Chapter 4

Delta shock wave solution of the Riemann Problem for the non-homogeneous modified Chaplygin gasdynamics *

"In mathematics, the art of proposing a question must be held of higher value than solving it"

–Georg Cantor

* "The contents of this chapter have been published in ***Journal of Dynamics and Differential Equations 1-18, 33 (2020)".***
DOI: https://doi.org/10.1007/s10884-020-09914-8

4.1 Introduction

This chapter investigates the classical and non-classical Riemann solution for the inhomogeneous system of partial differential equations which is of great interest to many researchers in applied mathematics and engineering sciences. The study of the hyperbolic systems have significant physical background which is interesting as it leads to diverse complex problems in Mathematics. For instance, authors in [97] have studied for the first time the homogeneous MCG equation model. In exotic background of fluid phenomenon, the MCG model plays an important role to describe the accelerated expansion of the universe and evolution of the perturbations of energy density. Also, It describes the dark energy and dark matter in the unified form. For further applications related to the modified Chaplygin gas equation, the interested reader is referred to [98, 4, 99, 100, 101] and references cited therein.

In this chapter, we propose to consider the following one-dimensional modified Chaplygin gas equation with constant external force term as,

$$\begin{cases} \partial_t \varrho + \partial_x (\varrho v) = 0, \\ \partial_t (\varrho v) + \partial_x (\varrho v^2 + P) = \eta \varrho, \end{cases} \tag{4.1}$$

where ϱ is the density and v is the velocity of the gas. The parameter η is a constant. Here the constant external force term appearing in the momentum equation of the model is treated as a coulomb-like friction term. The scalar $P = P(\varrho, \alpha)$ is known as the modified Chaplygin gas pressure defined as $P(\varrho, \alpha) = \alpha p(\varrho)$, where the parameter $\alpha > 0$ is constant.

The equation of state for the MCG equation is satisfied by the pressure p, given by

$$p(\varrho) = \varrho \left(A - \frac{B}{\varrho^2} \right),$$

where the parameter $A > 0$ and $B > 0$ are constants. The present chapter is mainly devoted to the Riemann Problem for the MCG model (4.1) with the initial data

$$(\varrho, v)(x,0) = \begin{cases} (\varrho_-, v_-), & x < 0, \\ (\varrho_+, v_+), & x > 0, \end{cases} \tag{4.2}$$

where $v_\pm$ and $\varrho_\pm$ are constants.

If, we take $\eta = 0$ in (4.1), it becomes the modified Chaplygin gas model which is studied by [46] when pressure vanishes. For $A = 0$, $B = 1$ and $\eta = 0$, the system (4.1) reduces into the homogeneous system of the Chaplygin gas equation as

$$\begin{cases} \partial_t \varrho + \partial_x (\varrho v) = 0, \\ \partial_t (\varrho v) + \partial_x \left(\varrho v^2 - \frac{1}{\varrho} \right) = 0, \end{cases} \tag{4.3}$$

which was proposed by [5], [86], and [87] to compute the lifting force in aerodynamics by using some mathematical approximations. The Chaplygin gas model is also utilized to discuss the concept of dark matter and dark energy related mathematical problems. Authors in [56] have studied the two dimensional Riemann Problem for the homogeneous Chaplygin gas model (4.3). The author in [89, 102] has studied the solution of the Riemann Problem with constant and variable initial data for non-homogeneous hyperbolic system. The governing equations

$$\begin{cases} \partial_t \varrho + \partial_x (\varrho v) = 0, \\ \partial_t (\varrho v) + \partial_x (\varrho v^2 + p(\varrho)) = 0, \end{cases} \tag{4.4}$$

with

$$p(\varrho) = -\frac{B}{\varrho^\alpha}, \qquad B > 0, 0 < \alpha \leq 1, \tag{4.5}$$

is known as the generalized Chaplygin gas dynamics. Authors in [103] have studied the Riemann Problem for the system (4.4) and (4.5) with initial conditions (4.2). An exact solution to the Riemann Problem (4.4) and (4.5) with friction is also developed in [91]. Further, authors in [104, 105] have introduced system (4.4) and (4.5) for dark energy because of its dark energy like evolution. Study of modified Chaplygin gas (MCG) model have advantage to unite the explanation of the dark matter and dark energy. Also, the Riemann Problem for hyperbolic model of two phase flow is studied in [106]. For more details of two phase model in conservative form the interested readers are referred to [107, 108, 109, 110]. In this study, it is shown that the Riemann Problem for the MCG in the presence of external force, has non-classical wave solution in a special situation. Delta shock is also a non-classical non-linear wave on which at least one of the state variables become a singular measure. The Rankine-Hugoniot conditions which are used to study the strength, position and propagation speed for the singular wave, are derived. The concept of $\delta-$ function in classical discontinuous solution was proposed by [111]. For more interesting results related to delta shock (non-classical wave solution), interested readers are referred to [34, 45, 46, 51]. The curved delta shock wave solution was firstly introduced in the study of the Riemann Problem for the pressureless Euler system with the Coulomb-like friction term. The constant external force term (Coulomb-like friction term) appearing in the momentum equation of the MCG model was used first time in [95]. The advantage of the source term appearing in the MCG model (4.1) is that the inhomogeneous model (4.1) can be reformulated in to the homogeneous conservation form which enable us to determine the solution of the Riemann Problem for the MCG model which causes to bent all the waves including shock wave, contact discontinuity, rarefaction wave and delta shock wave into the parabolic shape, and the solution of the Riemann Problem for the model (4.1) is not self-similar solution.

The motivation of this study is to obtain the exact solution (classical and non-classical wave solution both) to the Riemann Problem for the MCG equation with constant external force due to its wide applications in the area of aerodynamics, cosmology, astrophysics and engineering. Many researchers and scientists from several areas of science and engineering are working currently on the Riemann Problem for homogeneous and non-homogeneous model. From last two decades, study of the solution of the Riemann Problem for the homogeneous and non-homogeneous Chaplygin gas equation have great interest among the researchers. The Riemann Problem with classical and non-classical wave solution for the MCG model in the presence of constant external force is not studied by any researcher till now.

This chapter is structured into following sections as: In section (4.1), the inhomogeneous system (4.1) is modified into homogeneous conservative system by using new state variables and obtain the general properties of the modified system. Furthermore, the classical Riemann solution for modified system is discussed and non-classical solution involved in the Riemann solution for the modified system in certain situations is also obtained. The R-H relations are obtained for the delta shock and studied the exact location, strength and propagation speed of delta shock wave. In section (4.3), we obtain the solution of the Riemann Problem for the inhomogeneous system (4.1) with the help of result obtained in section (4.2) for the modified conservative system. The Rankine-Hugoniot conditions are derived. Section (4.4) contains conclusions of this study.

4.2 Riemann Problem for modified system

We study the Riemann solution for the modified homogeneous conservative form of (4.1) by introducing new variable for the velocity, $u(x,t) = v(x,t) - \eta t$.

