Chapter 5

The Phenomena of Concentration and Cavitation in the Riemann Solution for the Isentropic Zero-pressure Dusty Gasdynamics

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"Life is good for only two things, discovering mathematics and teaching mathematics"

–Simeon Poisson

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5.1 Introduction

The governing equations describing a planar isentropic dusty gas flow can be written as [94]

$$\begin{cases} \varrho_t + v\varrho_x + \varrho v_x = 0, \\ \varrho\left(v_t + v v_x\right) + P_x = 0, \end{cases} \tag{5.1}$$

where ϱ, P and v are the density, pressure and the velocity along the x-axis, respectively. x and t are the single spatial co-ordinate and time, respectively. The pressure P can be defined for ideal gas as $P(\varrho) = k\varrho^{\gamma}$, where k is positive constant and γ is specific heat ratio satisfying $1 \leq \gamma \leq 3$. The present chapter concerns with the dusty gas flow which is the mixture of ideal gas and small solid dust particles. The EoS for the dusty gas flow is defined by

$$P(\varrho) = \epsilon\left(\frac{\varrho}{(1-Z)}\right)^{\Gamma}, \tag{5.2}$$

where $\epsilon > 0$ is a constant and the parameters Z and Γ are called the volume fraction of the solid particles and the Grüneisen coefficient, defined as (See [112])

$$\Gamma = \frac{\gamma(1+\lambda\beta)}{(1+\lambda\beta\gamma)},$$

where $Z = V_{sp}/V_g$, $\beta = c_{sp}/c_p$, $\lambda = k_p/(1-k_p)$ and $\gamma = c_p/c_v$. Here, $k_p = m_{sp}/m_g$ is the mass fraction of dust particles, where m_{sp} and m_g are the mass of the dust particles and total mass of the mixture respectively. The parameters V_{sp}, V_g, c_{sp}, c_p and c_v are known as the volumetric extension of the solid particles, total volume of the mixture, the specific heat of solid particles, the specific heat of the gas at constant pressure and at constant volume, respectively. The mass fraction k_p and volume fraction Z are related by $Z = \theta\varrho$, $\theta = k_p/\varrho_{sp}$, where ϱ_{sp} is the specific density

of the dust particles. The dusty gas is a mixture of an ideal gas and small solid dust particles. In this composition the volume of the solid particle do not occupy more than five percent of the total volume of the mixture. The study of non-linear waves in dusty gas flow have significant role due to its applications to coal mines blast, volcanic and cosmic explosions, underground, metalized propellant, nozzle flow, lunar ash flow, supersonic flight in polluted air and many engineering science Problems (See [2], [113], [114], [115], [116], [117], [118]). Also, in many astrophysical phenomena, the composition of dust particles and gases play crucial role.
The system (5.1) with (5.2) can be written in the conservative form as

$$\begin{cases} \varrho_t + (\varrho v)_x = 0, \\ (\varrho v)_t + \left(\varrho v^2 + \epsilon \left(\frac{\varrho}{(1-\theta\varrho)} \right)^{\Gamma} \right)_x = 0, \end{cases} \tag{5.3}$$

For zero-pressure dusty gas flow, the system (5.3) can be written as when $\epsilon \to 0$

$$\begin{cases} \varrho_t + (\varrho v)_x = 0, \\ (\varrho v)_t + (\varrho v^2)_x = 0, \end{cases} \tag{5.4}$$

which is called the transport equation or pressureless Euler system. The author in [119] and [120] have used it to modal the motion of free particles which stick under collision. In [121], the author used the system (5.4) for the formation of large scale structures in the universe.

In the present study, we investigate the concentration and cavitation phenomenon in the solution of Riemann Problem by employing two-parameter flux approximation in the system (5.3) motivated by [34, 41], written as

$$\begin{cases} \varrho_t + (\varrho v - 2\alpha_1 v)_x = 0, \\ (\varrho v)_t + \left(\varrho v^2 - \alpha_1 v^2 + \epsilon\alpha_2 \left(\frac{\varrho}{(1-\theta\varrho)}\right)^{\Gamma}\right)_x = 0, \end{cases} \tag{5.5}$$

where $\varrho \geq 2\alpha_1$, $\alpha_1, \alpha_2 > 0$.

Physically, the perturbation technique is utilized to lead some dynamical process of fluids, therefore it is worthwhile to study the problem by flux approximation which plays a crucial role in computational and theoretical problem from its application point of view. Here, we employ a approximation in the flux which contain the pressure perturbation portion. We solve the Riemann Problem of the system (5.5) with the following initial data,

$$(\varrho, v)(x, 0) = \begin{cases} (\varrho_-, v_-), & x < 0, \\ (\varrho_+, v_+), & x > 0, \end{cases} \tag{5.6}$$

where $v_\pm$ and $\varrho_\pm$ are constants.

The motivation of this study is to analyze the phenomena of the concentration and cavitation in the solution to the Riemann Problem for the isentropic dusty gas flow by introducing two parameter flux approximation in the governing equations due to its wide applications in the area of aerodynamics, cosmology, astrophysics and engineering. Many researchers and scientists from several areas of science and engineering are working currently on the Riemann Problem for homogeneous and non-homogeneous models. From last two decades, study of the solution of the Riemann Problem for the homogeneous and non-homogeneous hyperbolic systems created great interest among the researchers. Recently, [89, 94, 102, 122, 123] have investigated the Riemann solutions for the homogeneous and non-homogeneous gas-dynamic models. Also, [56] have studied the two dimensional Riemann Problem for the homogeneous Chaplygin gas model. In recent years, many authors like as [93], [124] and [125] have studied the Riemann solution for the dusty gasdynamics.

Nowadays, we know that when $v_- < v_+$, the Riemann solution of (5.1) and (5.2) consists two contact discontinuities and a vacuum state between them and when $v_- > v_+$, only a δ- shock wave connecting the left state (ϱ_-, v_-) and the right state (ϱ_+, v_+) directly. Thus, it is of great interest to discuss the formation of δ- shock wave and vacuum state in the solutions to the Riemann Problem (5.1) and (5.2) by taking the parameters tending to zero. Now, we recall some main results in detail about the δ- shock wave in the theory of hyperbolic conservation laws. [34] investigated initially the formation of δ- shock and vacuum state for the Riemann solutions to the isentropic and non-isentropic pressureless Euler system by using the vanishing pressure approach, respectively. Initially, [126] have considered the formation of delta shock wave and vacuum state by using the two parameter flux approximation by the vanishing pressure and magnetic field. [52] have studied the limiting solution of the Riemann Problem for multidimensional zero pressure gasdynamic equation by employing vanishing viscosity approach. In addition, [47], [45] and [42] have investigated the formation of δ- shock and vacuum state for the Riemann solutions for the Aw–Rascle model, modified Chaplygin gas model and relativistic fluid by using the vanishing pressure approach, respectively. Also, the authors [46], [37] and [40] have discussed the concentration phenomenon in the vanishing pressure limit of the Riemann solution for the modified Chaplygin gas and extended Chaplygin gas models, respectively. Recently, the concentration and cavitation phenomenon in the solutions of the Riemann Problem have investigated by many authors like [41], [39], [38]. But the concentration and cavitation in the solution to the Riemann Problem for the isentropic dusty gas flow by introducing two parameter flux approximation technique is not studied by any researcher till now.

The complete structure of this chapter is summarized into sections as: In section (5.2), the delta shock and vacuum states for the transport equations are derived. The R-H relations are obtained for the delta shock and studied the exact location, strength and propagation speed of delta shock wave. In section (5.4), we structure the Riemann solution for the approximated system. Section contains the detailed study of the concentration phenomenon in the solution of approximated system under the flux approximation. Further, in section (5.5), we discuss the the phenomenon of cavitation in the solution of approximated system under the flux approximation. Ultimately, section (5.6) contains conclusions of this study.

5.2 Delta - shocks and vacuum states for the system (5.4)

The matrix form of the transport equation can be written as

$$U_t + M(U)U_x = 0, \tag{5.7}$$

where $U = \begin{pmatrix} \varrho \\ v \end{pmatrix}, M = \begin{pmatrix} v & \varrho \\ 0 & v \end{pmatrix}$.

