# Chapter 2

# Solution of generalized Riemann Problem for hyperbolic $p-$system with damping *

"Mathematics is the language with which
God has written the universe"

–Galileo Galilei

## 2.1 Introduction

In this chapter, we consider the model, hyperbolic $p-$system with linear damping and determine the exact solution of generalized Riemann Problem using differential

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constraint method. The governing equations describing the hyperbolic $p-$system with linear damping of a compressible flow, in the Lagrangian coordinate is given by [68, 69]

$$\begin{cases} \rho_t - v_x = 0, \\ v_t + p(\rho)_x = -\alpha v, \end{cases} \tag{2.1}$$

where $(x,t) \in R \times R_+$. $x$ and $t$, respectively, represent space and time coordinates. $v(x,t)$ and $\rho(x,t)>0$ are the velocity and the specific volume respectively. The function $p(\rho)$ (pressure) is the smooth function such that $p(\rho)>0$ and $p'(\rho)<0$. The term $-\alpha v$, which appears in the momentum equation represents a linear damping with $\alpha>0$. In the present Chapter, we consider a particular case that the pressure $p$ is given by a $\gamma$-law with $\gamma=1$ as [70, 71]

$$p(\rho) = \frac{1}{\rho}.$$

The system (2.1) is considered by several authors in past decades and various results have been examined. The authors Mei [68, 72], Nishihara [69] etc. have discussed the behavior of solution of system (2.1). In the present Chapter we use the method of differential constraints to obtain the exact solution of generalized Riemann Problem for the hyperbolic system (2.1). In past decades several types of mathematical methods have been used to obtain the exact or approximate analytical solution of the system of PDEs. For example, perturbation method, similarity method, differential constraints method etc. Among these methods the differential constraints method is of great interest because of its systematic procedure to find the exact solution of hyperbolic system of PDEs. This method is applicable to the system of PDEs and it was first used by Janenko [61] in the gasdynamics regime. The exact solution describing soliton-like interaction in a non dispersive medium is obtained by Seymour and Varley [73]. Further, C. Curro et al. [64] have studied the exact description of simple wave for multicomponent hyperbolic system.

Within this theoretical framework, differential constraint method is used for solving generalized Riemann Problem for hyperbolic $p-$system with linear damping. In recent years, Curro et al. [59, 60, 57], Manganaro [74], Sahoo et al. [67] have discussed the solution of Riemann Problem as well as generalized Riemann Problem for hyperbolic balance law. Chaiyasena et al. [75] have studied the generalized Riemann Problem and their adjoinment through a shock wave. Shekhar et al. [76], Singh et al. [77], Kuila et al. [78], Gupta et al. [79] have studied the classical Riemann Problem for several relevant cases. The study of generalized Riemann Problem (GRP) involves complexities and requires more attention. Since the analytical expressions for the exact or approximate solution of GRP, in the case of quasilinear hyperbolic system, are usually not available. In order to find the solution of GRP, authors have used this method for several models like $p-$system with relaxation ([58]), rate-type material ([67]), fast diffusion equations ([80]), traffic flow model ([59, 60]), ET6 model ([81]), non-linear transmission lines ([57]), non-linear diffusion-convention equations ([82]) etc.

The rest of the Chapter is organized as: section (2.2) contains the procedure of the method to find the exact solution of system (2.1). Section (2.3) contains the brief analysis of model taken in this Chapter. Section (2.4) sketches the phenomenon of generalized Riemann Problem and the exact solution of generalized Riemann Problem is obtained. In last section (2.5), we discuss the results and conclusions of the Problem.