Authors in [96] have introduced this new state variable to discuss the Riemann Problem for inhomogeneous shallow water equations. On insertion of this new velocity in (4.1), we obtain the following conservation form of the MCG model with constant external force

$$\begin{cases} \partial_t \varrho + \partial_x \left(\varrho(u+\eta t) \right) = 0, \\ \partial_t \left(\varrho u \right) + \partial_x \left(\varrho u(u+\eta t) + \alpha \left(A\varrho - \frac{B}{\varrho} \right) \right) = 0. \end{cases} \tag{4.6}$$

We consider the Riemann Problem for the modified conservative model (4.6) with the same initial data,

$$(\varrho, u)(x,0) = \begin{cases} (\varrho_-, v_-), & x < 0, \\ (\varrho_+, v_+), & x > 0. \end{cases} \tag{4.7}$$

Now, the Riemann solution for the original model (4.1) and (4.2) can be determined from the corresponding ones to the system (4.6) and (4.7) by utilizing the new state variables, $(\varrho, v)(x,t) = (\varrho, u+\eta t)(x,t)$. Reformulating (4.6) into the quasi-linear form as

$$MU_t + NU_x = 0, \tag{4.8}$$

where $U = \begin{pmatrix} \varrho \\ u \end{pmatrix}$, $M = \begin{pmatrix} 1 & 0 \\ u & \varrho \end{pmatrix}$ and $N = \begin{pmatrix} u+\eta t & \varrho \\ u(u+\eta t) + \alpha A + \frac{\alpha B}{\varrho^2} & \varrho(2u+\eta t) \end{pmatrix}$.
Let $\lambda_1(\varrho, u)$ and $\lambda_2(\varrho, u)$ are two eigenvalues of the matrix N given by

$$\lambda_1(\varrho, u) = u + \eta t - \left(\alpha(A + \frac{B}{\varrho^2}) \right)^{1/2}, \lambda_2(\varrho, u) = u + \eta t + \left(\alpha(A + \frac{B}{\varrho^2}) \right)^{1/2}, \tag{4.9}$$

and the right eigenvectors corresponding to both eigenvalues are

$$d_1 = \left[-\varrho \quad \left(\alpha(A + \frac{B}{\varrho^2}) \right)^{1/2} \right]^{Tr}, d_2 = \left[\varrho \quad \left(\alpha(A + \frac{B}{\varrho^2}) \right)^{1/2} \right]^{Tr}. \tag{4.10}$$

Thus, it leads to $\nabla\lambda_i \cdot d_i = A\alpha\left(\alpha(A+\frac{B}{\varrho^2})\right)^{-1/2} \neq 0, i = 1, 2,$ for $A > 0, \alpha > 0.$ Here, ∇ represents the gradient operator with respect to (ϱ, u). Since for $A > 0, \alpha > 0$, $\nabla\lambda_i \cdot d_i \neq 0$ which implies that the characteristic fields corresponding to the eigenvalues λ_1 and λ_2 are genuinely nonlinear. Hence the associated elementary waves are either rarefaction waves (continuous solution) or shock waves (bounded discontinuous solution) denoted by R and S, respectively. Along these characteristic fields the Riemann invariants are defined as

$$\begin{aligned} w &= u - \left(\alpha(A+\frac{B}{\varrho^2})\right)^{1/2} - \sqrt{A\alpha}\left(\ln\left(\left(A+\frac{B}{\varrho^2}\right)^{1/2} - \sqrt{A}\right) + \ln\varrho\right), \\ z &= u + \left(\alpha(A+\frac{B}{\varrho^2})\right)^{1/2} + \sqrt{A\alpha}\left(\ln\left(\left(A+\frac{B}{\varrho^2}\right)^{1/2} - \sqrt{A}\right) + \ln\varrho\right). \end{aligned} \tag{4.11}$$

Now, our attention is to study all the elementary waves composing of the Riemann solution for the modified model (4.6) in detail. Firstly, we study the rarefaction wave which is continuous solution satisfying (4.6) which may be computed by solving the integral curve of both characteristic fields. It is noticeable that 1-Riemann invariant (2-Riemann invariant) is conserved in the 1-rarefaction wave (2-rarefaction wave), respectively.

For a given left state (ϱ_-, u_-), the state (ϱ, u) can be connected to the state (ϱ_-, u_-) in the phase plane by the 1-rarefaction wave curve denoted by $R_1(\varrho_-, u_-)$ is

$$R_1(\varrho_-, u_-): \begin{cases} \frac{dx}{dt} = \lambda_1 = u + \eta t - \left(\alpha(A+\frac{B}{\varrho^2})\right)^{1/2}, \\ u - f(\varrho, \alpha) = u_- - f(\varrho_-, \varrho, \alpha) = w_-, \\ \lambda_1(\varrho_-, u_-) \leq \lambda_1(\varrho, u), \end{cases} \tag{4.12}$$

where

$$f(\varrho_-,\varrho,\alpha)=\int_{\varrho_-}^{\varrho}\frac{\left(\alpha(A+\frac{B}{\varrho^2})\right)^{1/2}}{\varrho}d\varrho=-\left(\alpha(A+\frac{B}{\varrho^2})\right)^{1/2}+\left(\alpha(A+\frac{B}{\varrho_-^2})\right)^{1/2}$$
$$+\sqrt{A\alpha}\left\{\ln\left(\left(A+\frac{B}{\varrho^2}\right)^{1/2}-\sqrt{A}\right)-\left(\ln\left(\left(A+\frac{B}{\varrho_-^2}\right)^{1/2}-\sqrt{A}\right)\right)\right\}$$
$$+\sqrt{A\alpha}\left(\ln\varrho-\ln\varrho_-\right). \tag{4.13}$$

On differentiating u of second equation of (4.12) with respect to ϱ, we obtain that $\frac{du}{d\varrho}<0$ and $\frac{d^2u}{d\varrho^2}>0$ which implies that the 1-rarefaction wave is made up of half branch of $R_1(\varrho_-,u_-)$ satisfying $u\geq u_-$ and $\varrho\leq\varrho_-$, which is convex in (ϱ,u) plane. From first equation of (4.12), we obtain

$$\frac{x}{t}-\frac{1}{2}\eta t=u-\left(\alpha(A+\frac{B}{\varrho^2})\right)^{1/2}. \tag{4.14}$$

Using (4.14) in second equation of (4.12), we obtain the solution (ϱ,u) at a point (x,t) inside the 1-rarefaction wave which is given by

$$\begin{aligned}\varrho(x,t)&=\left(\frac{B}{2\sqrt{A}}\left(\exp\left(\frac{w_--\frac{x}{t}+\frac{1}{2}\eta t}{\sqrt{\alpha A}}\right)\right)^{-1}\right),\\ u(x,t)&=\left(\frac{x}{t}-\frac{1}{2}\eta t+\sqrt{\alpha A}\left(1+\frac{4}{B}\exp 2\left(\frac{w_--\frac{x}{t}+\frac{1}{2}\eta t}{\sqrt{\alpha A}}\right)\right)^{1/2}\right).\end{aligned} \tag{4.15}$$

Analogously, the 2-rarefaction wave curve denoted by $R_2(\varrho_-, u_-)$ in the phase plane is

$$R_2(\varrho_-, u_-):\begin{cases}\frac{dx}{dt} = \lambda_2 = u + \eta t + \left(\alpha(A + \frac{B}{\varrho^2})\right)^{1/2}, \\ u + f(\varrho,\alpha) = u_- + f(\varrho_-, \varrho, \alpha) = z_-, \\ \lambda_2(\varrho_-, u_-) \leq \lambda_2(\varrho, u).\end{cases} \tag{4.16}$$

As before, we differentiate u of second equation of (4.16) with respect to ϱ, we obtain that $\frac{du}{d\varrho} > 0$ and $\frac{d^2u}{d\varrho^2} < 0$. Thus, the 2-rarefaction wave is made up of half branch of $R_2(\varrho_-, u_-)$ satisfying $u \geq u_-$ and $\varrho \geq \varrho_-$ which is concave in nature in the (ϱ, u) plane. From first equation of (4.16), we obtain

$$\frac{x}{t} - \frac{1}{2}\eta t = u + \left(\alpha(A + \frac{B}{\varrho^2})\right)^{1/2}. \tag{4.17}$$

