). Therefore, it is now essential to use the kinematic models earlier developed for the single-layered structure to model the deformation responses of multilayered structures. This is generally done by using the concept of Equivalent Single Layer (ESL), LayerWise (LW) and Zigzag (ZZ) approaches. A more detailed discussion on these approaches is presented in the subsequent sections.

1.4.4. Extension of the plate theories for the modeling of multilayered structures

In this section, the various approaches of modeling the behavior of multilayered composite plates with the available kinematic models in the literature namely, CPT, FSDT and HSDT are discussed. The modeling of the traditional homogeneous multilayered plate structures, smart structures and non-homogeneous structures is considered.

1.4.4.1. Equivalent Single Layer (ESL) Approach

The simplest way of extending the single-layered plate theories to multilayered structures without many computational complexities is by using the Equivalent Single Layer approach. In this approach, the total number of primary variables required to model the multilayered systems is kept constant and is equal to the total number of field variables in the original kinematic model. Though the extension of the kinematic model from a single-layered system to a multilayered system is straightforward, however, the various integrals for evaluating the rigidity matrices should be correctly executed keeping in mind the thickness of each layer, material properties. In a nutshell, it can be said that a multilayered laminated composite plate is replaced by a single-layered plate structure in an integrated sense whose stiffnesses are a weighted average of the stiffness of each individual layer through the thickness. A pictorial presentation of modeling a multilayered laminated plate structure using CPT following the ESL approach is shown in Figure 1.11. This approach has been extensively used in the literature for modeling the structural responses of traditional laminated composite plates and smart composite plate structures.

Wang and Rogers (1991) studied the bending responses of laminated composite plates with integrated piezoceramic patches using the CLPT. Moita *et al*. (2004) presented numerical solutions for the dynamic responses of laminated composite plates with PVDF actuator and sensor using the finite element method (FEM) in the framework of CLPT. CLPT does not consider the shear deformation in plates which is significant in the laminated composite structures. In this regard, Reddy (1982) presented a comparison of the CLPT and FSDT responses for the forced-vibration responses of traditional laminated composite plates using the Navier-based analytical method. The displacement-time responses obtained using CLPT are underestimated in comparison to the FSDT responses.

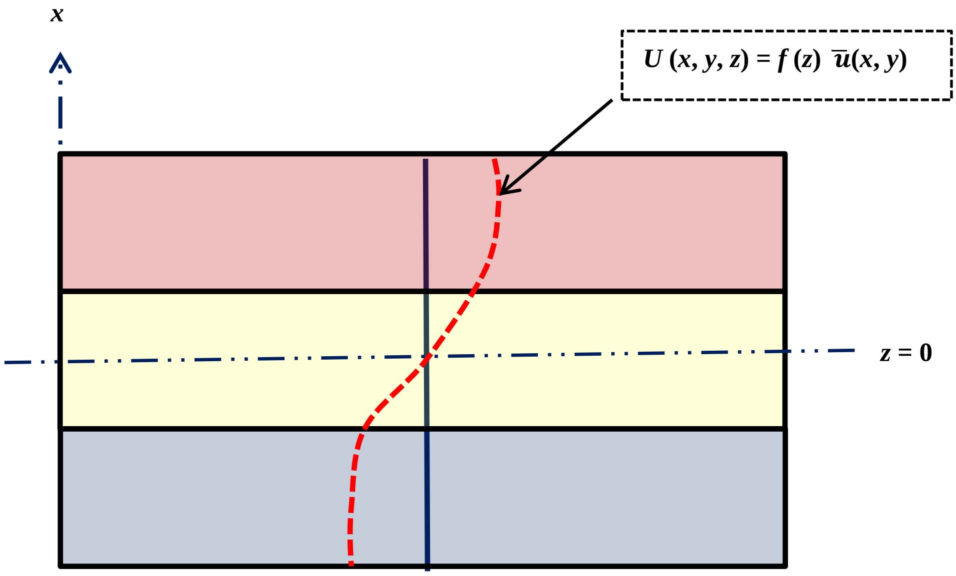

Figure 1.11. ESL representation of a plate theory

Chandrashekhara and Agarwal (1993) derived the vibration responses of smart composite plates with PZT patches in the framework of FSDT. Ray and Mallik (2004) derived the static responses of smart composite plate structures with PFRC actuator using a FE model based on FSDT. Later, Ray and Sachade (2006) extended the FE model for deriving the static responses of smart FG plate structures with a PFRC actuator. Shivakumar and Ray (2008) presented analytical solutions for the nonlinear static analysis of smart composite plates with PFRC actuator. Kerur and Ghosh (2011a, b) presented both linear and non-linear dynamic responses of smart composite plates with piezoelectric materials using FSDT. Behjat *et al*. (2011) carried out the static and dynamic analysis of smart FG plates subjected to various electrical and mechanical loadings using the FE method. The FSDT responses are satisfactory for thin systems only due to the linearly varying in-plane displacements across the thickness of the plate structures.

Lo *et al*. (1977) presented analytical solutions for the static analysis of traditional laminated composite plates using a PHSDT with eleven variables. The in-plane displacements and transverse displacement are assumed to have a cubic and quadratic expansion of the thickness coordinate, respectively. Reddy (1984) derived a five-variable PHSDT for the static analysis of laminated composite plate structures and also artificially imposed the zero transverse shear boundary conditions of transverse shear stresses at the top and bottom surface of the plate structure. Reddy's model discards the effect of the transverse normal strains as opposed to the PHSDT in Lo *et al*. (1977). In the literature, it can be found out that most of the HSDTs vary in the order of expansion of the in-plane and transverse displacement components. Kant and Swaminathan (2001, 2002) presented analytical solutions for the static and dynamic analysis of laminated composite and sandwich plate structures using various HSDTs with different order of expansions of the in plane and transverse displacements. Significant contributions are made by Phan and Reddy (1985), Khdeir and Reddy (1989), Pandya and Kant (1988), Kant and Manjunatha (1988) and Kant *et al*. (1992) to name but a few who presented static, free and forced vibration analysis of multilayered laminated composite plates. In addition to the above-mentioned PHSDTs, there is a two-variable PHSDT developed by Shimpi (2002) for deriving the bending responses of isotropic plate structures. The kinematic model is also extended for studying the deformation responses of laminated composite plates by Shimpi and Patel (2006). The proposed kinematic model does not consider the constant membrane deformations in the *x* and *y*-direction and with the membrane deformations, the total number of field variables increases to four. In all the aforementioned references, the nonlinear profile of the transverse shear strains/stresses are introduced with polynomial mathematical functions of the thickness coordinate. It is previously mentioned in the preceding section, that the nonlinear profile of the transverse shear strains/stresses can also be accommodated with non-polynomial functions. A non-polynomial function implicitly considers the higher-order terms of the Taylor Series expansion. Thus the number of primary variables in the kinematic model gets reduced in comparison to many PHSDTs in the literature like the HSDT of Lo *et al*. (1977), Kant and Manjunatha (1988), Pandya and Kant (1988), etc. In the framework of NHSDTs, several mathematical models are developed in the literature which exploits trigonometric functions (Touratier, 1991; Soldatos, 1992; Mantari *et al*., 2012a, 2012b and Grover *et al*., 2015), exponential (Karama *et al*., 2009 and Aydogdu, 2009) and hyperbolic (Mieche *et al*., 2011; Grover *et al*., 2013a; Mahi *et al*., 2015) for deriving the structural responses of multilayered composites and functionally graded plates.