## 2.2 Differential constraint method

In this section, for our convenience, we discuss the important steps of differential constraint method, described by Curro et al. [58], which will be helpful to obtain the solution of system (2.1). We consider the one-dimensional hyperbolic system of quasilinear PDEs

$$U_t + M(U)U_x = N(U), \tag{2.2}$$

where $U \in R^n$, $M(U)$ and $N(U)$ are column vector representing field variables, the coefficient matrix and column vector representing source terms respectively. Since the system (2.2) is hyperbolic in nature, therefore the matrix $M_{n\times n}$ has $n$ real eigenvalues $\lambda^{(i)}$, $i = 1, 2, ....n.$, and corresponding to each eigenvalue $\lambda^{(i)}$ we have $n$ linearly independent left eigenvectors $L^{(i)}$ and $n$ right eigenvectors $R^{(i)}$. Let us consider that the system (2.2) is strictly hyperbolic system i.e., $\lambda^{(i)} \neq \lambda^{(k)}, \forall i, k = 1, 2, ...n$, $i \neq k$. Let us assume that

$$L^{(i)}.R^{(k)} = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases} \tag{2.3}$$

We append the set of $n - 1$ differential constraints of first order as [57]

$$L^{(i)} \cdot U_x = Q^{(i)}(x, t, U), \qquad i = 1, 2, ....., n - 1, \tag{2.4}$$

where the function $Q^{(i)}$ is an unspecified function which will be determined during the reduction process.

The consistency conditions for the arbitrary function $Q^{(i)}$ can be determined from (2.2) and (2.4), given as

$$
\begin{aligned}
&Q_t^{(i)} + \lambda^{(i)} Q_x^{(i)} + \sum_{k=1}^{n-1} Q^{(k)} \left( L^{(i)} (\nabla R^{(k)} N - \nabla N R^{(k)}) + Q^{(i)} \nabla \lambda^{(i)} R^{(k)} \right) \\
&\quad + \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} Q^{(j)} Q^{(k)} (\lambda^{(j)} - \lambda^{(i)}) L^{(i)} \nabla R^{(j)} R^{(k)} \\
&\quad + \nabla Q^{(i)} \left( N - \sum_{j=1}^{n-1} Q^{(j)} (\lambda^{(j)} - \lambda^{(i)}) R^{(j)} \right) = 0,
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
&(\lambda^{(i)} - \lambda^{(n)}) \nabla Q^{(i)} R^{(n)} + \sum_{k=1}^{n-1} Q^{(k)} (\lambda^{(k)} - \lambda^{(n)}) L^{(i)} (\nabla R^{(k)} R^{(n)} - \nabla R^{(n)} R^{(k)}) \\
&\quad + L^{(i)} (R^{(n)} N - \nabla N R^{(n)}) + Q^{(i)} \nabla \lambda^{(i)} R^{(n)} = 0,
\end{aligned} \tag{2.6}
$$

where $\nabla = \partial/\partial U$, $i = 1, 2, ...n-1$.

Once the function $Q^{(i)}$ satisfies the consistency conditions (2.5) and (2.6), the systems (2.2) and (2.4) become compatible. Then we can obtain a class of exact solution of the given system of governing PDEs by direct integration of overdetermined system (2.2) and (2.4). The overdetermined constraint equations and consistency conditions help to rewrite the original system in to a partial decoupled form so that we can solve it and obtain exact solution which satisfies the original system and constraint equations. The reduced system is written in the following form

$$
U_t + \lambda^{(n)} U_x = N + \sum_{i=1}^{n-1} Q^{(i)} \left( \lambda^{(n)} - \lambda^{(i)} \right) R^{(i)}. \tag{2.7}
$$

Once $Q^{(i)}$ satisfies the consistency conditions, the solution of the systems (2.2) and (2.4) can be obtained by direct integration of the equations (2.7) along with the constraint equations (2.4). Furthermore, since the constraint equations (2.4) are involutive, they will characterize the set of initial conditions compatible with the class of solution under interest. Also, it is noticeable from (2.2) and (2.4) that when the source term $N(U) = 0$ and $Q^{(i)} = 0$, system (2.7) results in a classical wave solution to a hyperbolic system. Therefore, the obtained solution of (2.2) characterized by (2.7) generalizes the simple wave of the homogeneous case taking into account source-like effects in hyperbolic wave process. For more details one may refer [63, 59, 57, 58].