Using (4.17) in (4.16), we obtain the solution (ϱ, u) at a point (x, t) inside the 2-rarefaction wave which is given by

$$\begin{aligned}\varrho(x,t) &= \left(\frac{B}{2\sqrt{A}}\left(\exp\left(\frac{z_- - \frac{x}{t} + \frac{1}{2}\eta t}{\sqrt{\alpha A}}\right)\right)^{-1}\right), \\ u(x,t) &= \left(\frac{x}{t} - \frac{1}{2}\eta t + \sqrt{\alpha A}\left(1 + \frac{4}{B}\exp 2\left(\frac{z_- - \frac{x}{t} + \frac{1}{2}\eta t}{\sqrt{\alpha A}}\right)\right)^{1/2}\right).\end{aligned} \tag{4.18}$$

Now, we obtain the shock wave which is bounded discontinuous solution, satisfying the R-H relations and the entropy conditions. Let the propagating speed of the bounded discontinuity at $x = x(t)$ is denoted by $\xi(t) = x'(t)$, then R-H relations for (4.6) are

$$\begin{cases}-\xi(t)\,[\varrho] + [\varrho(u + \eta t)] = 0, \\ -\xi(t)\,[\varrho u] + \left[\varrho u(u + \eta t) + \alpha\left(A\varrho - \frac{B}{\varrho}\right)\right] = 0,\end{cases} \tag{4.19}$$

where $[\varrho] = \varrho_r - \varrho_l$ with $\varrho_l = \varrho(x(t) - 0, t)$, $\varrho_r = \varrho(x(t) + 0, t)$ represents the jump of ϱ across the discontinuity.

If $\xi(t) \neq 0$, then from (4.19) we obtain

$$(\varrho_r - \varrho_l)\left(\varrho_r u_r(u_r + \eta t) + \alpha\left(A\varrho_r - \frac{B}{\varrho_r}\right) - \varrho_l u_l(u_l + \eta t) - \alpha\left(A\varrho_l - \frac{B}{\varrho_l}\right)\right) \\ = (\varrho_r u_r - \varrho_l u_l)(\varrho_r(u_r + \eta t) - \varrho_l(u_l + \eta t)). \tag{4.20}$$

After simplifying (4.20), yields

$$u_r = u_l \pm (\varrho_r - \varrho_l)\sqrt{\frac{\alpha}{\varrho_l \varrho_r}\left(A + \frac{B}{\varrho_l \varrho_r}\right)}. \tag{4.21}$$

Thus, for a given left state (ϱ_-, v_-), the possible states can be connected to (ϱ_-, v_-) on the right by 1-shock wave curve, denoted by $S_1(\varrho_-, v_-)$, should satisfy

$$S_1(\varrho_-, v_-) : \begin{cases} \xi_1(t) = \frac{\varrho u - \varrho_- v_-}{\varrho - \varrho_-} + \eta t, \\ u - v_- = -\sqrt{\frac{\alpha}{\varrho_- \varrho}\left(A + \frac{B}{\varrho_- \varrho}\right)}(\varrho - \varrho_-), \\ v_- > u, \varrho_- < \varrho. \end{cases} \tag{4.22}$$

Analogously, the 2-shock wave curve, denoted by $S_2(\varrho_-, v_-)$, should satisfy

$$S_2(\varrho_-, v_-) : \begin{cases} \xi_2(t) = \frac{\varrho u - \varrho_- v_-}{\varrho - \varrho_-} + \eta t, \\ u - v_- = \sqrt{\frac{\alpha}{\varrho_- \varrho}\left(A + \frac{B}{\varrho_- \varrho}\right)}(\varrho - \varrho_-), \\ v_- > u, \varrho_- > \varrho. \end{cases} \tag{4.23}$$

It is clear that the set of possible states, connected on the right, consist of the 1-shock wave $S_1(\varrho_-, v_-)$, 1-rarefaction wave curve $R_1(\varrho_-, v_-)$, 2-Shock wave $S_2(\varrho_-, v_-)$ and the 2-rarefaction wave $R_2(\varrho_-, v_-)$. Thus, for the given state (ϱ_-, v_-), the phase plane can be divided into four regions by the curves of the elementary waves $R_1(\varrho_-, v_-)$,

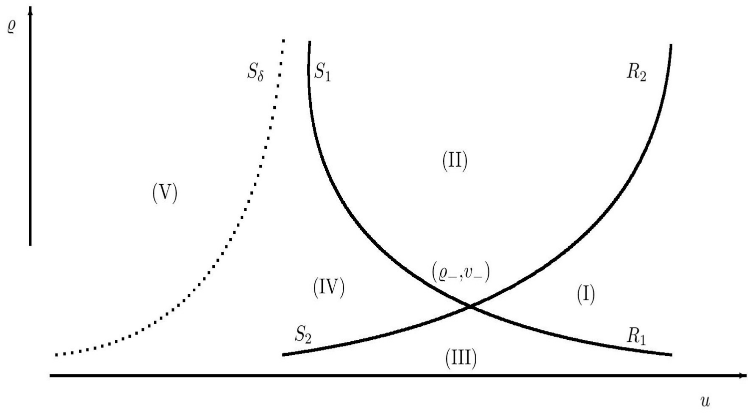


FIGURE 4.1: The (ϱ, u) phase plane for the model (4.6).

$R_2(\varrho_-, v_-)$, $S_1(\varrho_-, v_-)$ and $S_2(\varrho_-, v_-)$.

If $v_+ + g(\varrho_+, \alpha) \leq v_- - g(\varrho_-, \alpha)$ is satisfied, where $g(\varrho, \alpha) = \Big(\alpha(A + \frac{B}{\varrho^2})\Big)^{1/2}$, then the Riemann solution of (4.6) and (4.7) can not be structured by using only the rarefaction wave and shock wave, without introducing delta shock wave. In this non-classical situation, we introduce the phenomena of delta shock wave such as in [88, 103]. Thus we can draw the curve for the delta shock wave through the point $(\varrho_-, v_- - g(\varrho_-, \alpha))$ which is denoted by

$$S_\delta(\varrho_-, v_- - g(\varrho_-, \alpha)) : v + g(\varrho, \alpha) = v_- - g(\varrho_-, \alpha). \tag{4.24}$$

After discussing all the elementary waves, we obtain that the phase plane (ϱ, u) is divided into five regions $(I), (II), (III), (IV)$ and (V) by these curves as shown in figure 4.1.

Now, in view of the right state (ϱ_+, v_+) in the different regions, we can structure the unique global Riemann solution of the system (4.6) and (4.7) which connects two constant states (ϱ_-, v_-) and (ϱ_+, v_+).

If $(\varrho_+, v_+) \in (I) \cup (II) \cup (III) \cup (IV)$, then the Riemann solution of (4.6) and (4.7) consists 1-wave and 2-wave with a constant state (ϱ_*, v_*) (i.e. intermediate state) between them.

If $(\varrho_+, v_+) \in (I)$, then the solution of Riemann Problem of (4.6) and (4.7) consists of rarefaction waves R_1, R_2 and (ϱ_*, u_*) obtained by

$$\begin{cases} u_* - h(\varrho_*, \alpha) = v_- - h(\varrho_-, \alpha) = w_-, \\ u_* + h(\varrho_*, \alpha) = v_+ + h(\varrho_+, \alpha) = z_+, \end{cases} \tag{4.25}$$

where $h(\varrho, \alpha) = \left(\alpha(A + \frac{B}{\varrho^2})\right)^{1/2} + \sqrt{A\alpha}\left(\ln\left(\left(A + \frac{B}{\varrho^2}\right)^{1/2} - \sqrt{A}\right) + \ln \varrho\right)$.