The characteristic roots of the matrix $M(U)$ are $\lambda_1 = \lambda_2 = v$ and corresponding right eigenvectors are $R_{1,2} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ which satisfy $\nabla\lambda_i = 0, i = 1, 2$, where $\nabla = (\partial_\varrho, \partial_v)$. Therefore the characteristic fields for both characteristic roots are linearly degenerate. Hence the elementary waves associated with the characteristic fields are nothing but contact discontinuities.

The solutions of the Riemann Problem (5.4) with (5.6) can be described as: If $v_- < v_+$, then there is no characteristic passing through the region $\{(x, t) : v_- t < x < v_+ t\}$,

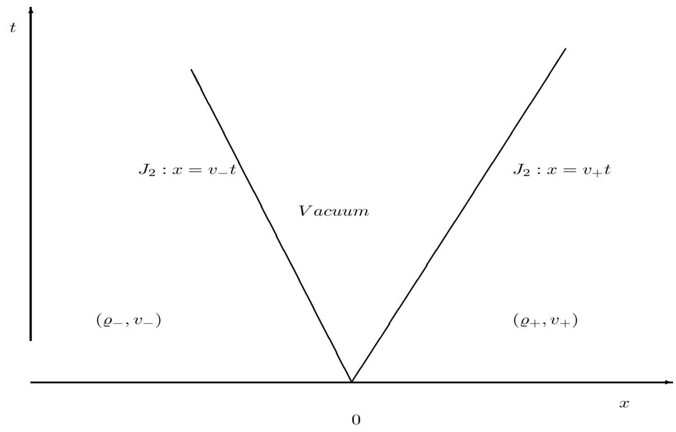

FIGURE 5.1: Structure of the Riemann solution for $v_- < v_+$.

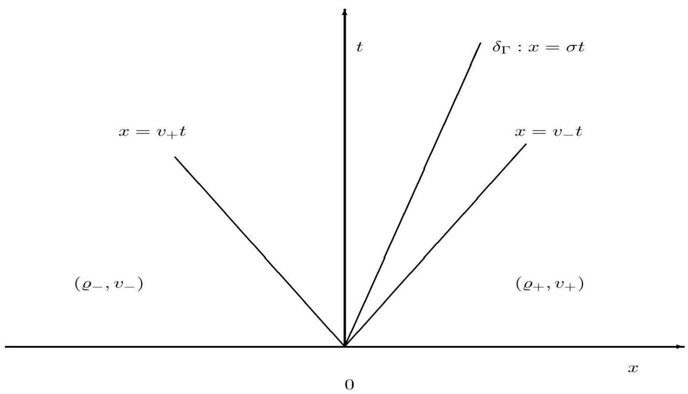

FIGURE 5.2: Characteristic analysis of delta shock.

so the vacuum appears in this region. The solution of Riemann Problem (5.4) with (5.6) can be written as (See Figure 5.1)

$$(\varrho, v)(x,t) = \begin{cases} (\varrho_-, v_-), & -\infty < x < v_- t, \\ vacuum, & v_- t < x < v_+ t, \\ (\varrho_+, v_+), & v_+ t < x < \infty. \end{cases} \tag{5.8}$$

If $v_- > v_+$, then the solution contains a delta shock (weighted δ measure) supported on a line due to overlap of characteristics in domain $\{(x,t) : v_+ t < x < v_- t\}$. The characteristic analysis of the delta shock is shown by Figure 5.2. To define the measure solution, we provide the following definition.

Definition 5.1. Let $\Gamma = \{(x,t)|x = x(t), t \geq 0\}$ is a smooth curve, then the two-dimensional weighted δ-measure function $f(t)\delta_\Gamma$ supported on Γ is defined as

$$\langle f(t)\delta_\Gamma, \phi(x(t),t)\rangle = \int_0^\infty f(t)\phi(x(t),t)dt,$$

where $\phi(x(t),t) \in C_0^\infty(R \times R_+)$.

From above definition, we can obtain a family of δ-measure solutions (delta shock solution) with the parameter σ in the case $v_- > v_+$ as

$$\varrho(x,t) = \varrho_0(x,t) + f(t)\delta_\Gamma, \qquad v(x,t) = v_0(x,t), \tag{5.9}$$

where $\varrho_0(x,t) = \varrho_- + [\varrho]\chi(x-\sigma t)$, $v_0(x,t) = v_- + [v]\chi(x-\sigma t)$, $f(t) = \frac{(\sigma[\varrho]-[\varrho v])}{1+\sigma^2}t$, and $\Gamma = \{(\sigma t, t) : t > 0\}$, in which $[\varrho] = \varrho_+ - \varrho_-$ is jump in ϱ across the discontinuity curve and $\chi(x)$ is characteristic function defined by $\chi(x) = 0$ when $x < 0$ and $\chi(x) = 1$ when $x > 0$.

Then the pair (ϱ, v) constructed above is called delta shock solution, it satisfies the following conditions

$$\begin{cases} \langle \varrho, \phi_t\rangle + \langle \varrho v, \phi_x\rangle = 0, \\ \langle \varrho v, \phi_t\rangle + \langle \varrho v^2, \phi_x\rangle = 0, \end{cases} \tag{5.10}$$

for every test function $\phi(x,t) \in C_0^\infty(R \times R_+)$, where

$$\begin{aligned} \langle \varrho, \phi\rangle &= \int_{R^+}\int_\infty \varrho_0\phi dxdt + \langle f(t)\delta_\Gamma, \phi\rangle, \\ \langle \varrho v, \phi\rangle &= \int_{R^+}\int_\infty (\varrho_0 v_0)\,\phi dxdt + \langle \sigma f(t)\delta_\Gamma, \phi\rangle. \end{aligned} \tag{5.11}$$

Also, the delta shock solution satisfies the following Rankine-Hugoniot jump conditions

$$\begin{cases} \frac{dx(t)}{dt} = \sigma = v_\delta, \\ \frac{df(t)}{dt} = \sigma[\varrho] - [\varrho v], \\ \frac{d(f(t)v_\delta)}{dt} = \sigma[\varrho v] - [\varrho v^2], \end{cases} \tag{5.12}$$

and must satisfy the delta entropy condition to ensure the uniqueness of the delta shock solution $\lambda(\varrho_+, v_+) < \sigma < \lambda(\varrho_-, v_-)$, i.e.

$$v_+ < \sigma < v_-. \tag{5.13}$$

On solving (5.9) with initial data $f(0) = 0, x(0) = 0$, we obtain the following results

$$\sigma = \frac{\sqrt{\varrho_+}v_+ + \sqrt{\varrho_-}v_-}{\sqrt{\varrho_+} + \sqrt{\varrho -}}, \quad x(t) = \sigma t, \quad f(t) = (v_- - v_+)\sqrt{\varrho_+\varrho_-}t. \tag{5.14}$$

Therefore, the delta shock wave solution for the transport equations with (5.4) and (5.6) is obtained by (5.9) with (5.12) and (5.14).

5.3 Solution of Riemann Problem (5.5) and (5.6)

The system (5.5) can be written in conservation form as

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0,$$

where

$$\mathbf{U} = \begin{pmatrix} \varrho \\ v \end{pmatrix}, \qquad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} v & \varrho - 2\alpha_1 \\ \epsilon\alpha_2\Gamma\left(\frac{\varrho}{1-\theta\varrho}\right)^{\Gamma-2} & v \end{pmatrix}.$$