## 2.3 Exact solution

In this section, we use the above mentioned method to obtain the exact solution of the hyperbolic system (2.1) describing $p-$system with linear damping. The system (2.1) can be written in the form of (2.2), where

$$U = \begin{pmatrix} \rho \\ v \end{pmatrix}, \qquad M(U) = \begin{pmatrix} 0 & -1 \\ p'(\rho) & 0 \end{pmatrix}, \qquad N(U) = \begin{pmatrix} 0 \\ -\alpha v \end{pmatrix}. \tag{2.8}$$

Consider $\lambda^{(i)}$, $i = 1, 2$ are eigenvalues of the coefficient matrix $M_{2\times 2}$, $R^{(i)}$ and $L^{(i)}$ are the right and left eigenvectors of the corresponding eigenvalues, $\lambda^{(i)}, i = 1, 2$, defined as

$$\lambda^{(1)} = -\sqrt{-p'(\rho)}, \qquad \lambda^{(2)} = \sqrt{-p'(\rho)},$$

which can be written as

$$\lambda^{(i)} = \lambda_0 \sqrt{-p'(\rho)}, \tag{2.9}$$

where $\lambda_0 = \pm 1$.

and corresponding eigenvectors are

$$\begin{aligned} L^{(1)} &= \begin{pmatrix} \sqrt{-p'(\rho)} & 1 \end{pmatrix}, L^{(2)} = \begin{pmatrix} -\sqrt{-p'(\rho)} & 1 \end{pmatrix}, \\ R^{(1)} &= \begin{pmatrix} \frac{1}{2\sqrt{-p'(\rho)}} & \frac{1}{2} \end{pmatrix}^{Tr}, R^{(2)} = \begin{pmatrix} -\frac{1}{2\sqrt{-p'(\rho)}} & \frac{1}{2} \end{pmatrix}^{Tr}, \end{aligned} \tag{2.10}$$

respectively. From equation (2.4), the required differential constraint associated to the system (2.1) can be written in the form

$$v_x + \lambda(\rho)\rho_x = Q(x, t, \rho, v). \tag{2.11}$$

Substituting equations (2.8), (2.9), (2.10) and (2.11) in the consistency conditions (2.5) and (2.6), we obtain the following consistency equations

$$Q_t + \lambda Q_x - \alpha v Q_v + Q(\lambda Q_v + Q_\rho) = -\alpha Q, \tag{2.12}$$

$$Q_\rho + \lambda Q_v = -\frac{1}{2\lambda}(\lambda\alpha + \lambda' Q), \tag{2.13}$$

where the prime denotes the differentiation. Once $Q(x, t, \rho, v)$ is obtained by (2.12) and (2.13), from (2.14) we obtain the solution of the system (2.1) and (2.11) by integrating the following set of constraints along with (2.11)

$$\begin{cases} \rho_t + \lambda \rho_x = Q, \\ v_t + \lambda v_x = -(\alpha v + \lambda Q). \end{cases} \tag{2.14}$$

Prior to determine the exact solution of the system (2.14), we determine the arbitrary function $Q(x, t, \rho, v)$ from the consistency equations (2.12) and (2.13). On integrating (2.13), we obtain the following general solution

$$Q = -\alpha\rho + \rho^{1/2}\phi(x, t, m), \tag{2.15}$$

where $m = v - \lambda_0 \log \rho$ and $\phi(x, t, m)$ is an arbitrary function.
Next the substitution of (2.15) into (2.12) yields

$$\phi_t + \lambda\phi_x - \alpha v \phi_m - \frac{1}{2}\rho^{-1/2}\phi^2 - \frac{\alpha}{2}\phi = 0. \tag{2.16}$$

The last step of the process is to determine $\phi(x,t,m)$ which satisfies (2.16) for all $\rho$. It is noticeable that $\phi = 0$ must satisfy the above equation (2.16) for all $\rho$ which implies that $\phi = 0$ must be a solution of equation (2.16). On the insertion in (2.15), we find that $Q = -\alpha\rho$.

Now we look for a special solution of the compatibility conditions (2.12) and (2.13) under the form $Q = Q(\rho)$. On substituting it in (2.12) and (2.13), and solving the resulting equations we obtain that $Q = -\alpha\rho$ is the solution of the consistency conditions (2.12) and (2.13) as well as $p(\rho) = c_1/\rho + c_2$, where $c_1$ and $c_2$ are arbitrary constants.