Thus, the Riemann solution of system (4.6) and (4.7) can be written as

$$(\varrho, u)(x,t) = \begin{cases} (\varrho_-, v_-), & x < X_1^-(t), \\ R_1, & X_1^-(t) \le x \le X_1^+(t), \\ (\varrho_*, u_*), & X_1^+(t) \le x \le X_2^-(t), \\ R_2, & X_2^-(t) \le x \le X_2^+(t), \\ (\varrho_+, v_+), & X_2^+(t) < x, \end{cases} \tag{4.26}$$

where

$$\begin{aligned} X_1^-(t) &= \left(v_- - \left(\alpha(A + \frac{B}{\varrho^2})\right)^{1/2}\right) t + \frac{\eta t^2}{2}, \\ X_1^+(t) &= \left(u_* - \left(\alpha(A + \frac{B}{\varrho_*^2})\right)^{1/2}\right) t + \frac{\eta t^2}{2}, \\ X_2^-(t) &= \left(u_* + \left(\alpha(A + \frac{B}{\varrho_*^2})\right)^{1/2}\right) t + \frac{\eta t^2}{2}, \\ X_2^+(t) &= \left(v_+ + \left(\alpha(A + \frac{B}{\varrho_+^2})\right)^{1/2}\right) t + \frac{\eta t^2}{2}. \end{aligned} \tag{4.27}$$

If $(\varrho_+, v_+) \in (II)$, then the solution of Riemann Problem of (4.6) and (4.7) consists of 1-shock wave S_1 and 2-rarefaction waves R_2, and (ϱ_*, u_*) obtained by

$$\begin{cases} u_* - v_- = -\sqrt{\frac{\alpha}{\varrho_- \varrho_*}\left(A + \frac{B}{\varrho_- \varrho_*}\right)}(\varrho_* - \varrho_-), \\ u_* + f(\varrho_*, \alpha) = v_+ + f(\varrho_+, \alpha), \end{cases} \tag{4.28}$$

then, the Riemann solution of (4.6) and (4.7) is,

$$(\varrho, u)(x,t) = \begin{cases} (\varrho_-, v_-), & x < X_1(t), \\ (\varrho_*, u_*), & X_1(t) \le x \le X_2^-(t), \\ R_2, & X_2^-(t) \le x \le X_2^+(t), \\ (\varrho_+, v_+), & X_2^+(t) < x, \end{cases} \tag{4.29}$$

where

$$X_1(t) = \left(\frac{\varrho_* u_* - \varrho_- v_-}{\varrho_* - \varrho_-}\right) t + \frac{\eta t^2}{2}. \tag{4.30}$$

If $(\varrho_+, v_+) \in (III)$, then the solution of Riemann Problem of (4.6) and (4.7) consists of 1-rarefaction wave R_1, 2-shock wave S_2, and the intermediate state (ϱ_*, u_*) obtained by

$$\begin{cases} u_* - f(\varrho_*, \alpha) = v_- - f(\varrho_-, \alpha), \\ v_+ - u_* = -\sqrt{\frac{\alpha}{\varrho_* \varrho_+}\left(A + \frac{B}{\varrho_+ \varrho_*}\right)}(\varrho_+ - \varrho_*), \end{cases} \tag{4.31}$$

then, the Riemann solution of system (4.6) and (4.7) can be written as

$$(\varrho, u)(x,t) = \begin{cases} (\varrho_-, v_-), & x < X_1^-(t), \\ R_1, & X_1^-(t) \le x \le X_1^+(t), \\ (\varrho_*, u_*), & X_1^+(t) \le x \le X_2(t), \\ (\varrho_+, v_+), & X_2(t) < x, \end{cases} \tag{4.32}$$

where

$$X_2(t) = \left(\frac{\varrho_+ v_+ - \varrho_* u_*}{\varrho_+ - \varrho_*} \right) t + \frac{\eta t^2}{2}. \tag{4.33}$$

If $(\varrho_+, v_+) \in (IV)$, then the Riemann solution of (4.6) and (4.7) consists of 1-shock wave S_1 and 2-shock wave S_2, and the intermediate constant state (ϱ_*, u_*) obtained by

$$\begin{cases} u_* - v_- = -\sqrt{\frac{\alpha}{\varrho_- \varrho_*}\left(A + \frac{B}{\varrho_- \varrho_*}\right)}(\varrho_* - \varrho_-), \\ v_+ - u_* = -\sqrt{\frac{\alpha}{\varrho_* \varrho_+}\left(A + \frac{B}{\varrho_+ \varrho_*}\right)}(\varrho_+ - \varrho_*), \end{cases} \tag{4.34}$$

then the Riemann solution of (4.6) and (4.7) can be written as,

$$(\varrho, u)(x,t) = \begin{cases} (\varrho_-, v_-), & x < X_1(t), \\ (\varrho_*, u_*), & X_1(t) \leq x \leq X_2(t), \\ (\varrho_+, v_+), & X_2(t) < x. \end{cases} \tag{4.35}$$

However, when $(\varrho_+, u_+) \in (V)$, then non-classical situation appears where the Cauchy problem usually does not own a weak L^∞-solution. In the framework of non-classical solution, we determine the solution of the Riemann Problem (4.6) and (4.7), containing a weighted δ-measure (i.e. delta shock) supported on a curve, should be defined as in [34, 90]. For more exact definition of the non-classical wave solution with delta measure initial data we recommend the reader to the ref. [34, 45, 111, 51].

Definition 4.1. If the primitive variables ϱ and u be a pair of distributions where $\varrho(x,t) = \hat{\varrho}(x,t) + y(x,t)\delta(\mathit{\Gamma})$ in which singular part is defined by $y(x,t)\delta(\mathit{\Gamma}) = \sum_{i \in \wedge} y_i(x,t)\delta(\gamma_i)$,
where $\wedge$ is finite index set and $\hat{\varrho}, u \in L^\infty(R \times R_+)$. Then the primitive variables (density and velocity) constitute the delta shock wave solution of the Riemann Problem

(4.6) and (4.7) if it satisfies (See [38]),

$$\begin{cases} \langle \varrho, \phi_t \rangle + \langle \varrho(u+\eta t), \phi_x \rangle = 0, \\ \langle \varrho u, \phi_t \rangle + \langle \varrho u(u+\eta t) + \alpha \left(A\varrho + \frac{B}{\varrho} \right), \phi_x \rangle = 0, \end{cases} \tag{4.36}$$

for all $\phi \in C_0^\infty(R \times R_+)$. The inner product is defined as

$$\begin{aligned} &\langle \varrho u(u+\eta t) + \alpha \left(A\varrho + \frac{B}{\varrho} \right), \phi_x \rangle \\ &= \int_0^\infty \int_{-\infty}^\infty \left(\hat{\varrho} u(u+\eta t) + \alpha \left(A\hat{\varrho} + \frac{B}{\hat{\varrho}} \right) \right) \phi_x dxdt + \langle y u_\delta (u_\delta + \eta t) \delta_S, \phi_x \rangle. \end{aligned} \tag{4.37}$$

Here we use the symbol S for the smooth delta shock wave curve, u_δ is the value of u and $\left(A\varrho + \frac{B}{\varrho} \right)$ vanishes on the curve S.

