Chapter 3

Riemann solutions to the Logotropic system with a Coulomb-type friction *

"Mathematics is the queen of science"

–Carl Friedrich Gauss

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3.1 Introduction

In this chapter, we find out the Riemann solutions of Euler system with a logarithmic state equation and a Coulomb-type friction. The logotropic system with a Coulomb-type friction can be written as

$$\begin{cases} \partial_t \varrho + \partial_x (\varrho v) = 0, \\ \partial_t (\varrho v) + \partial_x (\varrho v^2 + A \ln \varrho) = \eta \varrho, \end{cases} \tag{3.1}$$

where ϱ is the density and v is the velocity of the gas. The $A > 0$, $\eta > 0$ are constant parameters. For $\eta = 0$, the model has been proposed in the field of astrophysics to explain the certain properties of molecular clouds that could not be understood in terms of isothermal distributions. The authors in [83], [84] and [85] have studied the logarithmic equation of state to study the Logotropic dark fluid as a unification of dark energy (DE) and dark matter (DM). The Euler system with the logarithmic equation of state, named as "Logotropic model", provides more interesting cosmological behaviors as compared to the several generalized forms of the "Chaplygin gas model". This chapter concerns with the Riemann Problem for the non-homogeneous logotropic system with the following piecewise discontinuous initial data

$$\varrho(x,0) = \begin{cases} \varrho_-, & x < 0, \\ \varrho_+, & x > 0, \end{cases} , v(x,0) = \begin{cases} v_-, & x < 0, \\ v_+, & x > 0, \end{cases} \tag{3.2}$$

where $\varrho_\pm$ and $v_\pm$ are known constants.

The study of the hyperbolic systems have significant physical background, which is interesting but leads to complex problems in Mathematics. However, the study of the Riemann Problem in the area of non-linear hyperbolic conservation laws is also an interesting problem. It is well known that the Riemann Problem is an initial value problem with piecewise discontinuous initial data. In the study of the Riemann Problem (3.1) and (3.2), the system (3.1) is formulated into quasilinear form and it is obtained that the characteristic fields corresponding to the characteristic of the system (3.1) is genuinely nonlinear. This implies that the Riemann solution for (3.1) and (3.2) are composed of the elementary waves that is only rarefaction waves and shock waves. The study of the Riemann solution for the non-homogeneous Euler system is of great interest among the researchers in the subject of physics and mathematics. As to our observation, the study of δ-shock, rarefaction waves and shock waves in the solution of Riemann Problem has been increasingly active topic since few decades.

In last few decades, many authors have studied the solutions of the Riemann Problem for homogeneous and non-homogeneous hyperbolic p-system. [5], [86] and [87] proposed a p-system with the equation of state $p(\varrho) = A\varrho^{-1}$, known as Chaplygin gas model, to include the lifting force on the airplane wing in gas dynamics. [88] proposed the solution to the Chaplygin gas equation with concentration when the initial data lying in special domain of plane. [60], [89] have successfully applied differential constraint method to analyze the solution of non-homogeneous Riemann Problem. Further, [90] and [91] have studied the Riemann solution for homogeneous and non-homogeneous Chaplygin gas model, respectively. Recently, the limiting behavior of the solutions for vanishing pressure hyperbolic model have been discussed by [34], [45]. [92] have used the weak asymptotic method to deal the δ-shock wave problem. The authors in [77] and [93] have discussed the Riemann Problem for magnetogasdynamic flow and the polytropic dusty gas flow, respectively. Recently, [94] has studied the elementary wave interaction of the Riemann solution for the dusty gas flow.

The motivation of this study is to analyze the Riemann solutions to the non-homogeneous hyperbolic system with the logotropic equation of state due to its wide applications in the area of cosmology, astrophysics, aerodynamics and engineering. Also, the Riemann solutions have gained significant importance due to its practical and theoretical applications for the general mathematical theory of hyperbolic equations. Therefore, we analyze the Riemann solution for (3.1) and (3.2) case by case, and obtain the global structure of the solution for the Riemann Problem. The friction force term appearing in the momentum equation of the logotropic model was used first time in [95]. The advantage of the source term appearing in the logotropic model (3.1) is that the inhomogeneous model (3.1) can be reformulated in to the homogeneous conservation form which enable us to determine the solution of the Riemann Problem for the logotropic model which causes to bent all the waves including shock wave, contact discontinuity, rarefaction wave into the parabolic shape, and the solution of the Riemann Problem for the model (3.1) is not self-similar solution.