In the area of smart composites, the PHSDTs and NHSDTs are also widely used for their analysis. Ray *et al*. (1994) and Samanta *et al*. (1996) employed the PHSDT of Lo *et al*. (1977) for determining the static and dynamic responses of smart composite plates with PVDF actuator and sensor. Lee *et al*. (2004) used Reddy's third-order theory (Reddy, 1984) for investigating the suppressed forced-vibration responses of smart laminated composite plates integrated with magnetostrictive layers. Further, Moita *et al*. (2005) utilized the same kinematic model for determining the suppressed forced-vibration responses of smart composite plates with PVDF layers. Shiyekar and Kant (2010, 2011) extended the model of Kant and Manjunatha (1988) for studying the electromechanical responses of smart composite plates with PFRC actuator. A series of articles are presented by Rouzegar and Abad (2015), Rouzegar and Abbasi (2017, 2018), and Rouzegar *et al*. (2019) in which the four-variable plate theory of Shimpi (2002) is utilized for modeling the coupled electromechanical behavior of smart composite and FG plates. The membrane deformation modes are included in the original mathematical model of Shimpi (2002) which increased the number of field variables to four. Further, Wang *et al*. (2019) employed the NHSDT of Neves *et al*. (2011) which considers the thickness stretching effects for the modeling of FG plates with PFRC actuator. Some recent investigations have appeared like Barati *et al*. (2016, 2017), Zenkour and Alghanmi (2019a, 2019b, 2021), Zenkour and Hafed (2020), Zenkour and Shahrany (2020) and Joshan *et al*. (2020) in which the static and vibration responses of smart plate structures have been investigated using various PHSDTs and NHSDTs.

The main drawback of the ESL approach is that the variations of the transverse shear stresses are not consistent with the 3 D solutions as the transverse shear stresses are discontinuous at the interfaces of the plates. The ESL theories cannot satisfy the piecewise continuity requirements of the transverse displacements as the kinematic expansions consists of global mathematical functions of the thickness coordinate. As a result, the transverse shear strain and stress fields are continuous and discontinuous at the interface of the plates. The variation of the in-plane displacement (*U*) for a thick composite plate in the exact solutions of Pagano (1970) illustrates that the in-plane displacements have slope discontinuities at the interface of the plate structure. To accommodate the slope discontinuities the kinematic expansions of the 3 D displacements are required to be modified. As a refinement, the layerwise (LW) and zigzag (ZZ) approaches are developed in the literature to eradicate the drawbacks of the ESL models.

1.4.4.2. Layerwise (LW) Approach

In the LW approach, we assume separate kinematic field expansions within each layer of the multi-layered laminated composite plate. The various kinematic models can be CLPT, FSDT and HSDTs of both polynomial and non-polynomial form assumed in each discrete layer. Even different orders of expansion of the displacement components in each discrete layer can also be assumed depending on the type of problem. The field variables of the kinematic model are layer dependent and increase with the increase in the number of layers. Practically the field variables of a particular layer '*k*' cannot be independent with the field variables of the '$(k+1)^{th}$' layer. The relationships among the field variables of the adjacent layers can be established with the continuity conditions of the displacement components assuming that the layers are perfectly bonded. This kind of kinematic representation can generate highly accurate through-thickness variations of the stress/strain field in the multi-layered plate structures. Also, the estimations of the ply level stress sufficiently improve when the LW approach is employed instead of the ESL approach. The LW representation of a kinematic field is shown in Figure 1.12.

Srinivas (1973) developed a refined plate theory by assuming that the in-plane displacements (U, V) are varying piecewise linearly through the thickness of the plate and the transverse displacement is assumed to be constant for the entire thickness domain. The idea behind this kinematic representation came from the 3 D analysis presented earlier by Srinivas and Rao (1973) and Pagano (1970) in which the in-plane displacements are found to be piecewise linear and the transverse displacement is more or less constant for both thick laminated composite and sandwich plates. Reddy (1989) presented a generalized theory for studying the bending of plates. Apart from the usual mid-plane deformations, some mathematical functions are also assumed in the thickness coordinate. The functions used are the standard 1 D Lagrange shape functions used for discretizing the thickness domain so that the continuity of displacement components is automatically satisfied. The functions assumed in the thickness domain generate additional unknowns in each layer, thus the primary variables are layer-dependent. Reddy *et al*. (1989) extended the generalized theory of Reddy (1989) for studying the deformation responses of laminated composite plate structures. The standard 1D Lagrange shape functions are used for discretizing the thickness-domain for the in-plane displacement components. The transverse displacement is assumed to be constant. Robbins and Reddy (1993) presented the modeling of thick laminated composite plates in the framework of the generalized laminate theory of Reddy (1989). The approach presented is a good alternative to the conventional 3 D FE model of plate structures. The variation of the transverse displacement is not constant through the thickness of the plate in contrary to the work of Reddy *et al*. (1989). Further applications of the generalized theory of Reddy (1989) are presented by Davalos *et al*. (1994), Nosier *et al*. (1993), and Barbero *et al*. (1990) for the analysis of laminated beams, plates and shells, respectively. Plagianakos and Saravanos (2009) developed a LW laminated theory in which a piecewise linear distribution of the in-plane displacement components are assumed for each layer in the thickness coordinate to take into account the continuity of displacements. Further, the in-plane distributions are assumed to vary in a quadratic and cubic manner across the thickness of each layer. Ferreira *et al*. (1991) presented the modeling of sandwich plate structures in the framework of the LW approach.

As opposed to the ESL model which assumes a constant rotation of the transverse normal for the entire laminated plate in the thickness-domain, the model in this research assumes independent rotation of the normal for each layer of the sandwich plate in the framework of FSDT. Goswami and Becker (2015) employed two PHSDTs with nine and eleven variables in the framework of the LW approach for modeling soft-core-sandwich plates