If $(\varrho_+, u_+) \in (V)$, then the piecewise smooth Riemann solution of (4.6) and (4.7) is assumed as

$$(\varrho, u)(x,t) = \begin{cases} (\varrho_-, v_-), & x < x(t), \\ (y(t)\delta(x - x(t)), u_\delta), & x = x(t), \\ (\varrho_+, v_+), & x > x(t), \end{cases} \tag{4.38}$$

where $x = x(t)$ represents the delta shock wave curve, $y(t)$ is strength of delta shock wave and u_δ is the assignment of u on the delta shock curve.

Now, we check that the delta shock wave solution (4.38) to the Riemann Problem (4.6) and (4.7) should hold the generalized R-H conditions given as

$$\begin{cases} \frac{dx(t)}{dt} = \sigma(t) = u_\delta + \eta t, \\ \frac{dy(t)}{dt} = \sigma(t)[\varrho] - [\varrho(u+\eta t)], \\ \frac{d(y(t)u_\delta)}{dt} = \sigma(t)[\varrho u] - \left[\varrho u(u+\eta t) + \alpha \left(A\varrho + \frac{B}{\varrho} \right) \right], \end{cases} \tag{4.39}$$

with the initial data $x(0) = 0, y(0) = 0$.

Let Q be any point on the delta shock wave curve $\Gamma : (x,t)|x = x(t)$ and let ζ be a small ball with center P. Then we consider that ζ intersects to the curve Γ at the point $Q_1 = (x(t_1), t_1)$ and $Q_2 = (x(t_2), t_2)$, where $t_1 < t_2$, which cuts the ball into two part ζ_- (left hand part) and ζ_+ (right hand part). Then, we have

$$\begin{aligned}
I_1 &= \int\int_\zeta (\varrho\phi_t + \varrho(u+\eta t)\phi_x)\, dxdt \\
&= \int\int_{\zeta_-} (\varrho_-\phi_t + \varrho_-(v_- + \eta t)\phi_x)\, dxdt \\
&+ \int\int_{\zeta_+} (\varrho_+\phi_t + \varrho(v_+ + \eta t)\phi_x)\, dxdt \\
&+ \int_{t_1}^{t_2} (y(t)u_\delta\,(\phi_t(x(t),t) + (u_\delta + \eta t)\phi_x(x(t),t)))\, dt \\
&\quad = \int\int_{\zeta_-} ((\varrho_-\phi)_t + (\varrho_-(v_- + \eta t)\phi)_x)\, dxdt \\
&\quad + \int\int_{\zeta_+} ((\varrho_+\phi)_t + (\varrho(v_+ + \eta t)\phi)_x)\, dxdt \\
&\quad + \int_{t_1}^{t_2} y(t)d\phi(x(t),t),
\end{aligned}$$

where $\phi(x,t) \in C_0^\infty(\zeta)$. Using divergence theorem, we get

$$\begin{aligned}
I_1 &= \int_{\partial\zeta_-} -\varrho_-\phi dx + \varrho_-(v_- + \eta t)\phi dt \\
&+ \int_{\partial\zeta_+} -\varrho_+\phi dx + \varrho_+(v_+ + \eta t)\phi dt + \int_{t_1}^{t_2} y(t)d\phi(x(t),t) \\
&= \int_{t_1}^{t_2} \left((\varrho_+ - \varrho_-)\frac{dx}{dt} + \varrho_-(v_- + \eta t) - \varrho_+(v_+\eta t) \right) \phi(x(t),t)dt \\
&+ \int_{t_1}^{t_2} y(t)d\phi(x(t),t),
\end{aligned}$$

where $\partial\zeta_\pm$ is the boundary of $\zeta_\pm$.

Hence, the second equation of (4.39) holds when I_1 vanishes for any test function

$\phi(x,t) \in C_0^\infty(\zeta)$. Similarly we check third equality of (4.39)

$$
\begin{aligned}
I_2 = & \int\int_\zeta \left(\varrho u \phi_t + \left(\varrho u(u+\eta t) + \alpha A \varrho - \frac{\alpha B}{\varrho} \right) \phi_x \right) dxdt \\
= & \int\int_{\zeta_-} \left(\varrho_- v_- \phi_t + \left(\varrho_- v_- (v_- + \eta t) + \alpha A \varrho_- - \frac{\alpha B}{\varrho_-} \right) \phi_x \right) dxdt \\
+ & \int\int_{\zeta_+} \left(\varrho_+ v_+ \phi_t + \left(\varrho_+ v_+ (v_+ + \eta t) + \alpha A \varrho_+ - \frac{\alpha B}{\varrho_+} \right) \phi_x \right) dxdt \\
+ & \int_{t_1}^{t_2} y(t) u_\delta \left(\phi_t + (u_\delta + \eta t) \phi_x \right) dt \\
= & \int_{t_1}^{t_2} \left((\varrho_+ v_+ - \varrho_- v_+) \frac{dx}{dt} + \varrho_- v_- (v_- + \eta t) + \alpha A (\varrho_- - \varrho_+) \right) \\
- & \left(\alpha B \left(\frac{1}{\varrho_-} - \frac{1}{\varrho_+} \right) - \varrho_+ v_+ (v_+ + \eta t) \right) \phi(x(t), t) dt \\
+ & \int_{t_1}^{t_2} y(t) u_\delta d\phi(x(t), t).
\end{aligned}
$$

Thus, the third equation of (4.39) holds when I_2 vanishes for any test function $\phi(x,t) \in C_0^\infty(\zeta)$.

Now, to guarantee the uniqueness of the solution for the system (4.6) it is necessary that it should obey the over-compressive entropy condition for the delta shock wave as

$$\lambda_1(\varrho_+, u_+) < \lambda_2(\varrho_+, u_+) < \sigma(t) < \lambda_1(\varrho_-, u_-) < \lambda_2(\varrho_-, u_-), \tag{4.40}$$

such that

$$v_+ + \sqrt{\alpha \left(A + \frac{B}{\varrho_+^2} \right)} < u_\delta < v_- - \sqrt{\alpha \left(A + \frac{B}{\varrho_-^2} \right)}. \tag{4.41}$$

The equation (4.39) shows the relation among the strength, location and propagating speed of the delta shock wave. The condition (4.40) is an over-compressive condition for the delta shock wave which implies that all the characteristic lines on the both side of the delta shock curve are not out-going.

From equation (4.39),

$$\frac{dy(t)}{dt} = u_\delta(\varrho_+ - \varrho_-) - (\varrho_+ v_+ - \varrho_- v_-), \tag{4.42}$$

$$\begin{aligned} u_\delta \frac{dy(t)}{dt} = &(u_\delta + \eta t)\left((\varrho_+ v_+ - \varrho_- v_-) - (\varrho_+ v_+^2 - \varrho_- v_-^2)\right) \\ &- \alpha(u_\delta + \eta t)(\varrho_+ - \varrho_-)\left(A + \frac{B}{\varrho_- \varrho_+}\right). \end{aligned} \tag{4.43}$$

On substituting (4.42) into (4.43), we get

$$\begin{aligned} (\varrho_+ - \varrho_-)u_\delta^2 - &2u_\delta(\varrho_+ v_+ - \varrho_- v_-) + (\varrho_+ v_+^2 - \varrho_- v_-^2) \\ &- \alpha(\varrho_+ - \varrho_-)\left(A + \frac{B}{\varrho_- \varrho_+}\right) = 0. \end{aligned} \tag{4.44}$$