This chapter is structured into following sections as: In section (3.2), the inhomogeneous system (3.1) is modified into homogeneous conservative system by using new state variables and obtain the general properties of the modified system. Furthermore, the classical Riemann solution for modified system is discussed. We obtained the Riemann invariants corresponding to these characteristic fields. Also, we obtain the solution of the Riemann Problem for the inhomogeneous system (3.1) with the help of results obtained for the modified conservative system. Section (3.3) contains conclusions of this study.

3.2 Riemann solution of modified homogeneous system

We study the Riemann solution for the modified homogeneous conservative form of (3.1) by introducing new variable for the velocity, $u(x,t) = v(x,t) - \eta t$.

Authors in [96] have introduced this new state variable to discuss the Riemann Problem for inhomogeneous shallow water equations. On insertion of this new velocity in (3.1), we obtain the following conservation form of the logotropic system with friction term

$$\begin{cases} \partial_t \varrho + \partial_x \left(\varrho(u+\eta t)\right) = 0, \\ \partial_t \left(\varrho u\right) + \partial_x \left(\varrho u(u+\eta t) + A \ln \varrho\right) = 0. \end{cases} \tag{3.3}$$

We consider the Riemann Problem for the modified conservative model (3.3) with the same initial data,

$$\varrho(x,0) = \begin{cases} \varrho_-, & x<0, \\ \varrho_+, & x>0, \end{cases}, u(x,0) = \begin{cases} v_-, & x<0, \\ v_+, & x>0. \end{cases} \tag{3.4}$$

Now, the Riemann solution for the original model (3.1) and (3.2) can be determined from the corresponding ones to the system (3.3) and (3.4) by utilizing the new state variables, $(\varrho, v)(x,t) = (\varrho, u+\eta t)(x,t)$. Reformulating (3.3) into the quasi-linear form as

$$MU_t + NU_x = 0, \tag{3.5}$$

where $U = \begin{pmatrix} \varrho \\ u \end{pmatrix}, M = \begin{pmatrix} 1 & 0 \\ u & \varrho \end{pmatrix}$ and $N = \begin{pmatrix} u+\eta t & \varrho \\ u(u+\eta t) + \frac{A}{\varrho} & 2\varrho(u+\eta t) \end{pmatrix}$.

Let λ_1 and λ_2 are two eigenvalues of the system (3.3) which can be directly obtained from (3.5), given by

$$\lambda_1 = u+\eta t - \sqrt{\frac{A}{\varrho}}, \quad \lambda_2 = u+\eta t + \sqrt{\frac{A}{\varrho}} \tag{3.6}$$

and the right eigenvectors corresponding to both characteristic roots are

$$d_1 = \begin{pmatrix} -\varrho & \sqrt{\frac{A}{\varrho}} \end{pmatrix}^{Tr}, \quad d_2 = \begin{pmatrix} \varrho & \sqrt{\frac{A}{\varrho}} \end{pmatrix}^{Tr}. \tag{3.7}$$

Thus, on straight calculation it leads to $\nabla\lambda_i \cdot d_i \neq 0, i = 1, 2$, for $A > 0$. Here, $\nabla = \left(\frac{\partial}{\partial\varrho}, \frac{\partial}{\partial u}\right)$. Since for $A > 0$, $\nabla\lambda_i \cdot d_i \neq 0$ which implies that the characteristic fields corresponding to the characteristic roots λ_1 and λ_2 are genuinely nonlinear. Hence the waves for each of them is either rarefaction waves (continuous solution) or shock waves (bounded discontinuous solution) denoted by R and S, respectively. Along these characteristic fields the Riemann invariants are defined as

$$\begin{aligned} 1 - \textit{Riemann invariant} &= w = u - 2\sqrt{\frac{A}{\varrho}}, \\ 2 - \textit{Riemann invariant} &= z = u + 2\sqrt{\frac{A}{\varrho}}. \end{aligned} \tag{3.8}$$

3.2.1 Rarefaction wave solution

Here, we study the rarefaction wave solution satisfying (3.3) which is a continuous solution satisfying the system (3.3) can be computed by solving the integral curve of both characteristic fields. Also, it is worthwhile to notice that K- Riemann invariant ($K = 1, 2$) is conserved in the K- rarefaction wave. Now, for a given left state (ϱ_-, u_-) in terms of 1-Riemann invariant, the state (ϱ, u) can be connected to the state (ϱ_-, u_-) in the phase plane by the 1- rarefaction wave curve denoted by $R_1(\varrho_-, u_-)$ and can be written as (See [90, 91])

$$R_1(\varrho_-, u_-) : \begin{cases} \frac{dx}{dt} = \lambda_1 = u + \eta t - \sqrt{\frac{A}{\varrho}}, \\ u - 2\sqrt{\frac{A}{\varrho}} = u_- - 2\sqrt{\frac{A}{\varrho_-}} = w_-, \\ \lambda_1(\varrho_-, u_-) \leq \lambda_1(\varrho, u). \end{cases} \tag{3.9}$$