The eigenvalues μ_1 and μ_2 are given as follows

$$\mu_1 = v - \sqrt{\frac{\epsilon\alpha_2\Gamma\varrho^{\Gamma-2}}{(1-\theta\varrho)^{\Gamma+1}}(\varrho - 2\alpha_1)}, \quad \mu_2 = v + \sqrt{\frac{\epsilon\alpha_2\Gamma\varrho^{\Gamma-2}}{(1-\theta\varrho)^{\Gamma+1}}(\varrho - 2\alpha_1)}, \tag{5.15}$$

and the associated right eigenvectors and left eigenvectors are

$$\mathbf{R_1} = \begin{pmatrix} 1 & -\sqrt{\frac{\epsilon\alpha_2\Gamma\varrho^{\Gamma-2}}{(1-\theta\varrho)^{\Gamma+1}(\varrho-2\alpha_1)}} \end{pmatrix}^{\mathbf{Tr}}, \quad \mathbf{R_2} = \begin{pmatrix} 1 & \sqrt{\frac{\epsilon\alpha_2\Gamma\varrho^{\Gamma-2}}{(1-\theta\varrho)^{\Gamma+1}(\varrho-2\alpha_1)}} \end{pmatrix}^{\mathbf{Tr}}, \tag{5.16}$$

$$\mathbf{L_1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{\frac{(1-\theta\varrho)^{\Gamma+1}(\varrho-2\alpha_1)}{\epsilon\alpha_2\Gamma\varrho^{\Gamma-2}}} \end{pmatrix}, \quad \mathbf{L_2} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{\frac{(1-\theta\varrho)^{\Gamma+1}(\varrho-2\alpha_1)}{\epsilon\alpha_2\Gamma\varrho^{\Gamma-2}}} \end{pmatrix}, \tag{5.17}$$

Since, $\nabla\mu_i.\mathbf{R_i} \neq \mathbf{0}, \mathbf{i} = \mathbf{1}, \mathbf{2}$, where $\nabla = (\partial_\varrho, \partial_v)$. This implies that the characteristic fields corresponding to eigenvalues μ_1 and μ_2 are genuinely nonlinear and, therefore the associated elementary waves are shock wave and rarefaction waves.

5.3.1 Smooth solution

For the self-similar solution of the system (5), we substitute $(\varrho, v)(x,t) = (\varrho, v)(\xi)$, where $\xi = \frac{x}{t}$ in (5.5) and (5.6), we obtain the following boundary value Problem

$$\begin{cases} -\xi\varrho_\xi + (\varrho v - 2\alpha_1 v)_\xi = 0, \\ -\xi\left(\varrho v\right)_\xi + \left(\varrho v^2 - \alpha_1 v^2 + \epsilon\alpha_2\left(\frac{\varrho}{(1-\theta\varrho)}\right)^\Gamma\right)_\xi = 0, \end{cases} \tag{5.18}$$

and $(\varrho, v)(\pm\infty) = (\varrho_\pm, v_\pm)$.

For any smooth solution, the system (5.18) can be written in the following form

$$\begin{pmatrix} -\xi + v & \varrho - 2\alpha_1 \\ \xi v + v^2 + \epsilon\alpha_2\Gamma\left(\frac{\varrho}{1-\theta\varrho}\right)^{\Gamma-2} & -\xi\varrho + 2v(\varrho - \alpha_1) \end{pmatrix} \begin{pmatrix} d\varrho \\ dv \end{pmatrix} = 0. \tag{5.19}$$

The system (5.19) provides either a constant state solution

$$(\varrho, v)(\xi) = constant, \tag{5.20}$$

or, 1-rarefaction wave solution, denoted by R_1 defined as

$$R_1(\varrho_-, v_-) : \begin{cases} \frac{dx}{dt} = \xi = \mu_1 = v - \sqrt{\frac{\epsilon\alpha_2\Gamma\varrho^{\Gamma-2}}{(1-\theta\varrho)^{\Gamma+1}}(\varrho - 2\alpha_1)}, \\ v = v_- - \int_{\varrho_-}^{\varrho} \sqrt{\frac{\epsilon\alpha_2\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z-2\alpha_1)}} dz, \\ \varrho < \varrho_-, \end{cases} \tag{5.21}$$

or, 2-rarefaction wave solution, denoted by R_2, defined as

$$R_2(\varrho_-, v_-) : \begin{cases} \frac{dx}{dt} = \xi = \mu_2 = v + \sqrt{\frac{\epsilon\alpha_2\Gamma\varrho^{\Gamma-2}}{(1-\theta\varrho)^{\Gamma+1}}(\varrho - 2\alpha_1)}, \\ v = v_- + \int_{\varrho_-}^{\varrho} \sqrt{\frac{\epsilon\alpha_2\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z-2\alpha_1)}} dz, \\ \varrho > \varrho_-. \end{cases} \tag{5.22}$$

For 1-rarefaction wave, we differentiate v with respect to ϱ in second equality of (5.21), it gives that

$$\frac{dv}{d\varrho} = -\sqrt{\frac{\epsilon\alpha_2\Gamma\varrho^{\Gamma-2}}{(1-\theta\varrho)^{\Gamma+1}(\varrho - 2\alpha_1)}} < 0,$$

and for 2-rarefaction wave, we differentiate v with respect to ϱ in second equality of (5.22), yields

$$\frac{dv}{d\varrho} = \sqrt{\frac{\epsilon\alpha_2\Gamma\varrho^{\Gamma-2}}{(1-\theta\varrho)^{\Gamma+1}(\varrho - 2\alpha_1)}} > 0.$$

Taking the limit $\varrho \to 2\alpha_1$, in the second equality of (5.21) we have

$$\lim_{\varrho \to 2\alpha_1} v = v_- + \int_{2\alpha_1}^{\varrho_-} \sqrt{\frac{\epsilon\alpha_2\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z - 2\alpha_1)}} dz. \tag{5.23}$$

Since

$$\lim_{z\to 2\alpha_1}\left((z-2\alpha_1)\sqrt{\frac{\epsilon\alpha_2\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z-2\alpha_1)}}\right)=\sqrt{\frac{\epsilon\alpha_2\Gamma(2\alpha_1)^{\Gamma-2}}{(1-2\theta\alpha_1)^{\Gamma+1}}}.$$

This implies that the integral $\int_{2\alpha_1}^{\varrho_-}\sqrt{\frac{\epsilon\alpha_2\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z-2\alpha_1)}}dz$ is convergent by Cauchy criterion.

Thus from (5.23), we find that 1-rarefaction wave curve intersects the line $\varrho=2\alpha_1$ at the point $\left(2\alpha_1, v_- + \int_{2\alpha_1}^{\varrho_-}\sqrt{\frac{\epsilon\alpha_2\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z-2\alpha_1)}}dz\right)$.

Checking the limit $\varrho\to+\infty$ in (5.22), yields

$$\lim_{\varrho\to-\infty} v = v_- + \int_{\varrho_-}^{\infty}\sqrt{\frac{\epsilon\alpha_2\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z-2\alpha_1)}}dz.$$

Since

$$\sqrt{\frac{\epsilon\alpha_2\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z-2\alpha_1)}} > \sqrt{\frac{\epsilon\alpha_2\Gamma z^{\Gamma-2}}{z}},$$

we have

$$\int_{\varrho_-}^{+\infty}\sqrt{\frac{\epsilon\alpha_2\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z-2\alpha_1)}}dz > \int_{\varrho_-}^{+\infty}\left(\sqrt{\frac{\epsilon\alpha_2\Gamma z^{\Gamma-2}}{z}}\right)dz.$$

Thus, we conclude that $\lim_{\varrho\to+\infty} v=+\infty$.

5.3.2 Bounded discontinuous solution

Now our attention is to discuss the bounded discontinuous solution which is called shock wave solution satisfying the R-H conditions and entropy condition. The R-H conditions for the shock wave at $\xi=\sigma^{\alpha_1\alpha_2}$ are derived as

$$\begin{cases}-\sigma^{\alpha_1\alpha_2}[\varrho]+[\varrho v-2\alpha_1 v]=0,\\ -\sigma^{\alpha_1\alpha_2}[\varrho v]+\left[\varrho v^2-\alpha_1 v^2+\epsilon\alpha_2\left(\frac{\varrho}{(1-\theta\varrho)}\right)^{\Gamma}\right]=0,\end{cases}\tag{5.24}$$

where $[\varrho] = \varrho_+ - \varrho_-$ denotes jump in ϱ across the shock.