Equation (4.44) is quadratic in u_δ, so if $\varrho_+ \neq \varrho_-$ we have

$$u_\delta = \frac{\varrho_+ v_+ - \varrho_- v_- + \mu}{\varrho_+ - \varrho_-}, \tag{4.45}$$

where

$$\mu = \sqrt{\varrho_- \varrho_+ (v_+ - v_-)^2 - \alpha(\varrho_+ - \varrho_-)^2 \left(A + \frac{B}{\varrho_- \varrho_+}\right)}, \tag{4.46}$$

which enables to get,

$$\sigma(t) = u_\delta + \eta t, \quad x(t) = u_\delta t + \frac{1}{2}\eta t^2, \quad y(t) = \mu t. \tag{4.47}$$

If $\varrho_+ = \varrho_-$, then from (4.44) we get

$$u_\delta = \frac{v_+ - v_-}{2}, \tag{4.48}$$

which implies that

$$\sigma(t) = \frac{v_+ - v_-}{2} + \eta t, \quad x(t) = \frac{v_+ - v_-}{2}t + \frac{1}{2}\eta t^2, \quad y(t) = -(\varrho_+ v_+ - \varrho_- v_-)t. \tag{4.49}$$

4.3 Riemann Problem for the original system (4.1)

In this section, we construct the Riemann solution for the original system (4.1) and (4.2) for all cases. First we discuss the cases that the solution of the Riemann Problem does not consist delta shock wave.

If $(\varrho_+, v_+) \in (I) \cup (II) \cup (III) \cup (IV)$, then we can obtain the solution of the Riemann Problem for the original system (4.1) and (4.2) directly corresponding to the system (4.6) and (4.7) by applying the change of variables $(\varrho, v)(x,t) = (\varrho, u + \eta t)$, where the position of the elementary waves (shock wave and rarefaction wave) is unchanged. For example, let $(\varrho_+, v_+) \in (I)$,then the solution of the Riemann Problem for the system (4.1) and (4.2) can be written as

$$(\varrho, v)(x,t) = \begin{cases} (\varrho_-, v_- + \eta t), & x < X_1^-(t), \\ (\varrho_1, u_1 + \eta t), & X_1^-(t) \le x \le X_1^+(t), \\ (\varrho_*, u_* + \eta t), & X_1^+(t) \le x \le X_2^-(t), \\ (\varrho_2, u_2 + \eta t), & X_2^-(t) \le x \le X_2^+(t), \\ (\varrho_+, v_+ + \eta t), & X_2^+(t) < x, \end{cases} \tag{4.50}$$

where $X_1^-(t), X_1^+(t), X_2^-(t), X_2^+(t)$ are defined by (4.27). This situation is illustrated in Figure 4.2.

If $(\varrho_+, v_+) \in (V)$, then the weak solution, in the sense of distribution, of the Riemann Problem for the original system (4.1) and (4.2) is defined similarly.

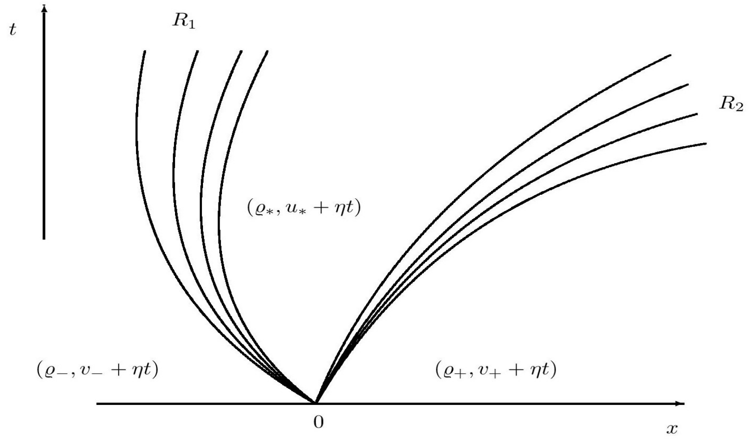


FIGURE 4.2: The Riemann solution to the system (4.1) and (4.2).

Definition 4.2. If the primitive variables ϱ and u be a pair of distributions where $\varrho(x,t) = \hat{\varrho}(x,t) + y(x,t)\delta(\Gamma)$. Then (ϱ, u) is called the delta shock wave solution of the RP (4.1) and (4.2) if it holds,

$$\begin{cases} \langle \varrho, \phi_t \rangle + \langle \varrho v, \phi_x \rangle = 0, \\ \langle \varrho v, \phi_t \rangle + \langle \varrho v^2 + A\varrho - \frac{B}{\varrho}, \phi_x \rangle = -\langle \eta \varrho, \phi \rangle, \end{cases} \tag{4.51}$$

for all $\phi \in C_0^\infty(R \times R_+)$. In which the inner product is defined as

$$\langle \varrho v^2 + A\varrho - \frac{B}{\varrho}, \phi \rangle = \int_{R^+} \int_{\infty} \left(\hat{\varrho} v^2 + A\hat{\varrho} - \frac{B}{\hat{\varrho}} \right) \phi dxdt + \langle y(t) u_\delta^2 \delta_S, \phi \rangle. \tag{4.52}$$

If $v_+ + g(\varrho_+, \alpha) \leq v_- - g(\varrho_-, \alpha)$ is satisfied, from above definition the delta shock wave solution for the system (4.1) and (4.2) may be considered in the following form

$$(\varrho, v)(x,t) = \begin{cases} (\varrho_-, v_- + \eta t), & x < x(t), \\ (y(t)\delta(x - x(t)), u_\delta), & x = x(t), \\ (\varrho_+, v_+ + \eta t), & x > x(t), \end{cases} \tag{4.53}$$

in which $(v_\delta + \eta t)$ be assumed as constant. As before we discuss the concept of delta shock wave solution for the modified system here again. Now we discuss the delta shock wave solution to the RP for the original system (4.1) and (4.2). Therefore the delta shock wave solution (4.53) must satisfy the R-H conditions

$$\begin{cases} \frac{dX(t)}{dt} = \sigma(t) = v_\delta, \\ \frac{dy(t)}{dt} = \sigma(t)[\varrho] - [\varrho v], \\ \frac{d(y(t)v_\delta)}{dt} = \sigma(t)[\varrho v] - \left[\varrho v^2 + \alpha\left(A\varrho + \frac{B}{\varrho}\right)\right] + \eta y(t), \end{cases} \tag{4.54}$$

in which the jump is defined as

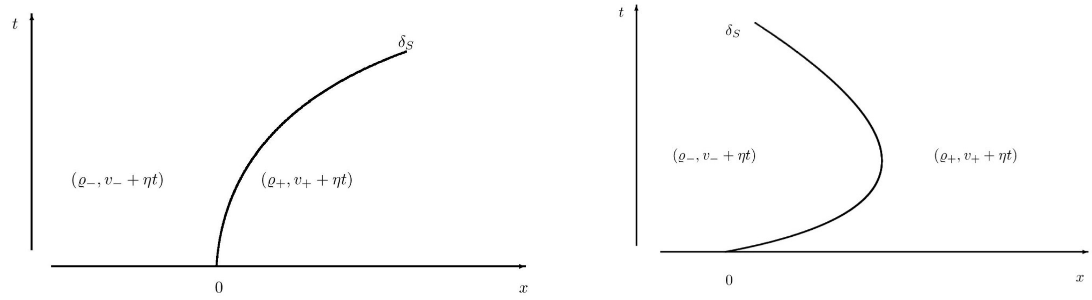
$$[\varrho v] = \varrho_+(v_+ + \eta t) - \varrho_-(v_- + \eta t),$$

$$\begin{aligned} \left[\varrho v^2 + \alpha\left(A\varrho - \frac{B}{\varrho}\right)\right] &= \varrho_+(v_+ + \eta t)^2 + \alpha\left(A\varrho_+ - \frac{B}{\varrho_+}\right) \\ &\quad -\varrho_-(v_- + \eta t)^2 - \alpha\left(A\varrho_- - \frac{B}{\varrho_-}\right). \end{aligned} \tag{4.55}$$