On taking derivative of u with respect to ϱ of second equation of (3.9), we have

$$\frac{du}{d\varrho} = -\sqrt{A}\varrho^{-3/2} < 0,$$
$$\frac{d^2u}{d\varrho^2} = \frac{3\sqrt{A}}{2}\varrho^{-5/2} > 0.$$

Thus, the 1-rarefaction wave is convex in (ϱ, u) phase plane and satisfying $u \geq u_-, \varrho \leq \varrho_-$.

Let us consider that the state (ϱ_1, u_1) is 1-rarefaction wave solution at a point (x, t) inside of the 1-rarefaction wave curve $R_1(\varrho_-, u_-)$. To express the (ϱ_1, u_1) in the interior of 1-rarefaction wave curve, we solve the initial value problem

$$\frac{dx}{dt} = \lambda_1(\rho_1, u_1),\ x(0) = 0.$$

On integrating, yields

$$\frac{x}{t} - \frac{1}{2}\eta t = u_1 - \sqrt{\frac{A}{\varrho_1}}. \tag{3.10}$$

Besides, ϱ_1, u_1 also satisfies the following equality,

$$u_1 - 2\sqrt{\frac{A}{\varrho_1}} = u_- - 2\sqrt{\frac{A}{\varrho_-}} = w.$$

wave solution (ϱ_1, u_1) as

Then on insertion of (2.10) in to the above equality, we obtain the 1-rarefaction

$$(\varrho_1, u_1)(x, t) = \left(A\left(\frac{x}{t} - \frac{1}{2}\eta t - w_-\right)^{-2}, 2\left(\frac{x}{t} - \frac{1}{2}\eta t\right) - w_- \right). \tag{3.11}$$

Similarly, for a given left state (ϱ_-, u_-) in terms of 2-Riemann invariant, the state (ϱ, u) can be connected to the state (ϱ_-, u_-) in the phase plane by the 2-rarefaction wave curve denoted by $R_2(\varrho_-, u_-)$ and can be written as

$$R_2(\varrho_-, u_-) : \begin{cases} \frac{dx}{dt} = \lambda_2 = u + \eta t + \sqrt{\frac{A}{\varrho}}, \\ u + 2\sqrt{\frac{A}{\varrho}} = u_- + 2\sqrt{\frac{A}{\varrho_-}} = z_-, \\ \lambda_2(\varrho_-, u_-) \leq \lambda_2(\varrho, u). \end{cases} \tag{3.12}$$

On taking derivative of u with respect to ϱ in second equation of (3.12), we have

$$\frac{du}{d\varrho} = \sqrt{A}\varrho^{-3/2} > 0,$$
$$\frac{d^2u}{d\varrho^2} = -\frac{3\sqrt{A}}{2}\varrho^{-5/2} < 0.$$

Thus, the 2-rarefaction wave is concave in (ϱ, u) phase plane and satisfying $u \geq u_-, \varrho \geq \varrho_-$.

Let us consider that the state (ϱ_2, u_2) is 2-rarefaction wave solution at a point (x, t) inside of the 2-rarefaction wave curve $R_2(\varrho_-, u_-)$. To express the (ϱ_2, u_2) in the interior of 2-rarefaction wave curve, we solve the initial value problem

$$\frac{dx}{dt} = \lambda_2(\rho_2, u_2), \;\; x(0) = 0.$$

On integrating, yields

$$\frac{x}{t} - \frac{1}{2}\eta t = u_2 - \sqrt{\frac{A}{\varrho_2}}. \tag{3.13}$$

Besides, ϱ_2, u_2 also satisfies the following equality,

$$u_2 + 2\sqrt{\frac{A}{\varrho_2}} = u_- + 2\sqrt{\frac{A}{\varrho_-}} = z_-.$$

On substituting (3.10) in to the above equality, we obtain the 2-rarefaction wave solution (ϱ_2, u_2) as

$$(\varrho_2, u_2)(x,t) = \left(A\left(z_- - \frac{x}{t} + \frac{1}{2}\eta t \right)^{-2}, 2\left(\frac{x}{t} - \frac{1}{2}\eta t \right) - z_- \right). \tag{3.14}$$