On eliminating $\sigma^{\alpha_1\alpha_2}$ from (5.24), we obtain

$$[\varrho v][\varrho v - 2\alpha_1 v] = [\varrho]\left[\varrho v^2 - \alpha_1 v^2 + \epsilon\alpha_2\left(\frac{\varrho}{(1-\theta\varrho)}\right)^{\Gamma}\right].$$

Simplifying above relation, we have

$$(\varrho_+\varrho_- - \alpha_1(\varrho_+ + \varrho_-))(v_- - v_+)^2 = \epsilon\alpha_2(\varrho_+ - \varrho_-)\left(\left(\frac{\varrho_+}{(1-\theta\varrho_+)}\right)^{\Gamma} - \left(\frac{\varrho_-}{(1-\theta\varrho_-)}\right)^{\Gamma}\right).$$

Since $(\varrho_+\varrho_- - \alpha_1(\varrho_+ + \varrho_-)) > 0$, which implies that

$$(v_- - v_+) = \pm\sqrt{\frac{\epsilon\alpha_2(\varrho_+ - \varrho_-)\left(\left(\frac{\varrho_+}{(1-\theta\varrho_+)}\right)^{\Gamma} - \left(\frac{\varrho_-}{(1-\theta\varrho_-)}\right)^{\Gamma}\right)}{(\varrho_+\varrho_- - \alpha_1(\varrho_+ + \varrho_-))}}. \tag{5.25}$$

The Lax entropy inequalities imply that 1-shock wave satisfies the following relation

$$\mu_1(\varrho_+, v_+) < \sigma^{\alpha_1\alpha_2} < \mu_2(\varrho_+, v_+), \qquad \mu_1(\varrho_-, v_-) > \sigma^{\alpha_1\alpha_2}, \tag{5.26}$$

and 2-shock wave satisfies

$$\mu_1(\varrho_-, v_-) < \sigma^{\alpha_1\alpha_2} < \mu_2(\varrho_-, v_-), \qquad \mu_2(\varrho_+, v_+) < \sigma^{\alpha_1\alpha_2}. \tag{5.27}$$

From (26), we have

$$\mu_1(\varrho_+, v_+) < \sigma^{\alpha_1\alpha_2} < \mu_1(\varrho_-, v_-),$$

$$-\sqrt{\frac{\epsilon\alpha_2\Gamma\varrho_+^{\Gamma-2}(\varrho_+ - 2\alpha_1)}{(1-\theta\varrho_-)^{\Gamma+1}}}\frac{1}{(\varrho_- - 2\alpha_1)} < \frac{v_+ - v_-}{(\varrho_+ - \varrho_-)} < -\sqrt{\frac{\epsilon\alpha_2\Gamma\varrho_-^{\Gamma-2}(\varrho_- - 2\alpha_1)}{(1-\theta\varrho_-)^{\Gamma+1}}}\frac{1}{(\varrho_+ - 2\alpha_1)}. \tag{5.28}$$

Similarly, for 2-shock wave, we have

$$\sqrt{\frac{\epsilon\alpha_2\Gamma\varrho_+^{\Gamma-2}(\varrho_+ - 2\alpha_1)}{(1-\theta\varrho_-)^{\Gamma+1}}}\frac{1}{(\varrho_- - 2\alpha_1)} < \frac{v_+ - v_-}{(\varrho_+ - \varrho_-)} < \sqrt{\frac{\epsilon\alpha_2\Gamma\varrho_-^{\Gamma-2}(\varrho_- - 2\alpha_1)}{(1-\theta\varrho_-)^{\Gamma+1}}}\frac{1}{(\varrho_+ - 2\alpha_1)}. \tag{5.29}$$

From (28) and (29), we conclude that $\varrho_- < \varrho_+, v_+ < v_-$, and $\varrho_- > \varrho_+, v_+ < v_-$, respectively.

For a given left state (ϱ_-, v_-), the shock curves in the phase plane, which are the sets of states that can be connected on the right by 1-shock or a 2-shock are defined as,

1-shock wave curve denoted by $S_1(\varrho_-, v_-)$ is defined as

$$S_1(\varrho_-, v_-) = \begin{cases} v = v_- - \sqrt{\frac{\epsilon\alpha_2(\varrho-\varrho_-)\left(\left(\frac{\varrho}{(1-\theta\varrho)}\right)^\Gamma - \left(\frac{\varrho_-}{(1-\theta\varrho_-)}\right)^\Gamma\right)}{(\varrho\varrho_- - \alpha_1(\varrho+\varrho_-))}}, \\ \varrho > \varrho_-, \end{cases} \tag{5.30}$$

and 2-shock wave curve, denoted by $S_2(\varrho_-, v_-)$, is defined as

$$S_2(\varrho_-, v_-) = \begin{cases} v = v_- - \sqrt{\frac{\epsilon\alpha_2(\varrho-\varrho_-)\left(\left(\frac{\varrho}{(1-\theta\varrho)}\right)^\Gamma - \left(\frac{\varrho_-}{(1-\theta\varrho_-)}\right)^\Gamma\right)}{(\varrho\varrho_- - \alpha_1(\varrho+\varrho_-))}}, \\ \varrho < \varrho_-. \end{cases} \tag{5.31}$$

In addition, on differentiating v with respect to ϱ in (5.30) and (5.31), we obtain that $v_\varrho < 0$ and $v_\varrho > 0$, respectively, which implies that the shock curves are monotone. When $\varrho \to 1/\theta$ in (5.30) we have $v \to -\infty$ and when $\varrho \to 2\alpha_1$ in (5.31), we get

$$v \to v_- - \sqrt{\frac{\epsilon\alpha_2}{-\alpha_1}\left(\left(\frac{2\alpha_1}{(1-2\theta\alpha_1)}\right)^\Gamma - \left(\frac{\varrho_-}{(1-\theta\varrho_-)}\right)^\Gamma\right)},$$

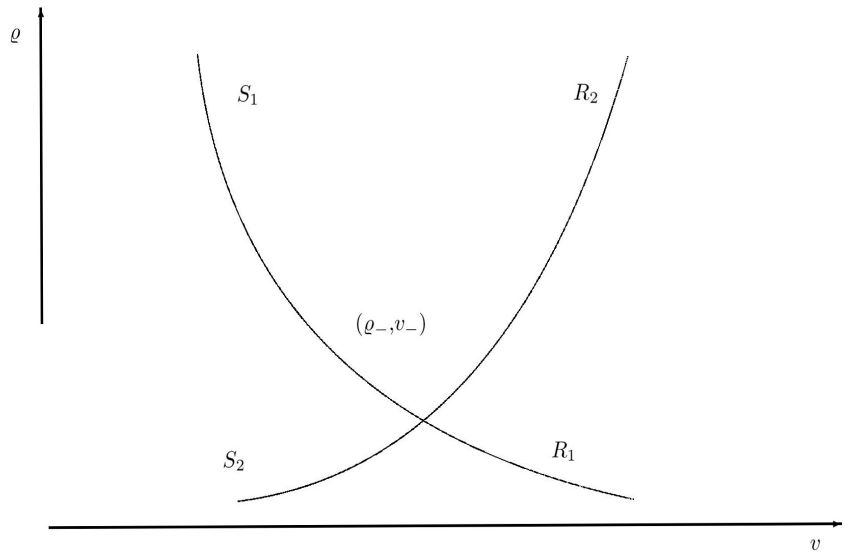

FIGURE 5.3: The (ϱ, v) phase plane for the model (1).

this implies that 2-shock wave curve intersects the line $\varrho = 2\alpha_1$ at the point

$$\left(2\alpha_1, v_- - \sqrt{\frac{\epsilon\alpha_2}{-\alpha_1}\left(\left(\frac{2\alpha_1}{(1-2\theta\alpha_1)}\right)^{\Gamma} - \left(\frac{\varrho_-}{(1-\theta\varrho_-)}\right)^{\Gamma}\right)}\right).$$

From above analysis, we notice that for a given left state (ϱ_-, v_-), the set of states that can be connected on the right by a rarefaction wave or a shock wave in the phase plane consists of 1-rarefaction wave curve $R_1(\varrho_-, v_-)$, 2-rarefaction wave curve $R_2(\varrho_-, v_-)$, 1-shock wave curve $S_1(\varrho_-, v_-)$ and 2-shock wave curve $S_2(\varrho_-, v_-)$. These elementary curves divide phase plane into four regions as $R_1R_2(\varrho_-, v_-)$, $R_2S_1(\varrho_-, v_-)$, $S_1S_2(\varrho_-, v_-)$ and $S_2R_1(\varrho_-, v_-)$ which is shown in Figure 5.3.

5.4 Concentration in Riemann solution to (5.5) and (5.6) under flux approximation

In this section, we study the phenomenon of concentration in the Riemann solution of (5.5) and (5.6) for isentropic dusty gas flow, when $(\varrho_+, v_+) \in S_1S_2(\varrho_-, v_-)$ with $v_- > v_+$.