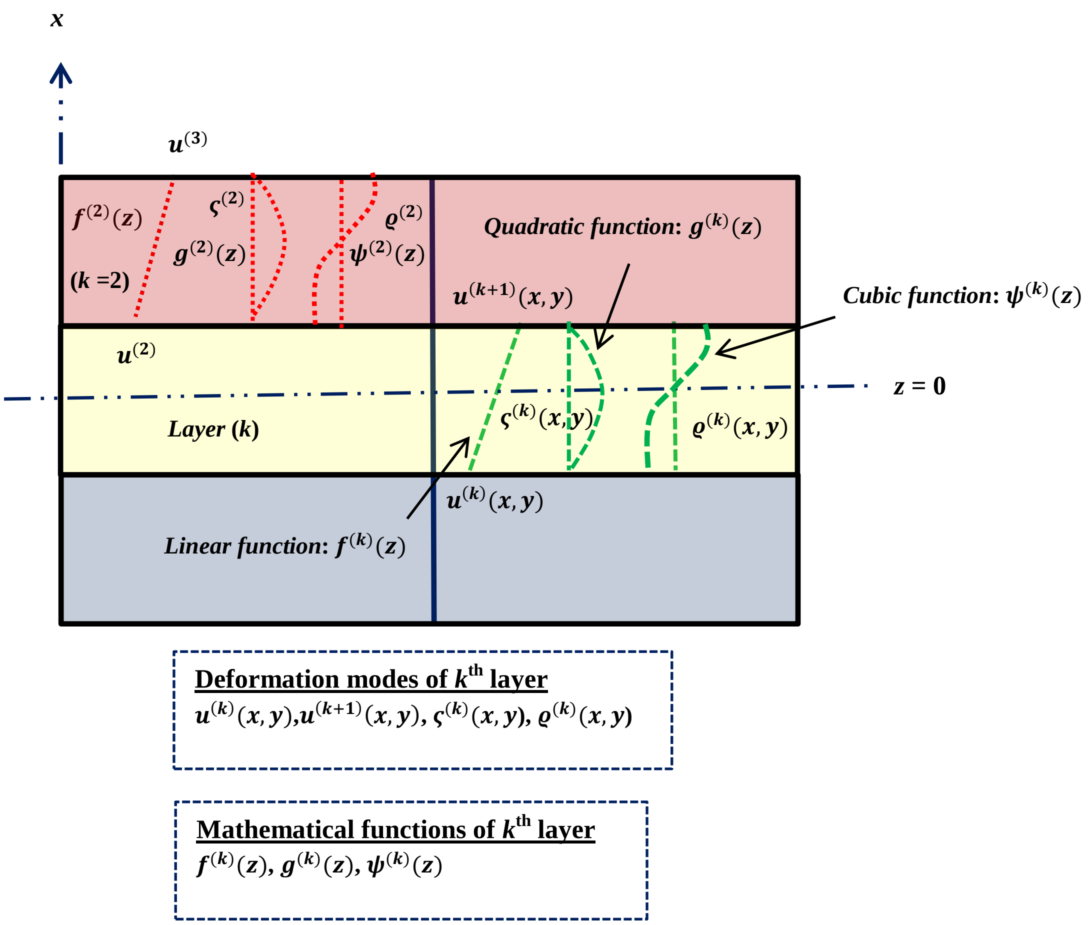

Figure 1.12. LW representation of a kinematic field

The kinematic description of the face-sheets is described with the nine-variable theory (Pandya and Kant, 1988) without taking into account the extensibility of the transverse normal and for the displacement components of the soft-core layer, the eleven-variable model (Lo *et al*., 1977) is assumed. This kind of hybrid kinematic representation can be used for modeling soft-core sandwich plates with thin face-sheets by taking into account the core-compressibility effect in the core layer only. Similar LW representation of transverse displacement only is also noticed in the works of Pandit *et al*. (2008) and Chalak *et al*. (2012a). Though the kinematic representation of the in-plane displacement components followed in the two works are based on the ZZ approach which is different from the LW approach used in Goswami and Becker (2015) yet these works share the same vision of assuming different orders of expansion for the transverse displacement in the face-sheets and the core for a soft-core sandwich system. Also, some recent research works like Cetkovic (2015), Mantari *et al*. (2012c, 2013), Pandey and Pradyumna (2015) and Sarangan and Singh (2018) have appeared in the literature in which the PHSDTs and NPHSDTs are used for modeling the laminated composites and sandwich plates using the LW approach. Saravanos and Heyliger (1995) presented a coupled LW analysis of smart composite beam structures with piezoelectric layers. Two LW theories are presented in which one assumes that the transverse displacement is constant through thickness of the plate while the other accounts for the piecewise continuous variations. Lee and Saravanos (1996) extended the same model to investigate the thermal effects in the coupled responses of smart composite plates with piezoelectric actuators. Robins and Reddy (1991) derived the dynamic responses of piezoelectric actuated beams using the LW theory of Reddy (1989). A comparison of responses obtained from the LW theories with the ELS-based classical beam theory (CBT) and the first-order beam theory (FOBT) is also presented in the work. Saravanos *et al*. (1997) presented the dynamic analysis of laminated composite plate structures with piezoelectric layers using the LW approach. The mechanical field variables and the electric field variables are assumed according to the LW representations across the thickness of the smart composite plates. Han and Lee (1998) employed the generalized LW theory of Reddy (1989) for modeling smart composite plates with distributed piezoelectric actuators. The vibrational analysis of the smart composite plate is presented in the work. Sheikh *et al*. (2001) carried out a FE analysis of smart composite plates using the FSDT model. However, the model included the electric potential distribution across the thickness of the plate with piecewise linear mathematical functions. The model presented is hybrid in the sense that the mechanical field variables are based on the ESL approach whereas the electric variables are based on the LW approach. Zabihollah *et al*. (2007) presented the vibration control of smart composite plates with piezoelectric layers using the LW theory of Reddy (1989). Moita *et al*. (2011a) presented the vibrational analysis of multi-layered smart sandwich plates using a refined model. The core layer is modeled using the FSDT while the thin piezoelectric layers are modeled using the CPT. Continuity conditions of the displacements are then artificially enforced to reduce the number of field variables. Further, Moita *et al*. (2011b) presented an improved model for the vibrational analysis of smart sandwich plates in which the deformation behavior of the core layer is modeled using the PHSDT of Reddy (1984) and the deformation of the face-sheets is modeled with CPT. The responses in the frequency domain and time-domain, are presented in the work. More articles like Kapuria *et al*. (2003), Garção *et al*. (2004), Lage *et al*. (2004), Kapuria and Hagedorn (2007), Beheshti-Aval *et al*. (2013), Naji *et al*. (2016, 2018) and Li and Shen (2018) have appeared in the literature in which the modeling of smart structures are carried out exploiting the LW approach.

In a nutshell, it can be found that there are three different categories of LW theories that are very popular among the researchers namely, Reddy's generalized LW theory (Reddy, 1989), Carrera's Unified Formulation (Carrera *et al*., 2016), and Ferreira's LW theory (Ferreira, 2005). The Carrera Unified Formulation (CUF) is not discussed here and a separate section has been made later in which detailed discussions on CUF are presented. In Reddy's LW theory, each layer of the multi-layered composite plate structure is modeled as a single layer and the continuity of the displacements is achieved through C^0 Lagrange shape-function in FE. On the other hand, CUF writes expansions for both displacement components and transverse stresses in the thickness direction for the top, bottom, and some other external surfaces of a *k*th layer. Legendre polynomials are sometimes considered to the mathematical functions in the thickness-domain and the parameters of the polynomials are obtained using the compatibility conditions at the interfaces. Ferreira's LW theory assumes that the deformation of each layer follows the kinematic field of FSDT and the transverse normal of each layer rotates independently, *i.e*, having a unique rotation of the transverse normal. The displacement continuity conditions at the interfaces are enforced at the beginning of the formulation. Most of the advancements in the LW theories are based on the above-mentioned LW theories. The LW representation provides an accurate description of the primary variables and the secondary variables in the thickness domain however, the computational efforts are very heavy, and finding solutions for multi-layered composites and smart composites become prohibitively expensive as the number of primary variables dramatically increases with the layers. Also, in the linear LW theories, the inter-laminar transverse stresses turn out to be constant and therefore, requires refinement in the discretization of the thickness-domain by taking higher-order Lagrange shape functions consequently, increasing the computational costs. In the next section, we are going to discuss the Zigzag (ZZ) representation of the plate theories.

1.4.4.3. Zigzag (ZZ) Approach

In the ZZ approach, the total number of primary variables is fixed like that of the ESL approach and contrary to the LW approach. To generate the slope discontinuities of the displacement components at the interfaces, some unknowns are assumed at the interfaces of the multilayered laminated composite pate structures in addition to the usual kinematic representation followed in the ESL approaches. The unknowns are related to the 3 D displacement components with a piecewise linear interpolation function of the thickness coordinate and are useful to create discontinuities of transverse shear strains. Therefore an opportunity is created of making the transverse shear stresses continuous at the interfaces. After enforcing the continuity conditions of the transverse stresses, the unknown variables can be expressed in terms of the primary variables defined at the mid-plane. Therefore, the total number of variables is equal to the number of primary variables defined at the mid-plane. Overall the approach expresses the 3 D displacements in terms of the 2 D deformation modes defined at the mid-plane in conjunction with some unknowns defined at the interfaces of the plate structures. The ZZ representation of a kinematic model is shown in Figure 1.13. The mathematical functions relating the 3 D displacements with the unknowns are piecewise continuous at the interfaces while the global mathematical functions assumed for the deformation modes are anyway continuous. Therefore, the displacement continuity is automatically satisfied. The continuity of the displacements is also satisfied in the ESL-based theories however, there are no provisions for generating continuous transverse stresses unless some post-processing techniques are applied for calculating the stresses. Moreover, the piecewise continuous variations of the in-plane displacement components observed in the 3 D results of laminated composite plates (Pagano, 1970 and Srinivas and Rao, 1970) cannot be achieved with the ESL approach only and some modifications are required in the kinematic representation as discussed above in this section.