3.2.2 Shock wave solution

Now, our attention to study the discontinuous solution of modified conservative system (3.3) which is called shock wave solution satisfying Rankine -Hugoniot jump relations and entropy condition. Let the jump speed of the shock wave, denoted by $\xi(t) = x'(t)$, connects the two states (ϱ, u) and (ϱ_-, u_-), then the Rankine-Hugoniot jump relations for (3.3) are

$$\begin{cases} -\xi(t)\,[\varrho] + [\varrho(u+\eta t)] = 0, \\ -\xi(t)\,[\varrho u] + [\varrho u(u+\eta t) + A\ln\varrho] = 0, \end{cases} \tag{3.15}$$

where $[h] = h_r - h_l$ with $h_l = h(x(t)-0,t), h_r = h(x(t)+0,t)$ represents the jump of h across the shock.

If $\xi(t) = 0$, from (3.15) we obtain trivial solution $(\varrho, u) = (\varrho_-, u_-)$. Otherwise, if $\xi(t) \neq 0$, we get

$$[\varrho u]^2 - [\varrho]\,[\varrho u(u+\eta t) + A\ln\varrho] = 0. \tag{3.16}$$

On simplifying, we get

$$u_r = u_l \pm \sqrt{A\left(\frac{1}{\varrho_l} - \frac{1}{\varrho_r}\right)(\ln\varrho_r - \ln\varrho_l)}. \tag{3.17}$$

For a given left state (ϱ_-, u_-), 1-shock wave curve, denoted by $S_1(\varrho_-, u_-)$, should satisfy

$$S_1(\varrho_-, u_-):\begin{cases}\xi_1(t)=\frac{\varrho u-\varrho u_-}{\varrho-\varrho_-}+\eta t,\\ u-u_-=-\sqrt{A\left(\frac{1}{\varrho_-}-\frac{1}{\varrho}\right)(\ln\varrho-\ln\varrho_-)},\\ u_->u, \varrho_-<\varrho.\end{cases}\tag{3.18}$$

Analogously, the 2-shock wave curve, denoted by $S_2(\varrho_-, u_-)$, should satisfy

$$S_2(\varrho_-, u_-):\begin{cases}\xi_2(t)=\frac{\varrho u-\varrho u_-}{\varrho-\varrho_-}+\eta t,\\ u-u_-=-\sqrt{A\left(\frac{1}{\varrho_-}-\frac{1}{\varrho}\right)(\ln\varrho-\ln\varrho_-)},\\ u_->u, \varrho_->\varrho.\end{cases}\tag{3.19}$$

It follows from (3.18) that

$$\frac{du}{d\rho}=-\frac{\sqrt{A}\left(\frac{1}{\rho^2}\ln\frac{\rho}{\rho_-}+\left(\frac{1}{\rho_-}-\frac{1}{\rho}\right)\frac{1}{\rho}\right)}{2\sqrt{\left(\frac{1}{\rho_-}-\frac{1}{\rho}\right)\left(\ln\rho-\ln\rho_-\right)}}<0 \quad \text{for} \quad \rho>\rho_-,$$

and

$$\frac{du}{d\rho}=\frac{\sqrt{A}\left(\frac{1}{\rho^2}\ln\frac{\rho}{\rho_+}+\left(\frac{1}{\rho_+}-\frac{1}{\rho}\right)\frac{1}{\rho}\right)}{2\sqrt{A\left(\frac{1}{\rho}-\frac{1}{\rho_+}\right)\left(\ln\rho_+-\ln\rho\right)}}>0 \quad \text{for} \quad \rho<\rho_-,$$

Thus, through tedious calculations, we obtain that the concavity of 1-shock wave curve (or convexity of 2-shock wave curve) is similar to that for 1-rarefaction curve (or 2-rarefaction curve). Now, it is clear that the set of possible states, connected on the right, consist of the 1-shock wave $S_1(\varrho_-, v_-)$, 1-rarefaction wave curve $R_1(\varrho_-, v_-)$, 2-Shock wave $S_2(\varrho_-, v_-)$ and the 2-rarefaction wave $R_2(\varrho_-, v_-)$. Thus, for the given left state (ϱ_-, v_-), the curves of $R_1(\varrho_-, v_-)$, $R_2(\varrho_-, v_-)$, $S_1(\varrho_-, v_-)$ and $S_2(\varrho_-, v_-)$ divide the phase plane into four regions which are denoted with I, II, III and IV (See Fig. 3.1).