Substituting this to the system (2.14) specialize to

$$\begin{cases} \rho_t + \lambda\rho_x = -\alpha\rho, \\ v_t + \lambda v_x = -\alpha v + \lambda\alpha\rho. \end{cases} \tag{2.17}$$

along with (2.11)

$$v_x + \lambda(\rho)\rho_x = -\alpha\rho. \tag{2.18}$$

Now, we integrate the system (2.18) with initial data $\rho(x,0) = \rho_0(x)$ and $v(x,0) = v_0(x)$ gives the following solution of the system (2.1)

$$\begin{cases} \rho(z,t) = \rho_0(z)e^{-\alpha t}, \\ v(z,t) = v_0(z)e^{-\alpha t} + \lambda_0(1 - e^{-\alpha t}), \\ z = x + \dfrac{\lambda_0}{\alpha\rho_0}\left(1 - e^{\alpha t}\right). \end{cases} \tag{2.19}$$

Within the context of differential constraint method, we have obtained the solution (2.19) of the system (2.1) for arbitrary initial data $\rho_0(x)$ and $v_0(x)$. Now we are interested to introduce the initial conditions in the next section which are known as generalized Riemann Problem. In next section, we solve the generalized Riemann

Problem to determine the exact solution for the model (2.1).

## 2.4 Generalized Riemann Problem

The purpose of this section is to discuss how the differential constraint method helps to solve the generalized Riemann Problem of interest in $2 \times 2$ hyperbolic $p-$system with damping. The generalized Riemann Problem for the system (2.1) with the initial data can be written as

$$U(x,0) = \begin{cases} (\rho_l(x), v_l(x)), & \text{for } x < 0, \\ (\rho_r(x), v_r(x)), & \text{for } x > 0, \end{cases} \tag{2.20}$$

where $\rho_r(x)$, $v_r(x)$, $\rho_l(x)$ and $v_l(x)$ are arbitrary functions. We set

$$\begin{aligned} \rho_L &= \lim_{x \longrightarrow 0^-} \rho_l(x), \qquad & v_L &= \lim_{x \longrightarrow 0^-} v_l(x), \\ \rho_R &= \lim_{x \longrightarrow 0^+} \rho_r(x), \qquad & v_R &= \lim_{x \longrightarrow 0^+} v_r(x), \end{aligned} \tag{2.21}$$

where $\rho_L \neq \rho_R$ and $v_L \neq v_R$. In order to solve the generalized Riemann Problem (2.20), we consider the solution (2.19) and substitute this into the constraint (2.18) which gives the following constraint equation for the initial conditions

$$\frac{dv_0(z)}{dz} + \lambda_0 \rho_0^{-1} \frac{d\rho_0(z)}{dz} = -\alpha \rho_0(z). \tag{2.22}$$

According to method of differential constraints, the initial data (2.20) satisfies the constraint equation (2.22). Then on integrating (2.22) along with the initial data

(2.20) yields the following implicit forms

$$\begin{cases} v_l(x) = v_L(x) - \lambda_0 \left(\ln \rho_l(x) - \ln \rho_L\right) - \alpha \displaystyle\int_0^x \rho_l(\xi) d\xi, \\ v_r(x) = v_R(x) - \lambda_0 \left(\ln \rho_r(x) - \ln \rho_R\right) - \alpha \displaystyle\int_0^x \rho_r(\xi) d\xi. \end{cases} \tag{2.23}$$

Now we characterize the solution (2.19) to solve the governing system (2.1) along with (2.20).

**Case I-** For $x < 0$, we get

$$\begin{cases} \rho(z,t) = \rho_l(z) e^{-\alpha t}, \\ v(z,t) = v_l(z) e^{-\alpha t} - \lambda_0 (1 - e^{-\alpha t}), \end{cases} \tag{2.24}$$

along with

$z = x + \frac{\lambda_0}{\alpha \rho_l} (1 - e^{\alpha t})$.