For the unique solution of the Riemann Problem for the model (4.1) and (4.2), the over-compressive entropy condition for the delta shock wave

$$v_+ + \sqrt{\alpha\left(A + \frac{B}{\varrho_+^2}\right)} + \eta t < v_\delta < v_- - \sqrt{\alpha\left(A + \frac{B}{\varrho_-^2}\right)} + \eta t. \tag{4.56}$$

should also be assumed. Similarly we can determine $y(t)$, $\sigma(t)$ and $x(t)$ from (4.54-4.56) together as before determined. To study in detail, we consider the following theorem to depict the Riemann solution of the original system (4.1) and (4.2) when $v_+ + \sqrt{\alpha\left(A - \frac{B}{\varrho_+^2}\right)} < v_- - \sqrt{\alpha\left(A - \frac{B}{\varrho_-^2}\right)}$ and $\varrho_- \neq \varrho_+$ is satisfied which is illustrated in Figure 4.3.



(a) $\eta > 0$ (b) $\eta < 0$

FIGURE 4.3: Delta shock wave solution to the Riemann Problem (4.1) and (4.2) for (a) $\eta > 0$ and (b) $\eta < 0$.

Theorem 4.3. *If* $v_+ + \sqrt{\alpha\left(A - \frac{B}{\varrho_+^2}\right)} < v_- - \sqrt{\alpha\left(A - \frac{B}{\varrho_-^2}\right)}$ *and* $\varrho_- \neq \varrho_+$*, then the delta shock wave solution to the Riemann Problem (4.1) and (4.2) can be written as*

$$(\varrho, v)(x,t) = \begin{cases} (\varrho_-, v_- + \eta t), & x < x(t), \\ (y(t)\delta(x - x(t)), v_\delta + \eta t), & x = x(t), \\ (\varrho_+, v_+ + \eta t), & x > x(t), \end{cases} \tag{4.57}$$

where $y(t) = \mu t$ *and* $x(t) = u_\delta t - \frac{1}{2}\eta t^2$ *represent the strength and position of the delta shock wave solution (4.57) and (4.2),respectively. Here*

$$\mu = \sqrt{\varrho_-\varrho_+(v_+ - v_-)^2 - \alpha(\varrho_+ - \varrho_-)^2\left(A + \frac{B}{\varrho_-\varrho_+}\right)}, \tag{4.58}$$

$$u_\delta = \frac{\varrho_+ v_+ - \varrho_- v_- + \mu}{\varrho_+ - \varrho_-}, \tag{4.59}$$

Proof. Let $v_+ + \sqrt{\alpha\left(A - \frac{B}{\varrho_+^2}\right)} < v_- - \sqrt{\alpha\left(A - \frac{B}{\varrho_-^2}\right)}$ and $\varrho_- \neq \varrho_+$ is satisfied, then it is required to obtain the delta shock wave solution (4.57) to the Riemann Problem (4.1) and (4.2). Consider the Rankine-Hugoniot condition (4.54). According to the jump condition across the discontinuity, the second equality of (4.54) can be written as

$$\frac{dy(t)}{dt} = (\sigma(t) - \eta t)(\varrho_+ - \varrho_-) - (\varrho_+ v_+ - \varrho_- v_-). \tag{4.60}$$

Taking into account the assumption that $v_\delta - \eta t$ is constant and $\sigma(t) = v_\delta$, then the third equality of (4.54) is

$$\begin{aligned} \sigma(t)\frac{dy(t)}{dt} &= (\sigma(t) - 2\eta t)\big((\varrho_+ v_+ - \varrho_- v_-) - (\varrho_+ v_+^2 - \varrho_- v_-^2) \\ &\quad - \alpha(\varrho_+ - \varrho_-)\left(A + \frac{B}{\varrho_- \varrho_+}\right) + \sigma(t)\eta t(\varrho_+ - \varrho_-) - (\varrho_+ - \varrho_-)\eta^2 t^2. \end{aligned} \tag{4.61}$$

On substituting (4.60) in (4.61), we get

$$\begin{aligned} (\varrho_+ - \varrho_-)(\sigma(t) - \eta t)^2 - 2(\varrho_+ v_+ - \varrho_- v_-)(\sigma(t) - \eta t) + (\varrho_+ v_+^2 - \varrho_- v_-^2) \\ + \alpha(\varrho_+ - \varrho_-)\left(A + \frac{B}{\varrho_- \varrho_+}\right) = 0. \end{aligned} \tag{4.62}$$

Thus, if $\varrho_+ \neq \varrho_-$, then $\sigma(t) = v_\delta(t) = u_\delta + \eta t$ can be obtained by the condition (4.56), in which μ and u_δ can be determined from (4.58) and (4.59), respectively. The strength of the delta shock wave $y(t) = \mu t$ can be obtained directly from R-H conditions (4.54).

Now, we check that the delta shock wave solution for the Riemann Problem (4.1) and (4.2) must satisfy the given definition (4.50) in the sense of distribution. i.e., we verify that the equation (4.57) with (4.58) and (4.59) should satisfy

$$\begin{cases} D_1 = \langle \varrho, \phi_t \rangle + \langle \varrho v, \phi_x \rangle = 0, \\ D_2 = \langle \varrho v, \phi_t \rangle + \langle \varrho v^2 + \alpha A \varrho - \frac{\alpha B}{\varrho}, \phi_x \rangle = -\langle \eta \varrho, \phi \rangle, \end{cases} \tag{4.63}$$

for any test function $\phi \in C_0^\infty(R \times R_+)$. Which is a weak form of Riemann Problem (4.1) and (4.2). The proof of this theorem is completely similar to the proof given by the author [90]. Therefore, we only give the main steps of the proof for second equation of (4.51) for completeness.

Since the delta shock curve is given by

$$x(t) = u_\delta t + \frac{1}{2}\eta t^2. \tag{4.64}$$

If $u_\delta, \eta > 0$ or $u_\delta, \eta < 0$ (see figure $3(a)$ and $3(b)$), then $x(t)$ is invertible function for all time given by

$$t(x) = -\frac{u_\delta}{\eta} + \sqrt{\frac{u_\delta^2}{\eta^2} + \frac{2x}{\eta}}. \tag{4.65}$$

Since $x'(t)$ changes its sign across the point $\left(-\frac{u_\delta^2}{2\eta}, -\frac{u_\delta}{\eta}\right)$, the point is called critical point, on the delta shock wave curve. Thus, the inverse function of $x(t)$ is required to determine respectively for $t \leq -\frac{u_\delta}{\eta}$ and $t > -\frac{u_\delta}{\eta}$, as

$$t(x) = \begin{cases} -\frac{u_\delta}{\eta} - \sqrt{\frac{u_\delta^2}{\eta^2} + \frac{2x}{\eta}}, & t \leq -\frac{u_\delta}{\eta}, \\ -\frac{u_\delta}{\eta} + \sqrt{\frac{u_\delta^2}{\eta^2} + \frac{2x}{\eta}}, & t > -\frac{u_\delta}{\eta}. \end{cases} \tag{4.66}$$

For convenience, let $\eta > 0$, it follows from (4.64) that the location of delta shock wave must satisfy $x = x(t) > 0$ for all the time.