Now, in view of the right state (ϱ_+, v_+) in the different regions, we can construct

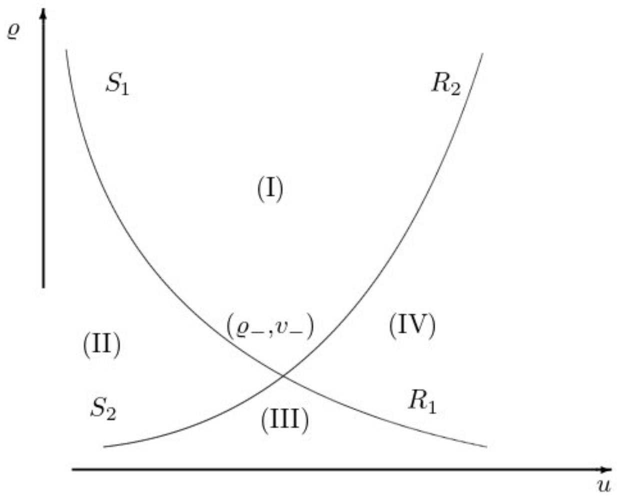

FIGURE 3.1: The (ϱ, u) phase plane for the model (1).

the unique global Riemann solution of the system (3.3) which connects two states (ϱ_-, v_-) and (ϱ_+, v_+).

If $(\varrho_+, v_+) \in$ I, then the solution of Riemann Problem of (3.3) and (3.4) is composed of 1-shock wave S_1 and 2-rarefaction waves R_2, and (ϱ_*, u_*) obtained as

$$\begin{cases} u_* = v_- - \sqrt{A\left(\frac{1}{\varrho_-} - \frac{1}{\varrho_*}\right)(\ln \varrho_* - \ln \varrho_-)}, \\ u_* + 2\sqrt{\frac{A}{\varrho_*}} = v_+ + 2\sqrt{\frac{A}{\varrho_+}} = z_+. \end{cases} \tag{3.20}$$

In this case the solution of the Riemann Problem of (3.3) and (3.4) is,

$$(\varrho, u)(x,t) = \begin{cases} (\varrho_-, v_-), & x < C_1(t), \\ (\varrho_*, u_*), & C_1(t) \leq x \leq C_2^-(t), \\ (\varrho_2, u_2), & C_2^-(t) \leq x \leq C_2^+(t), \\ (\varrho_+, v_+), & C_2^+(t) < x, \end{cases} \tag{3.21}$$

where the state (ϱ_2, u_2) in R_2 is given by (3.14) and the position of S_1 and R_2 curves are defined as

$$C_1(t) = \left(\frac{\varrho_* u_* - \varrho_- v_-}{\varrho_* - \varrho_-} \right) t + \frac{\eta t^2}{2}, \tag{3.22}$$

$$C_2^+(t) = \left(v_+ + \sqrt{\frac{A}{\varrho_+}} \right) t + \frac{\eta t^2}{2}, \tag{3.23}$$

$$C_2^-(t) = \left(u_* + \sqrt{\frac{A}{\varrho_*}} \right) t + \frac{\eta t^2}{2}. \tag{3.24}$$

If $(\varrho_+, v_+) \in \mathrm{II}$, then the Riemann solution of (3.3) and (3.4) consists of 1-shock wave S_1 and 2-shock wave S_2, and the intermediate constant state (ϱ_*, u_*) obtained by

$$\begin{cases} u_* = v_- - \sqrt{A\left(\frac{1}{\varrho_-} - \frac{1}{\varrho_*}\right)(\ln \varrho_* - \ln \varrho_-)}, \\ u_* = v_+ + \sqrt{A\left(\frac{1}{\varrho_*} - \frac{1}{\varrho_+}\right)(\ln \varrho_+ - \ln \varrho_*)}, \end{cases} \tag{3.25}$$

then the Riemann solution of (3.3) and (3.4) can be written as,

$$(\varrho, u)(x,t) = \begin{cases} (\varrho_-, v_-), & x < C_1(t), \\ (\varrho_*, u_*), & C_1(t) \leq x \leq C_2(t), \\ (\varrho_+, v_+), & C_2(t) < x, \end{cases} \tag{3.26}$$

where the position of 1-shock wave S_1 is given by (3.22) and 2-shock wave curve is defined as

$$C_2(t) = \left(\frac{\varrho_+ v_+ - \varrho_* u_*}{\varrho_+ - \varrho_*} \right) t + \frac{\eta t^2}{2}. \tag{3.27}$$