The starting step of any analysis with a plate theory is assuming some through-thickness variations of the 3 D displacements. Thus the accuracy responses of the system depend on how close the variations are with the exact responses. Substantial research has been carried out to circumvent the problems faced in the LW theories. In this regard, Di Sciuva (1986) presented a refined first-order ZZ model in which the FSDT model is used in conjunction with some unknowns defined at the interfaces. The unknowns are further defined in terms of the deformation modes of FSDT after satisfying the continuity conditions of the transverse shear stresses at the interfaces.

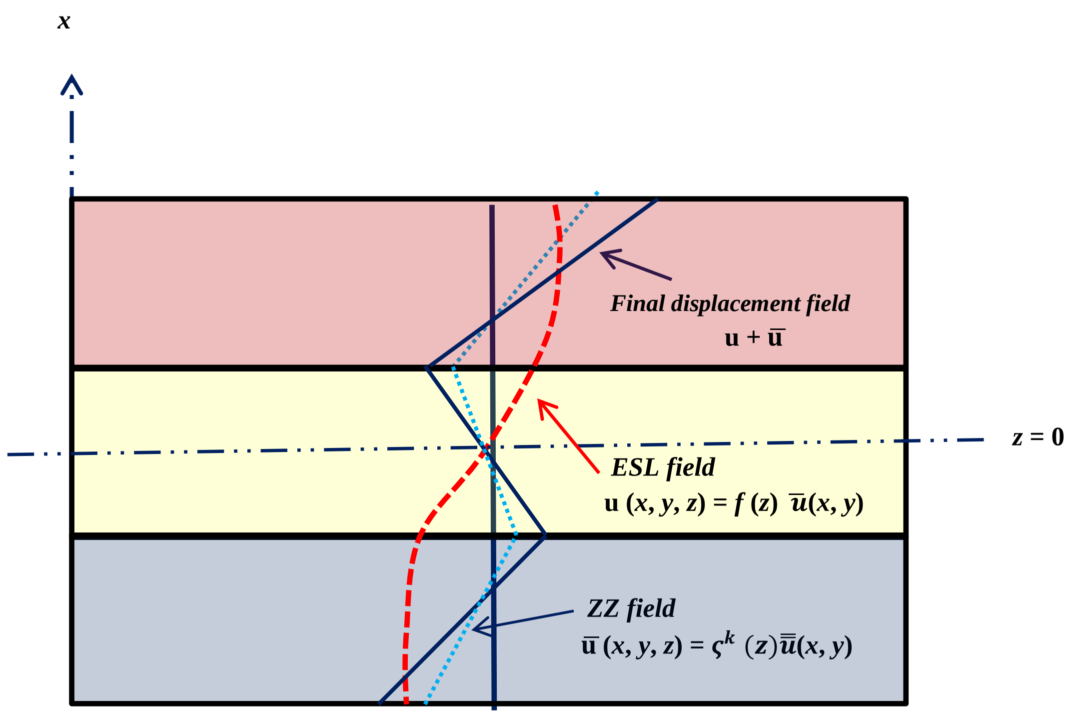

Figure 1.13. ZZ representation of a kinematic model

The first-order ZZ model is utilized in a variety of problems by Tessler *et al*. (2010), Gherlone (2013), Iurlaro *et al*. (2013) and Flores (2014). Further improvement in the first-order ZZ model is the refined higher-order ZZ theory (RHZZT) in which Reddy's third-order theory (Reddy, 1984) is utilized. Contributions are made by Di Sciuva (1992), Bhaskar and Varadan (1989) and Cho and Parmerter (1993) by developing RHZZTs for the analysis of laminated composite and sandwich plates. The RHZZTs satisfy the traction-free conditions of transverse shear along with the inter-laminar continuity conditions at the interfaces. Moreover, it also produces the parabolic profile of the transverse shear stresses across the thickness of the plate structures. The basic features of the aforementioned references are more or less the same, however, there are some modifications of one over the other. The model of Cho and Parmerter (1993) is further extended by many researchers for solving several problems of laminated composites and sandwich plates. To begin with, the extensive research works of Chakrabarti and Sheikh (2004a, 2004b) in which a refined triangular FE is proposed to address the problem of C^1 continuity of transverse displacement in the model of Cho and Parmerter (1993). Even the RHZZTs in Di Sciuva (1992) and Bhaskar and Varadan (1989) require the C^1 continuity of transverse displacement at the interface of the elements due to the presence of the second-order derivatives of transverse displacement in the strain components. Pandit *et al*. (2009, 2010a, 2010b) improved the model of Cho and Parmerter (1993) by including the core-compressibility effect. The FE modeling of soft-core sandwich plates is extensively carried out in these articles for the static and dynamic analysis including the random variations of the material properties. In these works, the C^1 continuous kinematic model is first transformed to a C^0 continuous model by enforcing some constraints. Then the constraint equations are variationally satisfied in the FE model by employing a penalty approach. More applications of the RHZZT are available in the articles of Chalak *et al*. (2012b, 2013) and Khandelwal *et al*. (2013a, 2013b) for the FE modeling of soft-core sandwich beams and plates including the core-compressibility effects. The above-mentioned references are all polynomial-based RHZZTs in which the nonlinearity of the transverse shear stresses/strains is introduced with the higher-order polynomial functions of the thickness coordinate.

Cho and Oh (2004) studied the thermo-electro-mechanical responses of smart composite plates using the RHZZT of Cho and Parmerter (1993) by taking into account the extensibility of the transverse normal. Oh and Cho (2004) carried out the FE modeling of smart composite plates for the coupled thermo-electro-mechanical analysis of smart composite plates using the RHZZT. Oh and Cho (2007) also extended the RHZZT for deriving the thermo-electro-mechanical responses of smart composite shells. Kapuria (2004) presented analytical solutions for the static analysis of piezoelectric laminated composite plates using a coupled third-order ZZ theory. Kapuria and Achary (2005) presented the dynamic analysis of piezoelectric laminated cross-ply plates using the coupled third-order ZZ theory. Kapuria and Kulkarni (2009) carried out the FE analysis of smart composites and sandwich plates using an improved quadrilateral element. The C^1 continuity requirements of the transverse displacement are circumvented using the improved discrete Kirchhoff constraint technique. Topdar *et al*. (2004, 2006, 2007) employed the RHZZT model of Cho and Parmerter (1993) and investigated the static and dynamic control of smart composite plates. A hybrid model is noticed in their analysis as the electric variables are expressed using the LW approach while the kinematics of deformation is represented using the ZZ approach. Similarly, Khandelwal *et a*l. (2013c, 2014) presented a hybrid mathematical model for the coupled electro-mechanical responses of smart composite plate structures by considering the LW and ZZ representation for describing the electrical and mechanical variables, respectively. More investigations on the smart composite plates and shell structures are available in Nath and Kapuria (2009, 2012), Mishra *et al*. (2019) and Das and Nath (2021) which have recently appeared in the literature. The above-mentioned ZZ theories used for modeling the smart systems are all polynomial-based.