$$\begin{aligned} \frac{d\phi(x,t)}{dt} &= \phi_t(x,t) + \frac{dx}{dt}\phi_x(x,t) \\ &= \phi_t(x,t) + (u_\delta + \eta t)\phi_x(x,t) \\ &= \phi_t(x,t) + v_\delta(t)\phi_x(x,t). \end{aligned}$$

Now, our attention is to prove the second equality D_2 which is given as

$$
\begin{aligned}
D_2 &= \int_0^\infty \int_{-\infty}^\infty \left(\varrho v \phi_t + \left(\varrho v^2 + \alpha A \varrho - \frac{\alpha B}{\varrho} \right) \phi_x \right) dxdt \\
&= \int_0^\infty \int_{-\infty}^{x(t)} \left(\varrho_-(v_- + \eta t)\phi_t + \left(\varrho_-(v_- + \eta t)^2 + \alpha A \varrho_- - \frac{\alpha B}{\varrho_-} \right) \phi_x \right) dxdt \\
&+ \int_0^\infty \int_{x(t)}^\infty \left(\varrho_+(v_+ + \eta t)\phi_t + \left(\varrho_+(v_+ + \eta t)^2 + \alpha A \varrho_+ - \frac{\alpha B}{\varrho_+} \right) \phi_x \right) dxdt \\
&+ \int_0^\infty y(t) v_\delta(t) \left(\phi_t(x(t), t) + v_\delta(t) \phi_x(x(t), t) \right) dt. \\
&= \int_0^\infty \int_{t(x)}^\infty \varrho_-(v_- + \eta t)\phi_t dtdx \\
&+ \int_0^\infty \int_{-\infty}^{x(t)} \left(\varrho_-(v_- + \eta t)^2 + \alpha A \varrho_- - \frac{\alpha B}{\varrho_-} \right) \phi_x dxdt \\
&+ \int_0^\infty \int_0^{t(x)} \varrho_+(v_+ + \eta t)\phi_t dtdx \\
&+ \int_0^\infty \int_{x(t)}^\infty \left(\varrho_+(v_+ + \eta t)^2 + \alpha A \varrho_+ - \frac{\alpha B}{\varrho_+} \right) \phi_x dxdt \\
&+ \int_0^\infty \mu t(u_\delta + \eta t) d\phi(x(t), t).
\end{aligned}
$$

We substitute the following equations in the above integral equation,

$$
\begin{aligned}
\int_{t(x)}^\infty & \varrho_-(v_- + \eta t)\phi_t dt + \int_0^{t(x)} \varrho_+(v_+ + \eta t)\phi_t dt \\
&= \left(\varrho_+(v_+ + \eta t(x)) - \varrho_-(v_- + \eta t(x)) \right) \phi(x, t(x)) \\
&- \int_0^{t(x)} \eta \varrho_+ \phi(x, t(x)) dt - \int_{t(x)}^\infty \eta \varrho_- \phi(x, t(x)) dt.
\end{aligned} \tag{4.67}
$$

$$
\int_0^\infty \mu t(u_\delta + \eta t) d\phi(x(t), t) = -\int_0^\infty \mu(u_\delta + 2\eta t)\phi(x(t), t) dt. \tag{4.68}
$$

We get,

$$\begin{aligned}D_2 = &\int_0^\infty \{(\varrho_+(v_+ + \eta t(x)) - \varrho_-(v_- + \eta t(x)))\}\, \phi(x, t(x))dx \\ &+ \int_0^\infty \left(\varrho_-(v_- + \eta t)^2 - \varrho_+(v_+ + \eta t)^2 - \alpha(\varrho_+ - \varrho_-)\left(A + \frac{B}{\varrho_-\varrho_+}\right)\right)\phi(x(t), t)dt \\ &- \int_0^\infty \int_0^{t(x)} \eta\varrho_+\phi(x, t(x))dt - \int_0^\infty \int_{t(x)}^\infty \eta\varrho_-\phi(x, t(x))dt \\ &- \int_0^\infty \mu(u_\delta + 2\eta t)\phi(x(t), t)dt.\end{aligned}$$

On applying the change of variables, we obtain

$$\begin{aligned}D_2 = \int_0^\infty E(t)\phi(x(t), t)dt &- \int_0^\infty \int_{-\infty}^{x(t)} \eta\varrho_-\phi(x(t), t)dxdt \\ &- \int_0^\infty \int_{x(t)}^\infty \eta\varrho_+\phi(x(t), t)dxdt,\end{aligned} \tag{4.69}$$

where,

$$\begin{aligned}E(t) = &(u_\delta + \eta t)\left(\varrho_+(v_+ + \eta t) - \varrho_-(v_- + \eta t)\right) + \varrho_-(v_- + \eta t)^2 - \varrho_+(v_+ + \eta t)^2 \\ &- \alpha A(\varrho_+ - \varrho_-) - \alpha B\left(\frac{1}{\varrho_-} - \frac{1}{\varrho_+}\right) - \mu(u_\delta + 2\eta t).\end{aligned}$$

On the insertion of u_δ from (4.45) in the above expression, yields

$$E(t) = \frac{\varrho_-\varrho_+(v_+ - v_-)^2}{(\varrho_+ - \varrho_-)} - \alpha(\varrho_+ - \varrho_-)\left(A + \frac{B}{\varrho_-\varrho_+}\right) - \frac{\mu^2}{(\varrho_+ - \varrho_-)} - \eta\mu t. \tag{4.70}$$

Using μ^2 from (4.46) in (4.70), we get

$$E(t) = -\eta\mu t = -\eta y(t). \tag{4.71}$$

Thus, we can conclude from the equations (4.69) and (4.71) together with second equality of (4.63) is satisfied in the sense of distributions. Hence, the proof is complete. □

Remark 4.4. If $v_+ + \sqrt{\alpha\left(A - \frac{B}{\varrho_+^2}\right)} < v_- - \sqrt{\alpha\left(A - \frac{B}{\varrho_-^2}\right)}$ and $\varrho_- = \varrho_+$ are satisfied, then $\sigma(t) = v_\delta$ can be obtained directly from (4.62) and the delta shock wave solution to the Riemann Problem (4.1) and (4.2) can also be expressed in the form of (4.57), where the propagating speed $\sigma(t)$, position $x(t)$ and the strength $y(t)$ of the delta shock for the case $\varrho_- = \varrho_+$ are defined as

$$\sigma(t) = \frac{v_+ - v_-}{2} + \eta t, \quad x(t) = \frac{v_+ - v_-}{2}t + \frac{1}{2}\eta t^2, \quad y(t) = -(\varrho_+ v_+ - \varrho_- v_-)t. \tag{4.72}$$

One can verify that the delta shock wave solution of Riemann Problem (4.1) and (4.2) satisfying (4.63) in the sense of distribution for the case $\varrho_- = \varrho_+$ is given by (4.57). The process of the proof is completely similar to the proof as given in theorem (4.3) for (4.63) and hence not repeated here.

4.4 Conclusions

In this study, we have discussed the classical and non-classical solution of the Riemann Problem for the one-dimensional MCG model with constant external force. It is noticed that the solution of the Riemann Problem for the non-homogeneous MCG model can be determined directly by introducing new variable for the velocity, discussed in section (4.2), which leads to the Riemann Problem for homogeneous model. In a special situation, we observed that the Riemann Problem for MCG model have delta shock wave solution which is worthwhile to analyze the concentration phenomenon for the dark energy and dark matter in the evolution of the universe. The exact location, strength and propagation speed of the delta shock wave is obtained.

Also, we analyzed that the presence of the external constant force (friction) causes to bend all the elementary waves in the parabolic shape like shock wave, rarefaction wave and delta shock wave in the Riemann solution of MCG model.

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