If $(\varrho_+, v_+) \in \mathrm{III}$, then the solution of Riemann problem of (3.3) and (3.4) consists of 1-rarefaction wave R_1, 2-shock wave S_2, and the intermediate state (ϱ_*, u_*) obtained by

$$\begin{cases} u_* - 2\sqrt{\frac{A}{\varrho_*}} = v_- - 2\sqrt{\frac{A}{\varrho_*}} = w_-, \\ u_* = v_+ + \sqrt{A\left(\frac{1}{\varrho_*} - \frac{1}{\varrho_+}\right)(\ln \varrho_+ - \ln \varrho_*)}, \end{cases} \tag{3.28}$$

then, the Riemann solution of system (3.3) and (3.4) can be written as

$$(\varrho, u)(x,t) = \begin{cases} (\varrho_-, v_-), & x < C_1^-(t), \\ (\varrho_1, u_1), & C_1^-(t) \le x \le C_1^+(t), \\ (\varrho_*, u_*), & C_1^+(t) \le x \le C_2(t), \\ (\varrho_+, v_+), & C_2(t) < x, \end{cases} \tag{3.29}$$

where the state (ϱ_1, u_1) in R_1 and the position of 2-shock wave S_2 are given by (3.11) and (3.27), and the position of 1-rarefaction wave curve are defined as

$$C_1^-(t) = \left(v_- - \sqrt{\frac{A}{\varrho}}\right) t + \frac{\eta t^2}{2}, \tag{3.30}$$

$$C_1^+(t) = \left(u_* - \sqrt{\frac{A}{\varrho_*}}\right) t + \frac{\eta t^2}{2}. \tag{3.31}$$

If $(\varrho_+, v_+) \in IV$, then the solution of Riemann Problem of (3.3) and (3.4) consists of rarefaction waves R_1, R_2 and (ϱ_*, u_*) is obtained by

$$\begin{cases} v_* - 2\sqrt{\frac{A}{\varrho_*}} = v_- - 2\sqrt{\frac{A}{\varrho_-}} = w_-, \\ v_* + 2\sqrt{\frac{A}{\varrho_*}} = v_+ + 2\sqrt{\frac{A}{\varrho_+}} = z_+. \end{cases} \tag{3.32}$$

Thus the solution of the Riemann Problem of (3.3) and (3.4) can be written explicitly as

$$(\varrho, u)(x,t) = \begin{cases} (\varrho_-, v_-), & x < C_1^-(t), \\ (\varrho_1, u_1), & C_1^-(t) \leq x \leq C_1^+(t), \\ (\varrho_*, u_*), & C_1^+(t) \leq x \leq C_2^-(t), \\ (\varrho_2, u_2), & C_2^-(t) \leq x \leq C_2^+(t), \\ (\varrho_+, v_+), & C_2^+(t) < x, \end{cases} \tag{3.33}$$

where the states (ϱ_1, u_1) and (ϱ_2, u_2) are given by (3.11) and (3.14), respectively, and the position of R_1 and R_2 wave curve are defined by (3.23), (3.24) and (3.30), (3.31),respectively.

As in [24], to ensure the uniqueness of the solution of the Riemann Problem for all regions, it suffices to show that $\frac{\partial v}{\partial \varrho_*} > 0$. Suppose $(\varrho_+, v_+) \in I$, then using equations

$$v = u_* + \sqrt{A\left(\frac{1}{\varrho_-} - \frac{1}{\varrho_*}\right)(\ln \varrho_* - \ln \varrho_-)}, \quad u_* + 2\sqrt{\frac{A}{\varrho_*}} = v_+ + 2\sqrt{\frac{A}{\varrho_+}},$$

we compute

$$\frac{\partial v}{\partial \varrho_*} = \frac{2A}{\varrho_*^2}\sqrt{\frac{\varrho_*}{A}} + \frac{A}{\sqrt{A\left(\frac{1}{\varrho_-} - \frac{1}{\varrho_*}\right)(\ln \varrho_* - \ln \varrho_-)}}\left(\frac{(\varrho_* - \varrho_-)}{\varrho_-} + (\ln \varrho_* - \ln \varrho_-)\right)\frac{1}{\varrho_*^2}.$$

Hence $\frac{\partial v}{\partial \varrho_*} > 0$ for $A > 0$, which implies the uniqueness result in region I. Similarly, we can prove for all regions.