5.4.1 Limiting behavior of the solution of Riemann Problem as α_1, α_2 tends to 0

For any $\alpha_1, \alpha_2 > 0$, we suppose $(\varrho_*^{\alpha_1\alpha_2}, v_*^{\alpha_1\alpha_2})$ be the intermediate state in the sense that (ϱ_-, v_-) and $(\varrho_*^{\alpha_1\alpha_2}, v_*^{\alpha_1\alpha_2})$ are connected by 1-shock curve $S_1(\varrho_-, v_-)$ with speed $\sigma_1^{\alpha_1\alpha_2}$, and the states $(\varrho_*^{\alpha_1\alpha_2}, v_*^{\alpha_1\alpha_2})$ and (ϱ_+, v_+) are connected by 2-shock curve $S_2(\varrho_-, v_-)$ with speed $\sigma_2^{\alpha_1\alpha_2}$. They have relations

$$\begin{cases} v_*^{\alpha_1\alpha_2} = v_- - \sqrt{\dfrac{\epsilon\alpha_2(\varrho_*^{\alpha_1\alpha_2} - \varrho_-)\left(\left(\frac{\varrho_*^{\alpha_1\alpha_2}}{(1-\theta\varrho_*^{\alpha_1\alpha_2})}\right)^\Gamma - \left(\frac{\varrho_-}{(1-\theta\varrho_-)}\right)^\Gamma\right)}{(\varrho_*^{\alpha_1\alpha_2}\varrho_- - \alpha_1(\varrho_*^{\alpha_1\alpha_2} + \varrho_-))}}, \\ \varrho_*^{\alpha_1\alpha_2} > \varrho_-, \end{cases} \tag{5.32}$$

on S_1 and

$$\begin{cases} v_*^{\alpha_1\alpha_2} = v_+ - \sqrt{\dfrac{\epsilon\alpha_2(\varrho_+ - \varrho_*^{\alpha_1\alpha_2})\left(\left(\frac{\varrho_+}{(1-\theta\varrho_+)}\right)^\Gamma - \left(\frac{\varrho_*^{\alpha_1\alpha_2}}{(1-\theta\varrho_*^{\alpha_1\alpha_2})}\right)^\Gamma\right)}{(\varrho_+\varrho_*^{\alpha_1\alpha_2} - \alpha_1(\varrho_+ + \varrho_*^{\alpha_1\alpha_2}))}}, \\ \varrho_*^{\alpha_1\alpha_2} > \varrho_+, \end{cases} \tag{5.33}$$

on S_2. Then, we propose some Lemmas to analyze the limit solution of the Riemann Problem (5.5) and (5.6) as $\alpha_1, \alpha_2 \to 0$.

Lemma 5.2. $\lim\limits_{\alpha_1,\alpha_2\to 0} \varrho_*^{\alpha_1\alpha_2} = +\infty.$

Proof: If $\lim\limits_{\alpha_1,\alpha_2\to 0} \varrho_*^{\alpha_1\alpha_2} = M$, where $M \in (max(\varrho_-, \varrho_+), +\infty)$, then from (5.32) and (5.33), we have (see [45])

$$v_+ - v_- = -\left(\sqrt{\frac{\epsilon\alpha_2(\varrho_*^{\alpha_1\alpha_2} - \varrho_-)\left(\left(\frac{\varrho_*^{\alpha_1\alpha_2}}{(1-\theta\varrho_*^{\alpha_1\alpha_2})}\right)^\Gamma - \left(\frac{\varrho_-}{(1-\theta\varrho_-)}\right)^\Gamma\right)}{(\varrho_*^{\alpha_1\alpha_2}\varrho_- - \alpha_1(\varrho_*^{\alpha_1\alpha_2} + \varrho_-))}}\right) + \left(\sqrt{\frac{\epsilon\alpha_2(\varrho_+ - \varrho_*^{\alpha_1\alpha_2})\left(\left(\frac{\varrho_+}{(1-\theta\varrho_+)}\right)^\Gamma - \left(\frac{\varrho_*^{\alpha_1\alpha_2}}{(1-\theta\varrho_*^{\alpha_1\alpha_2})}\right)^\Gamma\right)}{(\varrho_+\varrho_*^{\alpha_1\alpha_2} - \alpha_1(\varrho_+ + \varrho_*^{\alpha_1\alpha_2}))}}\right).$$

Letting $\alpha_1, \alpha_2 \to 0$ in above expression, we get $v_+ - v_- = 0$, i.e. $v_+ = v_-$, which is contradiction of our supposition $v_+ < v_-$. Therefore, Lemma 5.2 is true.

Lemma 5.3. $\lim\limits_{\alpha_1,\alpha_2\to 0} \epsilon\alpha_2\left(\frac{\varrho_*^{\alpha_1\alpha_2}}{(1-\theta\varrho_*^{\alpha_1\alpha_2})(1-\theta\varrho_-)}\right)^\Gamma = \varrho_-\varrho_+\left(\frac{v_- - v_+}{\sqrt{\varrho_-}+\sqrt{\varrho_+}}\right)^2.$

Lemma 5.4. *Set* $\sigma = \frac{v_-\sqrt{\varrho_-}+v_+\sqrt{\varrho_+}}{\sqrt{\varrho_-}+\sqrt{\varrho_+}}$. *Then*

$$\lim_{\alpha_1,\alpha_2\to 0} v_*^{\alpha_1\alpha_2} = \lim_{\alpha_1,\alpha_2\to 0} \sigma_1^{\alpha_1\alpha_2} = \lim_{\alpha_1,\alpha_2\to 0} \sigma_2^{\alpha_1\alpha_2} = \sigma.$$

Proof: Taking the limit as $\alpha_1, \alpha_2 \to 0$ in (5.33) and noticing Lemma 5.3, we have

$$\lim_{\alpha_1,\alpha_2\to 0} v_*^{\alpha_1\alpha_2} = v_- - \lim_{\alpha_1,\alpha_2\to 0} \frac{1}{\sqrt{\varrho_-}}\sqrt{\epsilon\alpha_2\left(\frac{\varrho_*^{\alpha_1\alpha_2}}{(1-\theta\varrho_*^{\alpha_1\alpha_2})(1-\theta\varrho_-)}\right)^\Gamma} = \sigma.$$

From R-H jump relations (5.24), we have

$$\sigma_1^{\alpha_1\alpha_2} = v_*^{\alpha_1\alpha_2} + \frac{(\varrho_- - 2\alpha_1)(v_*^{\alpha_1\alpha_2} - v_-)}{\varrho_*^{\alpha_1\alpha_2} - \varrho_-},$$

$$\sigma_2^{\alpha_1\alpha_2} = v_*^{\alpha_1\alpha_2} + \frac{(\varrho_+ - 2\alpha_1)(v_+ - v_*^{\alpha_1\alpha_2})}{\varrho_+ - \varrho_*^{\alpha_1\alpha_2}}.$$

Taking the limit as $\alpha_1, \alpha_2 \to 0$, we have

$$\lim_{\alpha_1,\alpha_2\to 0} \sigma_1^{\alpha_1\alpha_2} = \lim_{\alpha_1,\alpha_2\to 0} \sigma_2^{\alpha_1\alpha_2} = \lim_{\alpha_1,\alpha_2\to 0} v_*^{\alpha_1\alpha_2} = \sigma.$$

From Lemma 5.4, it is noticeable that when α_1, α_2 tend to 0, the speed of 1-shock S_1 and 2-shock S_2 and intermediate velocity coincide at σ, which determine the δ-shock speed for the transport equations, and the intermediate density $\varrho_*^{\alpha_1\alpha_2}$ becomes singular.

From Lemma 5.2 and Lemma 5.4, we have the following result.