The ZZ theories have received much attention, in particular, the polynomial ZZ theories because of their accuracy and less computational efforts in comparison to the LW models. However, the non-polynomial ZZ theories have not been extensively applied in the literature for modeling the traditional laminated composites and smart composite plates. It is observed in the literature that the non-polynomial functions are widely used in the ESL-based non-polynomial higher-order shear deformation theories (NHSDTs) for the structural analysis of beams, plates and shells. The non-polynomial shear strain functions in NHSDTs create additional advantages in comparison to the polynomial shear strain functions. To begin with, the number of primary variables gets reduced as with a single non-polynomial shear strain function the higher-order expansions of *z* can be implicitly accommodated. Also, the non-polynomial functions are much richer than the polynomial functions as shown below by comparing the shear-strain functions of Reddy (1984) and Touratier (1991).

Reddy's function (1984)

$$f(z) = \left(z - \frac{4z^3}{3h^2}\right) = z - \frac{4z^3}{3h^2}$$

Touratier's function (1991)

$$f(z) = \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right) = \frac{h}{\pi}\left(\frac{\pi z}{h} - \left(\frac{\pi z}{h}\right)^3 \frac{1}{3!} + \left(\frac{\pi z}{h}\right)^5 \frac{1}{5!} - \left(\frac{\pi z}{h}\right)^7 \frac{1}{7!} \ldots \ldots \infty\right)$$

When we expand the functions, we get the polynomial expansions of *z* for both the cases as shown above. We can see that Reddy's function has only two polynomial terms of *z* while the trigonometric function of Touratier contains an infinite number of odd terms of *z* therefore, much richer than Reddy's function. The other non-polynomial functions in the literature can also be expanded with infinite terms of *z*.

It is observed in the literature that the developments of the non-polynomial ZZ theories are mainly credited to Sahoo and Singh (2013, 2014) in which various non-polynomial shear-strain functions in the literature are employed, and the FE analysis of laminated composites and sandwich plates are carried out for the static and free vibration responses. Though the FE method can handle a wide range of complex problems, the solutions of the governing PDEs are approximate and the efficiency of the kinematic model cannot be ascertained with the FE solutions. In the FE method, the solutions for the primary variables are assumed over an element in terms of the interpolation functions and unknown generalized coordinates. The variational principle generates the governing equations in the algebraic form in terms of the generalized coordinates. After solving the algebraic equations, the generalized coordinates provide the magnitude of the primary variables at some of the discrete points on the elements, also known as the nodes. It is important to note that any plate theory in the literature is approximate as the 3 D displacements are reduced to 2 D deformation modes after making some assumptions. Based on the assumptions, suitable mathematical functions are chosen and the 3 D displacements are written in terms of the assumed functions and surface dependent variables. The efficiency of the solutions obtained from the plate theories depends on the accuracy of the mathematical functions, *i.e*, how consistent are the functions with the exact variations. Therefore some amount of error has already been initiated in the process of reducing the 3 D displacement to 2 D deformation modes. If the responses of the system are now obtained using FE in the framework of a plate theory, then the total error in the responses will be the cumulative error already incurred due to the initial approximations and some additional error generated due to the discretization, choice of the interpolation functions, and integration schemes adopted in the FE method, etc. In addition to this, the calculation of stresses in FEM involves additional approximation in the process and more error gets accumulated in the response. In general, the stresses are obtained at the gauss points of an element as the stiffness matrix is numerically evaluated at the gauss points by numerical integration in the isoparametric formulations. While comparing the stresses at the gauss points with the 3 D results of stresses, it is likely to be that the gauss points and the point in which the 3 D responses are evaluated are different. At this stage, either we compare the 3 D results with the stresses at the nearest gauss points or apply some method by which the stresses at the gauss points are extrapolated to the nodes. In the second case, some extrapolation functions are defined to extrapolate the stresses to the nodes. In the FE mesh, nodes are shared by the adjacent elements and it is found that the extrapolated stresses at the shared nodes from the adjacent elements are different and one needs to further apply a nodal averaging scheme to get unique stress at the nodes. There are methods discussed in Cook *et al*. (2007) like the nodal averaging methods and patch recovery by which the stresses at the nodes can be calculated. Though the stresses are now obtained at the nodes, yet the approximations involved in the methods invoke an additional error in the results. In the analysis of multilayered composites, the 3 D equilibrium equations of elasticity are useful to estimate the transverse shear stresses to get a better estimation compared to the stresses obtained using the constitutive model. While calculating the transverse shear stresses using the equilibrium equations (EE) of elasticity, one needs to first determine the results of the in-plane stresses accurately. As the results of the in-plane stresses have already some error due to the extrapolations and nodal averaging techniques, therefore the estimations of the transverse shear stresses are likely to be not efficient. Thus the use of the equilibrium equations for accurate estimations of the transverse shear stresses won't be useful anymore. It is now evident that the estimation of the stresses using the FEM involves more approximations in the steps and results in larger discrepancies than the results of the displacements. Therefore the efficiency of a plate theory cannot be fully ascertained with the FEM results only as the error accumulated in the responses is not only due to the assumptions in the plate theory but also due to the approximations in the method used to find the solutions. The analytical methods assume closed-form solutions for the primary variables that are valid for the entire domain and the solutions exactly satisfy the boundary conditions and the PDEs at every point in the domain. There are no additional approximations involved in the analytical method while calculating the displacements and the stresses. The error accumulated in the responses obtained analytically is entirely due to the assumptions in the plate theory and can exactly justify the accuracy of the kinematic model. It is therefore essential to find the analytical solutions for the structural analysis of laminated composite plates using the non-polynomial ZZ theories developed by Sahoo and Singh (2013, 2014). The solutions will serve as a basis for determining the accuracy of the proposed theories.

1.4.4.4 Carrera Unified Formulation (CUF)

A unified approach is presented in Carrera (2003), popularly known as the Carrera Unified Formulation (CUF) for solving a variety of problems on beams, plates and shells. CUF is very versatile as the formulation can be used in the ESL, LW and ZZ approaches. The stiffness matrix, mass matrix, and load vector are obtained in a unified manner with CUF irrespective of the type of the plate theory considered in the formulation. CUF also expresses the 3 D displacements in terms of shear-strain functions of the thickness coordinates and 2 D deformation modes defined at the midplane.

The various elements of a FE formulation, *i.e*, stiffness matrix, load vector, and mass matrix are obtained in terms of the so-called fundamental nucleus (FN). The FN for a let's say, bar element is written as follows:

$$K^{ij} = \int_v \left(N_{j,x} E N_{i,x}\right) dv \quad 1.10$$

It is important to note that the FN in the above equation is not limited to a particular number of nodes of the considered bar element and interpolation functions of the element. The stiffness matrix of a bar element with N number of nodes can be constructed with Eq. 1.10. The general form of the stiffness matrix or the FN is derived by assuming the axial deformation of a bar in the following manner.

$$u_x(x) = N_1(x)u_{x1} + N_2(x)u_{x2} + N_3(x)u_{x3} + N_4(x)u_{x4} + \ldots\ldots\ldots\ldots\ldots N_{Nne}(x)u_{xNne} \quad 1.11a$$

The virtual variations of the deformation is written as

$$\delta u_x(x) = N_1(x)\delta u_{x1} + N_2(x)\delta u_{x2} + N_3(x)\delta u_{x3} + N_4(x)\delta u_{x4} + \ldots\ldots N_{Nne}(x)\delta u_{xNne} \quad 1.11b$$

In a condensed form, the above equations are expressed as

$u_x(x) = \sum_{i=1}^{Nne} N_i(x) u_{xi}$ and $\delta u_x(x) = \sum_{j=1}^{Nne} N_j(x) \delta u_{xj}$ 1.11c

where 'i' denote the displacement, 'j' denote the variations, Nne is the number of nodes in an element and N_i denote the shape functions. Using Eq. 1.11c, the strains are calculated with strain-displacement relations and with the virtual variation of the internal work, the FN in Eq. 1.10 is derived. In the above formulation, the deformation of the bar is assumed to be constant in the y and z-direction. However, if the deformation in the y and z-direction is not constant, then we need to modify Eq. 1.11a and the modified deformation is written as $u_x(x, y, z) = N_i(x) F_\tau(y, z) u_{xi}$, where $F_\tau(y, z)$ is some function that gives the variation of the axial deformation of the bar on the cross-section. Furthermore, we now introduce the two more displacement components in the y and z-direction as structural engineering problems are generally 3 D and the solution of the problem is aimed towards evaluating the deformation of any point along the three coordinate axes.