Now, we discuss the structure of Riemann solutions for the original system (3.1) and (3.2) by using the change of variables $(\varrho, v)(x,t) = (\varrho, u + \eta t)(x,t)$.These solutions can be structured from the corresponding solutions of the Riemann Problem (3.3) and (3.4) case by case as follows.

Case I: In this case the Riemann solutions of the original system of (3.1) and (3.2) is as (See fig.3.2),

$$(\varrho, v)(x,t) = \begin{cases} (\varrho_-, v_- + \eta t), & x < C_1(t), \\ (\varrho_*, u_* + \eta t), & C_1(t) \leq x \leq C_2^-(t), \\ (\varrho_2, u_2 + \eta t), & C_2^-(t) \leq x \leq C_2^+(t), \\ (\varrho_+, v_+ + \eta t), & C_2^+(t) < x, \end{cases} \tag{3.34}$$

where the state (ϱ_2, u_2) in R_2 is given by (3.14) and the position of 1-shock wave S_1 and 2-rarefaction wave curve are defined by (3.22) and (3.24).

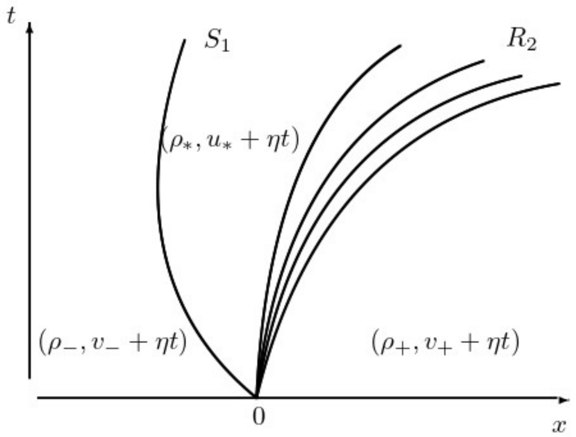

FIGURE 3.2: Solution structure of system (3.1) and (3.2) for case I.

Case II: If $(\varrho_+, v_+) \in \mathrm{II}$, then the Riemann solution of (3.3) and (3.4) consists of 1-shock wave S_1 and 2-shock wave S_2, then the Riemann solution of (3.1) and (3.2) can be written as (See fig. 3.3),

$$(\varrho, v)(x,t) = \begin{cases} (\varrho_-, v_- + \eta t), & x < C_1(t), \\ (\varrho_*, u_* + \eta t), & C_1(t) \le x \le C_2(t), \\ (\varrho_+, v_+ + \eta t), & C_2(t) < x, \end{cases} \tag{3.35}$$

where the position of 1-shock wave S_1 and 2-shock wave curve S_2 are given by (3.22) and (3.27).

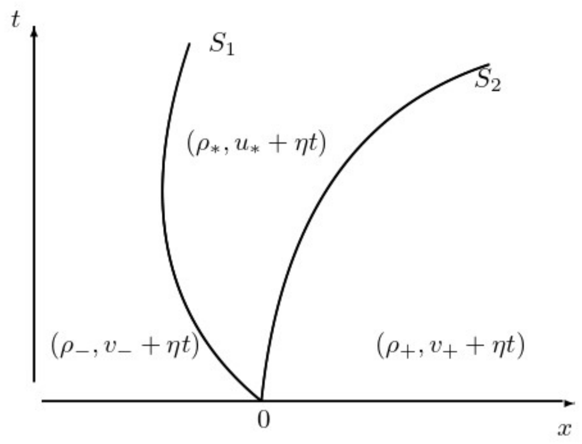

FIGURE 3.3: Solution structure of system (3.1) and (3.2) for case Ⅱ.

Case Ⅲ: If $(\varrho_+, v_+) \in$ Ⅲ, then the solution of Riemann Problem of (3.1) and (3.2) consists of 1-rarefaction wave R_1, 2-shock wave S_2, then the Riemann solution of system (3.1) and (3.2) can be written as (See fig.3.4)

$$(\varrho, v)(x,t) = \begin{cases} (\varrho_-, v_- + \eta t), & x < C_1^-(t), \\ (\varrho_1, u_1 + \eta t), & C_1^-(t) \le x \le C_1^+(t), \\ (\varrho_*, u_* + \eta t), & C_1^+(t) \le x \le C_2(t), \\ (\varrho_+, v_+ + \eta t), & C_2(t) < x, \end{cases} \tag{3.36}$$

where the state $(\varrho_1, u_1 + \eta t)$ in R_1, the position of 2-shock wave S_2 ad 1-rarefaction wave curve are given by (3.11), (3.30), (3.31) and (3.27), respectively.

