Chapter 1

Introduction

"Mathematics is the most beautiful and
most powerful creation of the human spirit."

–Stefan Banach.

1.1 Background

1.1.1 Linear and Nonlinear Waves

Understanding the behavior of nonlinear wave propagation is a matter of obvious importance. During the 19^{th} century, many mathematician and physicist like Stokes, Earnshaw, Riemann, Rankine, Lord Rayleigh, Hadamard, Von Neumann, Courant, Friedrichs, G. B. Whitham, and other developed fundamental concepts and wrote research papers and books inaugurating this field of research. A wave is any recognizable signal that is transferred from one part of the medium to another with a recognizable velocity of propagation. Waves occur in most scientific and engineering disciplines, for example: fluid mechanics, optics, electromagnetism, solid mechanics, structural mechanics, quantum mechanics, etc. The waves for all these applications are described by solutions to PDEs. The most important classification criterion is to distinguish PDEs as linear or nonlinear. Roughly, a homogeneous PDE is linear if the sum of two solutions is a solution, and a constant multiple of a solutions is a solution. Otherwise, it is nonlinear. The division of PDEs into these two categories is a significant one. The mathematical methods devised to deal with these two classes of equations are often entirely different, and the behavior of solutions differ substantially. One underlying cause is the fact that the solution space to a linear, homogeneous PDE is a vector space, and the linear structure of that space can be used with advantage in constructing solutions with desired properties that can meet diverse boundary and initial conditions. Such is not the case for nonlinear equations.

It is easy to find examples where nonlinear PDEs exhibit behavior with no linear counterpart. One is the breakdown of solutions and the formation of singularities; such as shock waves. Second is the existence of solutions, which are solutions to nonlinear dispersion equations. These solitary wave solutions maintain their shapes through collisions, in much the same was as linear equations do, even though the interactions are not linear.

1.1.2 Hyperbolic system of PDEs

The one-dimensional hyperbolic system of first order PDEs covers wide range of areas of scientific and technological interest. In particular, it has wide range of applications in gas dynamics, fluid dynamics, aerodynamics, multi phase flows, astrophysics, and plasma physics etc. The most interesting feature of quasilinear hyperbolic system of PDEs lies in the fact that a smooth solution breaks down after a finite time. The breaking of these smooth solutions give rise to one of the most interesting nonlinear phenomena that occurs in nature, i.e., the appearance of shock, which contains sudden jump in density, pressure and velocity. Another interesting feature of quasilinear hyperbolic system of PDEs is interaction of nonlinear waves. In order to present in mathematical description, let us consider the system of first order partial differential equations of the form

$$u_{i,t} + \sum_{j=1}^{m} a_{ij}(x,t,u_i)u_{i,x} + b_i(x,t,u_i) = 0 \tag{1.1}$$

for $i = 1, ..., m$. This is a system of m equations in m unknowns u_i that depend on space x and a time–like variable t. Here u_i are the dependent variables and x, t are the independent variables; this is expressed via the notation $u_i = u_i(x,t)$; $u_{i,t}$ denotes the partial derivative of u_i with respect to t; similarly $u_{i,x}$ denotes the partial derivative of u_i with respect to x. The system (1.1) can also be written in matrix form as

$$\mathbb{U}_t + \mathbb{A}\mathbb{U}_x + \mathbb{B} = 0, \tag{1.2}$$

where

$$\mathbb{U} = \begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ u_m \end{pmatrix}, \mathbb{B} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{pmatrix}, \mathbb{A} = \begin{pmatrix} a_{11} & . & . & a_{1m} \\ a_{21} & . & . & a_{2m} \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & . & . & a_{mm} \end{pmatrix}.$$

If the entries a_{ij} of the matrix $\mathbb{A}$ are all constant and the components b_j of the vector $\mathbb{B}$ are also constant then system (1.2) is linear with constant coefficients. If $a_{ij} = a_{ij}(x,t)$ and $b_i = b_i(x,t)$the system is linear with variable coefficients.The system is still linear if $\mathbb{B}$ depends linearly on $\mathbb{U}$ and is called quasi–linear if the coefficient matrix $\mathbb{A}$ is a function of the vector $\mathbb{U}$, that is $\mathbb{A}$=$\mathbb{A}(\mathbb{U})$. Note that quasi–linear systems are in general systems of non–linear equations. In system (1.2), $\mathbb{B} = 0$ corresponds to the homogeneous system.

Definition 1.1.1. (**Hyperbolic systems**) The system (1.2) is called a first-order hyperbolic system of partial differential equations if the matrix $\mathbb{A}(\mathbb{U})$ admits m real eigenvalues

$$\lambda_1(u_i) \leq \lambda_2(u_i) \leq \lambda_3(u_i) \leq ....., \leq \lambda_m(u_i).$$

together with set of m linearly independent eigenvectors L_1, L_2, L_3.....,L_m. The eigenvalues are also called the characteristic speeds or wave speeds associated with (1.2). The system is said to be strictly hyperbolic if its eigenvalues are distinct:

$$\lambda_1(u_i) < \lambda_2(u_i) < \lambda_3(u_i) < ....., < \lambda_m(u_i).$$

We now introduce some notions of linearity and nonlinearity for each j-wave family.

Definition 1.1.2. For each $i, j = 1, ..., m$, we say that the j-characteristic field corresponding to j-characteristic of (1.2) is **genuinely nonlinear** when

$$\nabla\lambda_j(u_i).L_j(u_i) \neq 0,$$

and **linearly degenerate** when

$$\nabla\lambda_j(u_i).L_j(u_i) = 0,$$

where $\nabla = \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, ...\frac{\partial}{\partial u_m}\right)$.