**Case II-** For $x > 0$, we get

$$\begin{cases} \rho(z,t) = \rho_r(z) e^{-\alpha t}, \\ v(z,t) = v_r(z) e^{-\alpha t} - \lambda_0 (1 - e^{-\alpha t}), \end{cases} \tag{2.25}$$

along with

$z = x + \frac{\lambda_0}{\alpha \rho_r} (1 - e^{\alpha t})$.

Since $v_l(x)$ and $v_r(x)$ are given by (2.23) and we obtain that $z = x$ at $t = 0$. Therefore the solution (2.24) exists for $z < 0$, i.e. $x < x_l(t)$, where

$$x_l(t) = \frac{\lambda_0}{\alpha \rho_l} \left(e^{\alpha t} - 1\right). \tag{2.26}$$

Similarly, the solution (2.25) exists for $x > x_r(t)$, where

$$x_r(t) = \frac{\lambda_0}{\alpha \rho_r} \left(e^{\alpha t} - 1\right). \tag{2.27}$$

In order to connect smoothly the left state (2.24) to the right state (2.25), we integrate the differential constraints (2.17) subject to the following initial conditions

$$\begin{aligned} &x(0) = 0, \rho = S(a), v = V(a), \quad \forall\, a \in [0,1], \\ &S(0) = \rho_L, S(1) = \rho_R, V(0) = v_L, V(1) = v_R. \end{aligned} \tag{2.28}$$

Then the solutions are

$$\begin{cases} \rho(x,t) = S(a)e^{-\alpha t}, \\ v(x,t) = V(a)e^{-\alpha t} - \lambda_0\left(1 - e^{-\alpha t}\right), \\ x = -\frac{\lambda_0}{\alpha S(a)}\left(e^{\alpha t} - 1\right). \end{cases} \tag{2.29}$$

On substitution of (2.29) in (2.18), yields

$$V' = \lambda_0 S^{-1} S'. \tag{2.30}$$

On integrating (2.30) along the conditions (2.28), we get

$$V(a) = v_L + \lambda_0\left(\ln S(a) - \ln \rho_L\right), \tag{2.31}$$

along with

$$v_L - \lambda_0 \ln \rho_L = v_R - \lambda_0 \ln \rho_R. \tag{2.32}$$

Now, in order to confirm the existence of the solution (2.29), we need $d\lambda/da > 0$ which implies that $\rho_L > \rho_R$ if $\lambda_0 = 1$ and $\rho_L < \rho_R$ if $\lambda_0 = -1$. Therefore the central state (2.29) exists in the region $x_l(t) \leq x(t) \leq x_r$ and it connects the left state (2.26) with right state (2.27) provided that the condition (2.32) is satisfied.

On eliminating $S(a)$ from first and third equations of (2.29), and substituting (2.31) in second equation of (2.29) yields the required result

$$\begin{cases} \rho(x,t) = \frac{\lambda_0}{\alpha x}\left(e^{-\alpha t} - 1\right), \\ v(x,t) = \left(v_L + \lambda_0\left(\ln S(a) - \ln \rho_L\right)\right)e^{-\alpha t} - \alpha x S(a) e^{-\alpha t}, \end{cases} \tag{2.33}$$

where $S(a) = \frac{\lambda_0}{\alpha x}\left(1 - e^{\alpha t}\right)$.

Equation (2.33) is required solution of generalized Riemann Problem for the governing system (2.1).

## 2.5 Conclusion

In this present study, we consider the hyperbolic system of first order PDEs describing $p-$system with linear damping and use differential constraint method to characterize the solution of this model. Within the framework of this method, we have obtained the consistency conditions for the governing model. Compatibility of the solution of the constraint equation and governing system is also discussed. The solution of the governing model is obtained in section (2.3) which is characterized for an arbitrary function as initial data to determine the exact solution for generalized Riemann Problem. The solution obtained for the generalized Riemann Problem consists of two states; left state and right state which are connected smoothly by the solution. This solution describes the rarefaction wave of the homogeneous case.

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