$u_x(x, y, z) = N_i(x)\, F_\tau(y, z)\, u_{xi}$
$u_y(x, y, z) = N_i(x)\, F_\tau(y, z)\, u_{yi}$
$u_z(x, y, z) = N_i(x)\, F_\tau(y, z)\, u_{zi}$ 1.12

The above deformation field in Eq. 1.12 can be easily extended to the plate and shell formulations. For 2 D domain, $N_i = N_i(x, y)$ and $F_\tau = F_\tau(z)$ and for a 3 D domain, $N_i = N_i(x, y, z)$ and $F_\tau = 1$. The FN can now be obtained using the variation of the internal virtual work and it is interesting to note that FN is a (3x3) array in the case of stiffness matrix and (3x1) vector for the force vector and its form is invariant to the 1 D, 2 D and 3 D formulations.

The same analogy can be extended for constructing a plate theory. According to CUF, the deformation of any point inside the plate (2 D) can be written as

$\tilde{u}(x, y, z) = \bar{F}_s(z)\ \tilde{u}_s(x, y)$ 1.13a

The corresponding variations of the above deformation field is expressed as

$$\delta\tilde{u}(x, y, z) = \bar{F}_{\tau}(z)\ \delta\tilde{u}_{\tau}(x, y) \quad 1.13b$$

where, $\tilde{u}$ is the vector containing the 3 D displacements and $\tilde{u}_s$ contains the 2 D deformation modes in the x, y and z-direction. 's, τ' are subscripts that can take the value 0,1.... M. M is the order of the expansion. The displacements can be explicitly written as

$$u_x = \bar{F}_0(z)\ u_{x0} + \bar{F}_1(z)\ u_{x1} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \bar{F}_N(z)\ u_{xn}$$
$$u_y = \bar{F}_0(z)\ u_{y0} + \bar{F}_1(z)\ u_{y1} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \bar{F}_N(z)\ u_{yn}$$
$$u_z = \bar{F}_0(z)\ u_{z0} + \bar{F}_1(z)\ u_{z1} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \bar{F}_N(z)\ u_{zn} \quad 1.14$$

$\bar{F}_0(z)$, $\bar{F}_1(z)$......$\bar{F}_N(z)$ are the shear-strain functions and can be polynomial or non-polynomial. The polynomial function of the Taylor series expansion can be used to substitute for the shear-strain function in Eq. 1.14

$$u_x = u_{x0} + zu_{x1} + z^2u_{x2} + \ldots\ldots\ldots\ldots\ldots\ldots.. z^N u_{xn}$$
$$u_y = u_{y0} + zu_{y1} + z^2u_{y2} + \ldots\ldots\ldots\ldots\ldots\ldots.. z^N u_{yn}$$
$$u_z = u_{z0} + zu_{z1} + z^2u_{z2} + \ldots\ldots\ldots\ldots\ldots\ldots.. z^N u_{zn} \quad 1.15$$

Similarly, non-polynomial functions can also be used as the shear-strain functions to construct the displacement field using CUF.

$$u_x = u_{x0} + zu_{x1} + \cosh(z)u_{x2} + \sinh(z)u_{x3} + \cosh(2z)u_{x4} + \sinh(2z)u_{x5} + \ldots\ldots$$
$$u_y = u_{y0} + zu_{y1} + \cosh(z)u_{y2} + \sinh(z)u_{y3} + \cosh(2z)u_{y4} + \sinh(2z)u_{y5} + \ldots\ldots$$
$$u_z = u_{z0} + zu_{z1} + \cosh(z)u_{z2} + \sinh(z)u_{z3} + \cosh(2z)u_{z4} + \sinh(2z)u_{z5} + \ldots\ldots. \quad 1.16$$

The fundamental nucleus can now be obtained with Eq. 1.15 or 1.16, strain-displacement relations, stress-strain constitutive model and the principle of virtual work. Carrera (1995) derived a class of 2 D models using CUF which is probably the first paper among the various research works on CUF that have addressed the issues related to 2 D problems. CUF is used to derive ESL-based models (Carrera, 1996), LW-based models (Carrera, 1999a; 1999b) for the analysis of layered structures. Carrera (2004) used the Murakami ZZ function (Murakami, 1984) in the framework of CUF for studying the deformation responses of layered structures. The thermal problems of multilayered composite structures are presented in Carrera (2005) and Robaldo *et al*. (2005) using CUF. CUF is extended to multiphysics problems of piezoelectricity by Robaldo *et al*. (2006) and Carrera *et al*. (2007). Carrera *et al*. (2008) and Brischetto *et al*. (2008) applied the CUF for modeling FG structures. Cinefra *et al*. (2012, 2013) employed the CUF for modeling the homogeneous and non-homogeneous shell structures. Several articles like Filippi *et al*. (2015), Moleiro *et al*. (2020), Natarajan *et al*. (2014a), Ferreira *et al*. (2013), Rodrigues *et al*. (2011) and Alesadi *et al*. (2017) have appeared in the literature which reflects the applicability of CUF in modeling beams, plates and shell structures.

1.4.5. Solution Schemes

In this section, various solution techniques are discussed which are frequently used for finding the solutions of the governing equations. The Navier-based analytical method is very popular in the research community as it produces exact solutions of the governing equations of beams/plates and shell structures with diaphragm supported boundary conditions. There are several articles in the literature like Kulkarni *et al*. (2015), Punera and Kant (2017), Singh and Sahoo (2020), and Soni *et al*. (2020) in which the Navier-based solutions are obtained for the static and dynamic analysis of multilayered composites, FG and CNT-reinforced plate and shell structures. Navier's solution is restricted to diaphragm supported boundary conditions only. Other analytical solution methods like the Galerkin method (Singh and Harsha, 2019 and Daikh and Zenkour, 2020), power series solution method (Shariyat and Alipour, 2013 and Alipour, 2016), and Ritz method (Aydogdu, 2005 and Nguyen *et al*., 2017) are available in the literature in which solutions for different boundary condition are obtained analytically.