Lemma 5.5.

$$\lim_{\alpha_1,\alpha_2\to 0} \int_{\sigma_1^{\alpha_1\alpha_2}}^{\sigma_2^{\alpha_1\alpha_2}} \varrho_*^{\alpha_1\alpha_2} d\eta = \sigma[\varrho] - [\varrho v],$$

and

$$\lim_{\alpha_1,\alpha_2\to 0} \int_{\sigma_1^{\alpha_1\alpha_2}}^{\sigma_2^{\alpha_1\alpha_2}} \varrho_*^{\alpha_1\alpha_2} v_*^{\alpha_1\alpha_2} d\eta = \sigma[\varrho v] - [\varrho v^2].$$

5.4.2 Delta - shock wave

We now show the following theorem characterizing the limit as $\alpha_1, \alpha_2 \to 0$ in the case $v_+ < v_-$ and $(\varrho_+, v_+) \in S_1S_2(\varrho_-, v_-)$.

Theorem 5.6. *Let $v_+ < v_-$ and $(\varrho_+, v_+) \in S_1S_2(\varrho_-, v_-)$ and let $(\varrho^{\alpha_1\alpha_2}, v^{\alpha_1\alpha_2})$ is a two shocks solution of Riemann Problem (5.5) and (5.6). Then, when $\alpha_1\alpha_2 \to 0$, $\varrho^{\alpha_1\alpha_2}$ and $\varrho^{\alpha_1\alpha_2}v^{\alpha_1\alpha_2}$ converge in the sense of distribution and limit functions of $\varrho^{\alpha_1\alpha_2}$ and $\varrho^{\alpha_1\alpha_2}v^{\alpha_1\alpha_2}$ are the sum of step functions and a δ-measure with weights*

$$\frac{(\sigma[\varrho] - [\varrho v])t}{\sqrt{1+\sigma^2}}, \qquad \frac{(\sigma[\varrho v] - [\varrho v^2])t}{\sqrt{1+\sigma^2}},$$

respectively, which forms a δ- shock wave solution of transport equations (5.4) with (5.6).

Proof: Take $\zeta = \frac{x}{t}$. Then, for each fixed $\alpha_1, \alpha_2 > 0$, the solution of Riemann Problem can be written as

$$(\varrho^{\alpha_1\alpha_2}, v^{\alpha_1\alpha_2}) = \begin{cases} (\varrho_-, v_-), & \zeta < \sigma_1^{\alpha_1\alpha_2}, \\ (\varrho_*^{\alpha_1\alpha_2}, v_*^{\alpha_1\alpha_2}), & \sigma_1^{\alpha_1\alpha_2} < \zeta < \sigma_2^{\alpha_1\alpha_2}, \\ (\varrho_+, v_+), & \zeta > \sigma_2^{\alpha_1\alpha_2}, \end{cases} \tag{5.34}$$

which satisfies the following weak formulation

$$-\int_{-\infty}^{\infty} \left((v^{\alpha_1\alpha_2} - \zeta)\varrho^{\alpha_1\alpha_2} - 2\alpha_1 v^{\alpha_1\alpha_2}\right)\phi' d\zeta - \int_{-\infty}^{\infty} \varrho^{\alpha_1\alpha_2}\phi d\zeta = 0, \tag{5.35}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} \left((\varrho^{\alpha_1\alpha_2} - \alpha_1)(v^{\alpha_1\alpha_2})^2 - \varrho^{\alpha_1\alpha_2} v^{\alpha_1\alpha_2}\zeta + \epsilon \left(\frac{\varrho^{\alpha_1\alpha_2}}{1 - \theta\varrho^{\alpha_1\alpha_2}} \right)^{\Gamma} \right) \phi' d\zeta \\ &- \int_{-\infty}^{\infty} \varrho^{\alpha_1\alpha_2} v^{\alpha_1\alpha_2}\phi d\zeta = 0, \end{aligned} \tag{5.36}$$

for any test function $\phi \in C_0^\infty(R^+ \times R)$.

Taking the first integral of (5.35) and on decomposing it, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \left((v^{\alpha_1\alpha_2} - \zeta)\varrho^{\alpha_1\alpha_2} - 2\alpha_1 v^{\alpha_1\alpha_2}\right)\phi' d\zeta \\ &= \left(\int_{-\infty}^{\sigma_1^{\alpha_1\alpha_2}} + \int_{\sigma_1^{\alpha_1\alpha_2}}^{\sigma_2^{\alpha_1\alpha_2}} + \int_{\sigma_2^{\alpha_1\alpha_2}}^{+\infty} \right) \left((v^{\alpha_1\alpha_2} - \zeta)\varrho^{\alpha_1\alpha_2} - 2\alpha_1 v^{\alpha_1\alpha_2}\right)\phi' d\zeta. \end{aligned} \tag{5.37}$$

The limit of the sum of first and third terms of RHS of (5.37), we have

$$\begin{aligned} &\lim_{\alpha_1,\alpha_2\to 0} \left(\int_{-\infty}^{\sigma_1^{\alpha_1\alpha_2}} + \int_{\sigma_2^{\alpha_1\alpha_2}}^{+\infty} \right) \left((v^{\alpha_1\alpha_2} - \zeta)\varrho^{\alpha_1\alpha_2} - 2\alpha_1 v^{\alpha_1\alpha_2}\right)\phi' d\zeta \\ &= \lim_{\alpha_1,\alpha_2\to 0} \left(\int_{-\infty}^{\sigma_1^{\alpha_1\alpha_2}} \left((v^{\alpha_1\alpha_2} - \zeta)\varrho^{\alpha_1\alpha_2} - 2\alpha_1 v^{\alpha_1\alpha_2}\right)\phi' d\zeta \right) \\ &+ \lim_{\alpha_1,\alpha_2\to 0} \left(\int_{\sigma_2^{\alpha_1\alpha_2}}^{+\infty} \left((v^{\alpha_1\alpha_2} - \zeta)\varrho^{\alpha_1\alpha_2} - 2\alpha_1 v^{\alpha_1\alpha_2}\right)\phi' d\zeta \right) \end{aligned}$$

$$= (\sigma[\varrho] - [\varrho v])\,\phi(\sigma) + \int_{-\infty}^{+\infty} h(\zeta - \sigma)\phi d\zeta,$$

where

$$h(\zeta - \sigma) = \begin{cases} \varrho_-, & \zeta < \sigma, \\ \varrho_+, & \zeta > \sigma. \end{cases}$$

Taking the limit of second integral of RHS of (5.37), we have

$$\lim_{\alpha_1,\alpha_2\to 0} \int_{\sigma_1^{\alpha_1\alpha_2}}^{\sigma_2^{\alpha_1\alpha_2}} \left((v_*^{\alpha_1\alpha_2} - \zeta)\varrho_*^{\alpha_1\alpha_2} - 2\alpha_1 v_*^{\alpha_1\alpha_2}\right) \phi' d\zeta = \lim_{\alpha_1,\alpha_2\to 0} \varrho_*^{\alpha_1\alpha_2} \left(\sigma_2^{\alpha_1\alpha_2} - \sigma_1^{\alpha_1\alpha_2}\right)$$
$$\times \left(\left(\frac{\phi(\sigma_2^{\alpha_1\alpha_2}) - \phi(\sigma_1^{\alpha_1\alpha_2})}{\sigma_2^{\alpha_1\alpha_2} - \sigma_1^{\alpha_1\alpha_2}} \right) v_* - \frac{\sigma_2^{\alpha_1\alpha_2}\phi(\sigma_2^{\alpha_1\alpha_2}) - \sigma_1^{\alpha_1\alpha_2}\phi(\sigma_1^{\alpha_1\alpha_2})}{\sigma_2^{\alpha_1\alpha_2} - \sigma_1^{\alpha_1\alpha_2}} + \frac{1}{\sigma_2^{\alpha_1\alpha_2} - \sigma_1^{\alpha_1\alpha_2}} \int_{\sigma_1^{\alpha_1\alpha_2}}^{\sigma_2^{\alpha_1\alpha_2}} \phi d\zeta \right)$$
$$- \lim_{\alpha_1,\alpha_2\to 0} 2\alpha_1 v_*^{\alpha_1\alpha_2} \left(\phi(\sigma_2^{\alpha_1\alpha_2}) - \phi(\sigma_1^{\alpha_1\alpha_2})\right) = (\sigma[\varrho] - [\varrho v]) \left(\sigma\phi'(\sigma) - \sigma\phi'(\sigma) - \phi(\sigma) + \phi(\sigma)\right) = 0.$$