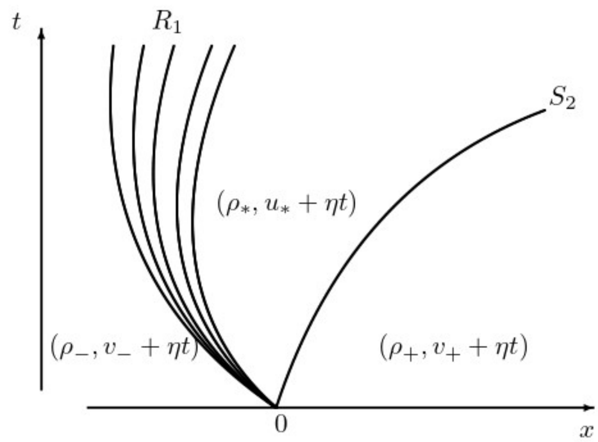

FIGURE 3.4: Solution structure of system (3.1) and (3.2) for case Ⅲ.

Case IV: If $(\varrho_+, v_+) \in$IV, then the solution of Riemann Problem of (3.1) and (3.2) consists of rarefaction waves R_1, R_2, then the solution of the Riemann Problem of (3.1) and (3.2) can be written as (See fig.3.5)

$$(\varrho, v)(x,t) = \begin{cases} (\varrho_-, v_- + \eta t), & x < C_1^-(t), \\ (\varrho_1, u_1 + \eta t), & C_1^-(t) \leq x \leq C_1^+(t), \\ (\varrho_*, u_* + \eta t), & C_1^+(t) \leq x \leq C_2^-(t), \\ (\varrho_2, u_2 + \eta t), & C_2^-(t) \leq x \leq C_2^+(t), \\ (\varrho_+, v_+ + \eta t), & C_2^+(t) < x, \end{cases} \tag{3.37}$$

where the states (ϱ_1, u_1) and $(\varrho_2, u_2 + \eta t)$ are given by (3.11) and (3.14), respectively, and the position of R_1 and R_2 are defined by (3.23), (3.24) and (3.30), (3.31), respectively.

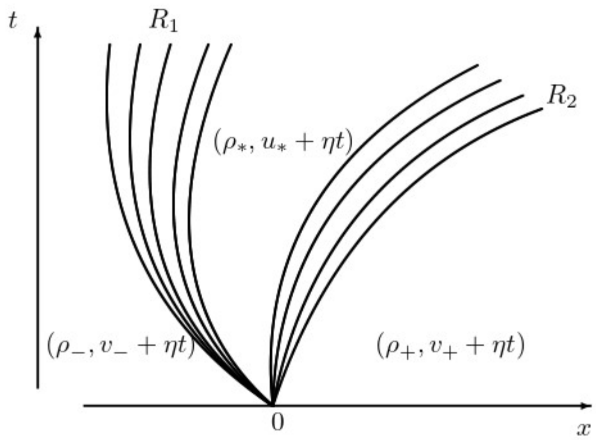

FIGURE 3.5: Solution structure of system (3.1) and (3.2) for case IV.

It is noticeable that the solutions of logotropic model with a source term (3.1) and (3.2) also consist rarefaction wave solution and shock wave solution but presence of Coulomb-type friction term, all the elementary waves in (x, t)-plane like rarefaction wave and shock wave present in the Riemann solutions of (3.1) and (3.2) do not remain straight. Hence, the solutions of logotropic model with a Coulomb-type friction term (3.1) and (3.2) are not self-similar.

3.3 Conclusions

In the current study, we concern with structure of solutions of the Riemann Problem for the logotropic model with a Coulomb-type friction term appearing in the momentum equation of the governing system. It is noticed that the Riemann solution for the non-homogeneous logotropic model can be determined directly by introducing new variable for the velocity, discussed in section (3.2), which leads to the Riemann Problem for homogeneous model. It is obtained that the modified model is strictly hyperbolic model and the characteristic fields corresponding to its characteristic roots are genuinely nonlinear. Also, the elementary waves associated with these characteristic fields are presented in the explicit forms. The solutions of the non-homogeneous logotropic model is obtained by four cases on the behalf of elementary waves appearing in the solutions. Also, we analyzed that the presence of Coulomb-type friction term causes to bend all the elementary waves in the parabolic shape like shock wave, rarefaction wave in the Riemann solution of the logotropic model. Particularly, the logotropic model have no self-similar solutions in (x, t)-plane due to the presence of time-dependent Coulomb-type friction term.

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