Khdeir and Reddy (1989, 1999) presented the free vibration and the forced-vibration responses of laminated composite plates using the state-space approach. A levy-type boundary condition is assumed in one direction and the assumed solutions for the primary variables reduce the system of PDEs to a system of higher-order ODEs in the other direction. Then the ODEs are further converted to a system of 1st order ODEs using the state-space approach and then solved to get the responses. There are many popular numerical approaches adopted in the literature for solving the governing equations of beams, plates, and shell structures. The main principle of any numerical approach is to reduce the governing PDEs and ODEs to a system of algebraic equations by making some approximations. This reduction helps to replace a continuous differential equation having a solution space that is infinite-dimensional with a finite system of algebraic equations whose solution space is now finite-dimensional. To begin with, the very popular and commonly used method in almost all disciplines of science and engineering, the finite element method (FEM). In the FEM, the field variables are assumed over an element as a linear combination of the polynomial shape functions and the nodal coordinates. The strong form of the governing equations is converted to an equivalent weak form and the assumed solutions are plugged in the weak form to get the elemental level equations. The elemental equations are then assembled to get the global discretized equations of the problem and then solved for the primary variables. Research works on structural analysis of homogeneous and non-homogeneous structures are available in Talha and Singh (2010), Bhar *et al*. (2010), Natarajan *et al*. (2012a, 2014), and Sarangi *et al*. (2014). The Extended Finite Element Method (XFEM) is another numerical method based on the FEM and it is especially used to treat crack discontinuities. The background of XFEM is the partition of the unity concept. By taking the advantage of the partition of unity concept, the FE approximation space is enriched with some enrichment functions and extra degrees of freedom in the nodes near the cracks. Natarajan *et al*. (2011), Nguyen-Vinh (2012), and Nasirmanesh and Mohammadi (2015) employed the XFEM to predict the dynamic responses and buckling of cracked composite and FG plates. The Isogeometric analysis (IGA), a modern approach in FEM, is also extensively used in many engineering problems where the geometry of the structure is complex. The approximation error in the mesh reduces significantly when IGA is employed in comparison to the conventional FEM as the geometry of the problem is accurately defined in IGA. Other than that, the problem with strong singularities and discontinuities can also be handled with IGA. The Non-Uniform Rational B-Splines (NURBS) are used in IGA as the basis functions for describing both geometry and the field variables like the isoparametric concept is FEM. NURBS are mainly used in computer graphics for representing complex 3 D geometry with arc, circle or curves, etc. Natarajan *et al*. (2012b), Thai *et al*. (2015), Phung-Van *et al*. (2017) and Gupta and Ghosh (2019) presented the static and dynamic analysis of FG plates, traditional laminated composites, and smart composite plates using IGA. The scaled boundary finite element method (SBFEM) is another numerical approach that is based on the FEM and boundary element method (BEM). In the conventional BEM, the boundaries of the domain are only discretized, thus the spatial dimension is reduced to one. This also leads to a reduction in the total number of degrees of freedom. In the SBFEM, the conventional coordinate system is transformed to a scaled boundary coordinate system in the radial and circumferential directions. The governing PDEs are reduced to a set of ODEs and the numerical solution is obtained in the circumferential direction using FEM or Meshless methods (He *et al*., 2012) while a smooth analytical solution is obtained in the radial direction. Song (2009) used the SBFEM for solving problems in structural dynamics. Further, Garg *et al*. (2020) extended the SBFEM for the modeling of laminated composite plates with weakly bonded interfaces. The conventional FEM requires that the mesh is frequently refined as the crack propagates in a crack propagation problem along with a conforming mesh. This is a limitation that is posed by the FEM when crack propagation problems are studied. As discussed above, the enrichment technique such as the XFEM proves to be very useful because the method does not require conforming mesh and mesh adaptation when the discontinuities, *i.e*, crack propagates. However, the enrichment functions near the cracks should be known a priori. In this regard, Natarajan and Song (2013) combined XFEM with the SBFEM to circumvent the need to know a priori the enrichment functions. Natarajan *et al*. (2014c) used the SBFEM in polygonal elements to model the crack propagation problems. Further, Li *et al*. (2013) employed the SBFEM for modeling fracture problems in piezoelectric materials. Recently, Natarajan *et al*. (2015) combined the IGA and SBFEM for studying fracture mechanics problems. Ray (2019) carried out the static analysis of smart composite plates using a Hybrid-Trefftz FEM. In the Trefftz method, some functions are derived known as the Trefftz function which satisfies the governing PDEs of the element. The final solution is the sum of the linear combination of the Trefftz function and the particular solution of the governing equations. Further, Ray and Dwibedi (2020) employed the Hybrid-Trefftz FEM for deriving the static responses of antisymmetric and angle-ply laminated composite plates. The Differential Quadrature Method (DQM) is employed by Alibeigloo and Liew (2015), Brischetto *et al*. (2016), Liu *et al*. (2016), and Sharma and Parashar (2016) for the modeling of smart structures, laminated composites, FG, and CNT-reinforced plate structures. Like any other numerical approach, the governing differential equations are transformed to a set of algebraic equations in terms of the discrete values of the field variables in DQM. This is achieved by expressing the derivatives of the primary variables at each grid point in a particular direction as the weighted linear sum of the values of the primary variables at all the discrete points in the same direction. The Discrete Singular Convolution (DSC) approach is employed by Civalek (2007, 2017) for the free vibration analysis of laminated composites and FG plates and shells. The method is somewhat identical to the DQM, and in DSC also, the derivatives of the primary variables at a grid point are approximated by a linear sum of the discrete values of the primary variables and approximation kernels in a narrow bandwidth.

1.5. Motivation and Literature Gap

Based on the literature survey, it is perceived that the ESL-based plate models are widely used to derive the structural responses of laminated composites and sandwich structures. ESL models are computationally easy to implement, however, the models cannot satisfy the piecewise continuous displacement requirements. Further, the transverse strain/stress fields are also not accurately represented. The LW and ZZ approaches are observed to eradicate the drawbacks of the ESL models. In the LW models, the number of primary variables dramatically increases with the increase in the number of layers, making it computationally very expensive. In the ZZ models, the displacement fields are developed by combining a global FSDT or HSDTs with a local ZZ function. The stress continuity conditions are then imposed at the interfaces of the plates which generate some relations between the local and global variables. As a result, the number of primary variables does not increase with the increase in the number of layers of the plates. In most of the ZZ models, the global HSDTs are of a polynomial type, and the application of non-polynomial HSDTs in the ZZ models is very rare in the literature. In the non-polynomial HSDTs, a single non-polynomial function is incorporated to model the non-linear variations of the transverse shear strains, resulting in a decrease in computational costs. Also, the ESL-based non-polynomial HSDTs have shown better performance over the polynomial HSDTs by predicting more accurate structural responses of the composite structures. The non-polynomial HSDTs when used in the framework of the ZZ approach can produce more accurate responses at the cost of moderate computational efforts.

Sahoo and Singh (2014) developed a non-polynomial ZZ model with a secant function for modeling the static responses of laminated composites and sandwich plates in the framework of the finite element method (FEM). The FEM has been firmly established in the literature as the most popular method for solving a variety of problems. However, the solutions are not free from numerical error as FEM is an approximate method. To produce responses that are free from numerical error, closed-form analytical solutions are required to be obtained for laminated composites and sandwich plates with the non-polynomial ZZ model in Sahoo and Singh (2014). These solutions would serve as benchmark results for verifying the accuracy of the solutions obtained with other numerical methods in the framework of the proposed non-polynomial ZZ model.

Smart composite structures are widely used in aerospace, mechanical and civil industries because the smart composites when coupled with a control strategy develop self-monitoring and self-controlling capabilities. The deflection, stresses and vibrations of the structures can be controlled by integrating traditional composite plates with piezoelectric materials. Similar to the laminated composite plate structures, the smart composite plates with piezoelectric materials are also multilayered structures. Therefore, it is essential to model the behavior of smart composites in the framework of the ZZ approach for producing efficient responses for both thick and thin systems. There are very few studies in the literature where the static responses of smart composite plates are modeled in the framework of ZZ kinematics. The efficiency of the newly developed non-polynomial ZZ theory of Sahoo and Singh (2014) is not yet utilized in the literature for modeling the coupled electro-mechanical responses of laminated composite plates with piezoelectric materials.