Hence

$$\int_{-\infty}^{\infty} \varrho^{\alpha_1\alpha_2}\phi d\zeta = (\sigma[\varrho] - [\varrho v])\,\phi(\sigma) + \int_{-\infty}^{+\infty} h(\zeta - \sigma)\phi d\zeta.$$

Now, from (5.36), we have

$$\int_{-\infty}^{\infty} \varrho^{\alpha_1\alpha_2} v^{\alpha_1\alpha_2}\phi d\zeta = \int_{-\infty}^{\infty} \left((\varrho^{\alpha_1\alpha_2} - \alpha_1)(v^{\alpha_1\alpha_2})^2 - \varrho^{\alpha_1\alpha_2} v^{\alpha_1\alpha_2}\zeta + \epsilon \left(\tfrac{\varrho^{\alpha_1\alpha_2}}{1-\theta\varrho^{\alpha_1\alpha_2}} \right)^{\Gamma} \right) \phi' d\zeta. \tag{5.38}$$

On decomposing the integral of RHS of (5.38), we get

$$\left(\int_{-\infty}^{\sigma_1^{\alpha_1\alpha_2}} + \int_{\sigma_1^{\alpha_1\alpha_2}}^{\sigma_2^{\alpha_1\alpha_2}} + \int_{\sigma_2^{\alpha_1\alpha_2}}^{+\infty} \right) \left((\varrho^{\alpha_1\alpha_2} - \alpha_1)(v^{\alpha_1\alpha_2})^2 - \varrho^{\alpha_1\alpha_2} v^{\alpha_1\alpha_2}\zeta + \epsilon \left(\frac{\varrho^{\alpha_1\alpha_2}}{1-\theta\varrho^{\alpha_1\alpha_2}} \right)^{\Gamma} \right) \phi' d\zeta. \tag{5.39}$$

Taking limit $\alpha_1, \alpha_2 \to 0$, of sum of first and second integral of RHS of (5.39), we obtain that

$$\lim_{\alpha_1,\alpha_2\to 0} \left(\int_{-\infty}^{\sigma_1^{\alpha_1\alpha_2}} + \int_{\sigma_1^{\alpha_1\alpha_2}}^{\sigma_2^{\alpha_1\alpha_2}} + \int_{\sigma_2^{\alpha_1\alpha_2}}^{+\infty} \right) \left((\varrho^{\alpha_1\alpha_2} - \alpha_1)(v^{\alpha_1\alpha_2})^2 - \varrho^{\alpha_1\alpha_2} v^{\alpha_1\alpha_2}\zeta \right) \phi' d\zeta$$
$$= \left(\sigma[\varrho v] - [\varrho v^2]\right) \phi(\sigma) + \int_{-\infty}^{+\infty} \hat{h}(\zeta - \sigma)\phi d\zeta,$$

where

$$\hat{h}(\zeta-\sigma)=\begin{cases}\varrho_-, & \zeta<\sigma,\\ \varrho_+, & \zeta>\sigma.\end{cases}$$

and the limit of third term of (5.39) on using Lemma 5.2 and Lemma 5.3, we have

$$\begin{aligned}
&\lim_{\alpha_1,\alpha_2\to 0}\left(\int_{-\infty}^{\sigma_1^{\alpha_1\alpha_2}}+\int_{\sigma_1^{\alpha_1\alpha_2}}^{\sigma_2^{\alpha_1\alpha_2}}+\int_{\sigma_2^{\alpha_1\alpha_2}}^{+\infty}\right)\epsilon\left(\frac{\varrho^{\alpha_1\alpha_2}}{1-\theta\varrho^{\alpha_1\alpha_2}}\right)^{\Gamma}\phi' d\zeta\\
&=\lim_{\alpha_1,\alpha_2\to 0}\epsilon\int_{-\infty}^{\sigma_1^{\alpha_1\alpha_2}}\left(\frac{\varrho_-^{\alpha_1\alpha_2}}{1-\theta\varrho_-^{\alpha_1\alpha_2}}\right)^{\Gamma}\phi' d\zeta+\lim_{\alpha_1,\alpha_2\to 0}\epsilon\int_{\sigma_1^{\alpha_1\alpha_2}}^{\sigma_2^{\alpha_1\alpha_2}}\left(\frac{\varrho_*^{\alpha_1\alpha_2}}{1-\theta\varrho_*^{\alpha_1\alpha_2}}\right)^{\Gamma}\phi' d\zeta\\
&+\lim_{\alpha_1,\alpha_2\to 0}\epsilon\int_{\sigma_2^{\alpha_1\alpha_2}}^{+\infty}\left(\frac{\varrho_+^{\alpha_1\alpha_2}}{1-\theta\varrho_+^{\alpha_1\alpha_2}}\right)^{\Gamma}\phi' d\zeta\\
&=\lim_{\alpha_1,\alpha_2\to 0}\epsilon\left(\left(\frac{\varrho_-^{\alpha_1\alpha_2}}{1-\theta\varrho_-^{\alpha_1\alpha_2}}\right)^{\Gamma}\phi(\sigma_1^{\alpha_1\alpha_2})-\left(\frac{\varrho_+^{\alpha_1\alpha_2}}{1-\theta\varrho_+^{\alpha_1\alpha_2}}\right)^{\Gamma}\phi(\sigma_2^{\alpha_1\alpha_2})\right)\\
&+\lim_{\alpha_1,\alpha_2\to 0}\epsilon\left(\frac{\varrho_*^{\alpha_1\alpha_2}}{1-\theta\varrho_*^{\alpha_1\alpha_2}}\right)^{\Gamma}\phi(\sigma_2^{\alpha_1\alpha_2}-\sigma_1^{\alpha_1\alpha_2})\\
&=0.
\end{aligned}$$

Hence,

$$\int_{-\infty}^{\infty}\varrho^{\alpha_1\alpha_2}v^{\alpha_1\alpha_2}\phi d\zeta=\left(\sigma[\varrho v]-[\varrho v^2]\right)\phi(\sigma)+\int_{-\infty}^{+\infty}\hat{h}(\zeta-\sigma)\phi d\zeta.$$

Now, we consider the limits of $\varrho^{\alpha_1\alpha_2}v^{\alpha_1\alpha_2}$ and $\varrho^{\alpha_1\alpha_2}$ depending on t, then for any test function $\phi\in C_0^\infty(R^+\times R)$, we have

$$\begin{aligned}
&\lim_{\alpha_1,\alpha_2\to 0}\int_0^\infty\int_{-\infty}^{\infty}\varrho^{\alpha_1\alpha_2}\left(\frac{x}{t}\right)\phi(x,t)dxdt\\
&=\lim_{\alpha_1,\alpha_2\to 0}\int_0^\infty t\left(\int_{-\infty}^{\infty}\varrho^{\alpha_1\alpha_2}(\zeta)\phi(\zeta t,t)d\zeta\right)dt\\
&=\int_0^\infty t\left(\left(\sigma[\varrho]-[\varrho v]\right)\phi(\sigma t,t)dt+\int_{-\infty}^{+\infty}h(\zeta-\sigma)\phi(\zeta t,t)d\zeta\right)dt\\
&=\int_0^\infty t\left(\sigma[\varrho]-[\varrho v]\right)\phi(\sigma t,t)dt+\int_0^\infty\int_{-\infty}^{+\infty}h(x-\sigma t)\phi(x,t)dxdt.
\end{aligned}$$

From definition 5.1, we have

$$\int_0^\infty t\,(\sigma[\varrho]-[\varrho v])\,\phi(\sigma t,t)dt = \langle f(.)\delta_\Gamma, phi(.)\rangle, \tag{5.40}$$

where $f(t)=\frac{(\sigma[\varrho]-[\varrho v])t}{\sqrt{1+\sigma^2}}$.

Similarly, we can get

$$\begin{aligned}\int_0^\infty\int_{-\infty}^\infty \varrho^{\alpha_1\alpha_2}v^{\alpha_1\alpha_2}\left(\frac{x}{t}\right)\phi(x,t)dxdt &= \langle f(.)\delta_\Gamma, phi(.)\rangle \\ &\quad+\int_0^\infty\int_{-\infty}^{+\infty}\hat{h}(x-\sigma t)\phi(x,t)dxdt,\end{aligned} \tag{5.41}$$

where $f(t)=\frac{(\sigma[\varrho v]-[\varrho v^2])t}{\sqrt{1+\sigma^2}}$.

Hence proof is complete.

5.5 Cavitation in Riemann solution to (5.5) and (5.6) under flux approximation

Here, we consider the flux approximation limit of the Riemann solution of (5.5) and (5.6) as $\alpha_1,\alpha_2\to 0$ in case $(\varrho_+,v_+)\in R_1R_2(\varrho_-,v_-)$ with $v_-<v_+$ and $\varrho_\pm>0$. Let $(\varrho_*^{\alpha_1\alpha_2},v_*^{\alpha_1\alpha_2})$ be intermediate states for $\alpha_1,\alpha_2>0$, which connects to the left state (ϱ_-,v_-) and right state (ϱ_+,v_+) by 1- rarefaction wave and 2- rarefaction wave curves, respectively. Then, 1-rarefaction wave satisfies

$$\begin{cases}\xi = v^{\alpha_1\alpha_2}-\sqrt{\frac{\epsilon\alpha_2\Gamma(\varrho^{\alpha_1\alpha_2})^{\Gamma-2}}{(1-\theta\varrho^{\alpha_1\alpha_2})^{\Gamma+1}}(\varrho^{\alpha_1\alpha_2}-2\alpha_1)},\\ v_- -\sqrt{\frac{\epsilon\alpha_2\Gamma\varrho_-^{\Gamma-2}}{(1-\theta\varrho_-)^{\Gamma+1}}(\varrho_- -2\alpha_1)}<\xi<v_*^{\alpha_1\alpha_2}-\sqrt{\frac{\epsilon\alpha_2\Gamma(\varrho_*^{\alpha_1\alpha_2})^{\Gamma-2}}{(1-\theta\varrho_*^{\alpha_1\alpha_2})^{\Gamma+1}}(\varrho_*^{\alpha_1\alpha_2}-2\alpha_1)},\\ \varrho_*^{\alpha_1\alpha_2}<\varrho_-,\end{cases} \tag{5.42}$$

and 2-rarefaction wave satisfies

$$\begin{cases} \xi = v^{\alpha_1\alpha_2} + \sqrt{\frac{\epsilon\alpha_2\Gamma(\varrho^{\alpha_1\alpha_2})^{\Gamma-2}}{(1-\theta\varrho^{\alpha_1\alpha_2})^{\Gamma+1}}(\varrho^{\alpha_1\alpha_2} - 2\alpha_1)}, \\ v_*^{\alpha_1\alpha_2} - \sqrt{\frac{\epsilon\alpha_2\Gamma(\varrho_*^{\alpha_1\alpha_2})^{\Gamma-2}}{(1-\theta\varrho_*^{\alpha_1\alpha_2})^{\Gamma+1}}(\varrho_*^{\alpha_1\alpha_2} - 2\alpha_1)} < \xi < v_+ - \sqrt{\frac{\epsilon\alpha_2\Gamma\varrho_+^{\Gamma-2}}{(1-\theta\varrho_+)^{\Gamma+1}}(\varrho_+ - 2\alpha_1)}, \\ \varrho_*^{\alpha_1\alpha_2} < \varrho_+, \end{cases} \tag{5.43}$$

Theorem 5.7. *Assuming $v_- < v_+$ and $(\varrho^{\alpha_1\alpha_2}, v^{\alpha_1\alpha_2})$ is a rarefaction wave solution of (5.5) and (5.6). Then there exist $\alpha_0 > 0$, such that the constant density solution occurs in the solution when $0 < \alpha_1 < \alpha_0$ and $0 < \alpha_2 < \alpha_0$. Also, the rarefaction waves tend to contact discontinuities ($x/t = v_\pm$) connecting the states $(\varrho_\pm, v_\pm)$ and the vacuum state ($\varrho = 0$), which form vacuum state solution for the transport equations.*

Proof: Let $\alpha_1 = \alpha_2 = \alpha_0$, since $(\varrho_*^{\alpha_1\alpha_2}, v_*^{\alpha_1\alpha_2})$ is on 1-rarefaction wave curve, we have $v_*^{\alpha_1\alpha_2} = v_- - \int_{\varrho_-}^{\varrho_*^{\alpha_1\alpha_2}} \sqrt{\frac{\epsilon\alpha_0\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z-2\alpha_0)}}dz \leq v_- - \int_{2\alpha_0}^{\varrho_-} \sqrt{\frac{\epsilon\alpha_0\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z-2\alpha_0)}}dz = B^{\alpha_0}$. If $v_- < v_+ < B^{\alpha_0}$, then there exist α_{01} such that $(\varrho_+, v_+) \in R_1R_2(\varrho_-, v_-)$ i.e. no constant density solution. However, the constant density solution occurs when $B^{\alpha_0} < v_+$ i.e. there exist α_{02} such that $(\varrho_+, v_+) \in R_1R_2(\varrho_-, v_-)$.
Let $g(\alpha) = \int_{2\alpha}^{\varrho_-} \sqrt{\frac{\epsilon\alpha\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z-2\alpha)}}dz - v_+ + v_-$, is a continuous function of α such that when $B^{\alpha_0} > v_+$, then $g(\alpha_{01}) > 0$ and when $B^{\alpha_0} < v_+$, then $g(\alpha_{02}) < 0$. Therefore, $g(\alpha_{01})g(\alpha_{02}) < 0$, which implies that there exist $\alpha_0 \in (\alpha_{01}, \alpha_{02})$ such that $g(\alpha_0) = 0$. Hence, when $0 < \alpha_1 < \alpha_0$ and $0 < \alpha_2 < \alpha_0$, then the intermediate state becomes constant state such that

$$(\varrho_*^{\alpha_1\alpha_2}, v_*^{\alpha_1\alpha_2})(\xi) = (2\alpha_1, \xi), \qquad v_1^{\alpha_1\alpha_2} \leq \xi \leq v_2^{\alpha_1\alpha_2},$$

where

$$v_1^{\alpha_1\alpha_2} = v_- + \int_{2\alpha_1}^{\varrho_-} \sqrt{\frac{\epsilon\alpha_1\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z-2\alpha_1)}}dz,$$

and

$$v_2^{\alpha_1\alpha_2} = v_+ - \int_{2\alpha_1}^{\varrho_+} \sqrt{\frac{\epsilon\alpha_1\Gamma z^{\Gamma-2}}{(1-\theta z)^{\Gamma+1}(z-2\alpha_1)}}dz.$$

Taking limit $\alpha_1, \alpha_2 \to 0$, we obtain

$$\lim_{\alpha_1,\alpha_2\to 0} \varrho_*^{\alpha_1,\alpha_2} = 0,$$

and

$$\lim_{\alpha_1,\alpha_2\to 0} v_{1,2}^{\alpha_1,\alpha_2} = v_\mp.$$

Hence, from above result it is clear that limit solution of (5.5) and (5.6) is a solution of transport equation containing vacuum state formed by two contact discontinuities ($x/t = v_\pm$). Theorem is proved.

5.6 Conclusions

In the present work, we investigated the phenomenon of concentration and cavitation in the solutions of the Riemann Problem for the dusty gasdynamics by employing two parameters flux approximation technique. It is obtained that the model considered in this study is genuinely nonlinear for both the characteristic fields. The Riemann solutions for the transport equations are constructed and obtained that it consists two contact discontinuities and a vacuum state between them and a delta shock wave solution when $v_- < v_+$ and $v_- > v_+$, respectively. Further, we obtained the self similar solution for the approximated system and later on, the Rankine-Hugoniot condition for the approximated system is derived to determine the bounded discontinuous solution. It is also obtained that for the case $v_- > v_+$, the Riemann solutions of approximated system converges to a delta shock wave solution of the transport equations as $\alpha_1\alpha_2$ tend to zero and for the case $v_- < v_+$, as $\alpha_1\alpha_2$ tend to zero, the Riemann solutions consisting two rarefaction waves of approximated system converges to the two contact discontinuities and vacuum state between them connecting the left state (v_-, ϱ_-) to the right state (v_+, ϱ_+) which is a vacuum state solution of the transport equations. Similar analysis of the Riemann solutions for different hyperbolic systems is investigated previously by [34, 41] and [45].

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