In the ESL models, the through-thickness variations of transverse shear stresses produce discontinuous stress values at the interfaces, and the variations disagree with the 3 D variations of the transverse shear stresses. The reason behind this is that the ESL models assume global functions of thickness coordinate for the entire thickness of the laminated composite plates in the kinematic expansions. The constitutive relations are used to predict the transverse shear stresses which produce discontinuous transverse stresses in the case of multi-layered composite structures. In most of the studies, it is observed that the transverse stresses are obtained using constitutive relations (CR) which cannot produce efficient results of transverse shear stresses. The equilibrium equations (EE) of elasticity can be used as supplementary equations along with the constitutive relations of the in-plane stresses to refine the results of the transverse shear stresses. The transverse shear stresses can be represented with first-order ordinary differential equations (ODEs). The ODEs are solved as a one-point boundary value problem (BVP) by using the traction-free conditions of transverse shear stresses at the top or bottom surfaces of the plates. The results obtained with this scheme are observed to be reliable and much better than the constitutive equations. Very few studies are available in the literature which employ the EE to derive the through-thickness variations of the transverse shear stresses. The results of transverse shear stresses for the laminated composite plates presented in Sahoo and Singh (2014) are obtained with the CR. Therefore, the accuracy of the stresses can be enhanced by using the EE.

It is observed in the exact solutions of Mallik and Ray (2004) that the variations of the transverse shear stresses of a smart composite plate with piezoelectric actuators are different from the traditional laminated composite plates under the action of combined electromechanical load. Some studies have reported the variations using the constitutive rule-based approach however the variations disagree with the exact solutions of Mallik and Ray (2004). It is evident from the studies that the use of constitutive rule produces the results of transverse stresses with significant error in the case of smart systems. It is therefore essential to employ the EE to enhance the accuracy of the results for the case of smart composite plates.

In the literature, the dynamic analysis of traditional laminated composite plates has not been carried out in detail using the non-polynomial ZZ theories, especially the transient responses under the action of time-dependent mechanical loads. The forced vibration responses of the laminated composite plates are essential to be obtained as the various structural components in the aerospace and mechanical industries are frequently exposed to time-dependent loads like gust, sonic boom pulses, etc. The laminated composite beams, plates, and shells are major structural components in engineering applications like supersonic flight vehicles, automobiles, space-station structures, and offshore structures, to name a few. The displacement-time responses under the action of mechanical loads give an idea of the amplitude of the dynamic responses experienced by the structures. Also, the free-vibration responses in the form of natural frequencies give information on the resonance condition of the structures. The forced vibration responses of laminated composite plates under the action of various blast loads generated from fuel and nuclear explosions and sonic boom pulses have also not been dealt with extensively. These loadings can create structural damage due to excessive mechanical vibrations unless efficient vibration characteristic research is carried out in the design stage.

The dynamic analysis of smart composite plates with piezoelectric actuators and sensors is necessary to understand the dynamic controlling capacity of the piezoelectric patch.

The dynamic response under the action of time-dependent electromechanical excitations is essential to determine the controlled vibrational amplitudes of the smart composite plates. A detailed investigation on the dynamic control is highly essential to understand the factors on which the electrical loads are dependent like the span-thickness ratio and aspect ratio of the smart composites, the magnitude of mechanical excitations, and thickness of the PFRC patch. There are very limited works in the literature in which a parametric study on the vibration control of the piezoelectric patches is carried out in the framework of non-polynomial ZZ theories. The Active Control of smart composite plate structures is also essential for developing the self-controlling capacities of the structures. In the Active Control, the electrical loads are not applied externally as an input and are obtained from the charges due to the mechanical strains in the sensor. The actuator and the sensor are coupled with a controller by which the electric voltages are fed back to the actuator. The vibrations get suppressed due to the electrical loads and a self-controlling capacity is generated.

Analysis of plate structures resting on an elastic foundation is very crucial in structural engineering applications. Concrete slabs (plates) supported by elastic soil is a very common construction form. It is widely used in residential buildings, institutional structures and commercial buildings, etc. In some of the structures, very heavy slab load occurs like libraries, warehouses and grain storage buildings. The efficient design of mat foundations, swimming pools and storage tanks require the analysis of elastic plates supported by the soil medium. It is essential to find out the structural displacements under a given set of loads and to ensure that the induced stresses are within acceptable limits for a safe and economical design. In the analysis of plates supported by elastic medium, the behavior of the plates under the action of external loads is influenced by the elastic medium and the behavior of the elastic medium is, in turn, influenced by the action of the plate under the external load. It is observed in the literature that there are very limited works which reports the static and dynamic analysis of plate structures resting on elastic foundation in the framework of a non-polynomial ZZ theory.

1.6. Objective and Scopes of the Present Work

In view of the observations made in the previous section, the following major scopes are identified that defines the overall objective of the present work.

Objective

To develop an efficient Analytical and Finite Element (FE) model for the static and dynamic analysis of smart composite plates supported on elastic foundation in the framework of non-polynomial Zigzag theory.

Scopes

1. Deriving analytical solutions for the structural responses (static, free vibration and transient) of laminated composite plates and smart composite plates under electromechanical loads.
2. To enhance the accuracy of the transverse shear stresses of traditional laminated composite plates and smart composite plates using a post-processing approach.
3. To develop a generalized FE formulation for the structural analysis of smart composite plates.
4. To investigate the dynamic behavior of multilayered composite plates under the action of various forms of blast loadings.
5. To derive the controlling capacity of the piezoelectric materials by evaluating the counteracting electrical loads that eradicates the unwanted mechanical vibrations from the system.
6. To study the Active Control of smart composite plates coupled with a feedback controller for the suppressed vibration responses of the plates.

7. To develop an efficient analytical model for the structural analysis of laminated composite and sandwich plates resting on an elastic foundation using non-polynomial ZZ theory.
8. To extend the analytical model for the structural analysis of smart composite plates resting on an elastic foundation using non-polynomial ZZ theory.

1.7. Organization of the thesis

The complete work presented in the thesis has been organized into five chapters. Chapter 1 contains a brief introduction to the traditional laminated composites, sandwich construction, smart materials and the applications of laminated composites and smart composite structures. It also contains a detailed discussion on the modeling of plate structures using plate theories, development of the plate theories and the underlying assumptions, various approaches used to extend the single-layered plate theories for multi-layered plate structures, literature review on the modeling of laminated composites, and smart composite plates using 3 D elasticity approach, 2 D plate models and the various solution techniques used for solving the governing equations. Chapter 2 is devoted to the detailed mathematical formulations of the problem. The fundamentals of traditional laminated composites and smart composites are discussed in this chapter. This chapter contains two major sections: the first section contains the analytical formulation and the closed-form solution scheme used to solve the governing equations. The second section is devoted to the finite element (FE) formulation and solution of the problem. Chapter 3 presents the discussions on the results obtained from the developed mathematical formulations for the static, free vibration, and transient analysis of traditional laminated composites and smart composite plate structures, validation of the solutions, and new results. This chapter contains seven major sections: the first section contains the static analysis of traditional laminated composites and sandwich plates, the second section is devoted to the static analysis of smart composite plates with piezoelectric actuators and sensors, the third section contains the dynamic analysis (free and forced vibration) of laminated composites and sandwich plates, the fourth section presents the dynamic analysis of smart composite plates, the fifth section contains the static responses of traditional laminated composites and sandwich structures supported on the elastic foundation, the sixth section contains the dynamic responses of traditional laminated composites supported by an elastic medium, the seventh section deals with the coupled static responses of smart composite plates supported by an elastic medium subjected to electromechanical loads and the eighth section contains the dynamic responses of smart composite plates supported by an elastic medium. In Chapter 4, important findings have been summarized, major contributions of the present research are stated and the recommendations for the future scope of the work are identified.