1.1.3 The Riemann Problem

The Riemann problem is one of the most fundamental problems in the field of hyperbolic conservation laws. In comparison with the Cauchy problem, it is easier to study but still reveals some basic properties of the Cauchy problem. Furthermore, the solutions of Riemann problem constitute the basic building blocks for the construction of solution to the Cauchy problem by using the random choice method. The Riemann problem was initiated and solved for one-dimensional Euler equations of isentropic flows in gas dynamics by Riemann [1] in his pioneering work on the mathematical theory of shock waves. It is an initial value problem with the simplest discontinuous initial data which is scale invariant and piecewise constant with one arbitrary discontinuity. There the concept of weak solution and the method of phase plane analysis for Riemann's solution which reveal the elementary waves of isentropic flows: shock waves and rarefaction waves.

The Riemann problem for the one–dimensional time–dependent Euler equations is the Initial Value Problem for the conservation laws

$$U_t + F(U)_x = 0. \tag{1.3}$$

Here, $U = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \end{pmatrix},$

where $(x,t) \in R \times R_+$.

The space and time coordinates are represented by x and t respectively. $u(x,t)$ and $\rho(x,t) > 0$ stand for velocity and density respectively. The initial conditions of the

Riemann problem for the system (1.3) is given by

$$U(x,0) = U_0(x) = \begin{cases} U_l = (\rho_l, v_l, p_l), & \text{if } x < 0 \\ U_r = (\rho_r, v_r, p_r), & \text{if } x > 0 \end{cases}. \tag{1.4}$$

Here, U_l and U_r denote the left and right constant state respectively which is separated by the jump discontinuity at $x = 0$.

Generalized Riemann problem is characterized by non-constant initial data while a Riemann problem is determined by initial constant data which are, in fact, equilibrium states of the governing system. The Generalized Riemann problem for the linear hyperbolic system provides a smooth solution for the system. The Generalized Riemann problem for non-linear hyperbolic system provides a bounded discontinuous solution depending on its characteristics. Usually, Generalized Riemann problem is considered as a perturbation of a Riemann problem, and it has been proved that the solution obtained in Generalized Riemann problem has the same behavior as the classical solution of the Riemann problem in the neighborhood of the origin.
The Generalized Riemann problem for governing system (1.3) with initial boundary condition

$$U(x,0) = \begin{cases} (\rho_l(x), u_l(x)), & \text{if } x < 0, \\ (\rho_r(x), u_r(x)), & \text{if } x > 0. \end{cases} \tag{1.5}$$

Here $\rho_l(x)$, $\rho_r(x)$, $u_l(x)$ and $u_r(x)$ are smooth arbitrary functions such that $\rho_l(0) \neq \rho_r(0)$, $u_l(0) \neq u_r(0)$.

For a scalar nonlinear case where $\lambda_1(U) = F'(U)$ is strictly convex, the weak solution to this problem will be either a shock wave or a continuous rarefaction wave. Generally, bounded, piecewise smooth weak solutions to a system of conservation laws satisfy the Rankine-Hugoniot condition at discontinuities. The Rankine-Hugoniot condition is

$$S[U] = [F],$$

where [.] represents the jump in a quantity across a shock of speed S, giving a relation between the speed S of the discontinuity and the constant states U_l and U_r on each side of the discontinuity. An additional condition, entropy condition, is required to pick out the physically correct weak solution. One such condition is the Lax entropy condition, which states that if $F(U)$ is convex, a shock wave would satisfy the entropy condition

$$\lambda_1(U_l) > S > \lambda_1(U_r).$$

More general weak solutions for (1.3) are the nonlinear wave solutions briefly described below:

- **Shock Wave:** If $\lambda_1(U_l) > \lambda_1(U_r)$, the entropy satisfying weak solution is a shock wave given by

$$U(x,t) = \begin{cases} U_l \quad for \quad \frac{x}{t} < S, \\ U_r \quad for \quad \frac{x}{t} \geq S, \end{cases}$$

  where S is the shock speed given by the Rankine-Hugoniot condition.

- **Rarefaction Wave:** If $\lambda_1(U_l) < \lambda_1(U_r)$, then the correct entropy solution is a rarefaction wave given by

$$U(x,t) = \begin{cases} U_l \quad for \quad \lambda_1(U_l)t > x, \\ U(\frac{x}{t}) \quad for \quad \lambda_1(U_l) \leq \frac{x}{t} \leq \lambda_1(U_r), \\ U_r \quad for \quad \lambda_1(U_r) \leq \frac{x}{t}, \end{cases}$$

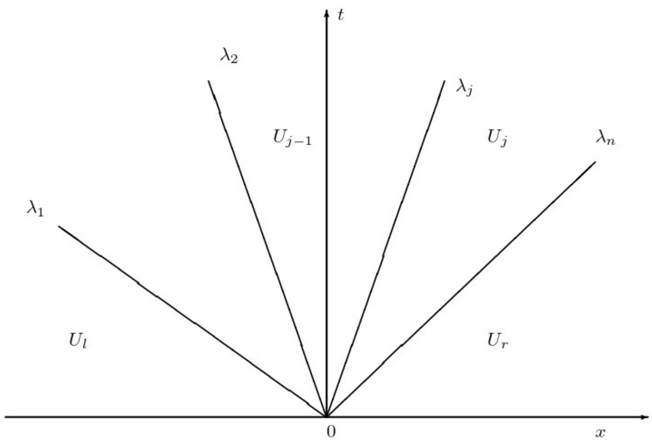

FIGURE 1.1: Structure of the Riemann solution for a system of conservation laws.

where $U(\frac{x}{t})$ is the solution of $F'(U(\frac{x}{t})) = \frac{x}{t}$.

These elementary wave solutions also will be the key elements to describe the structure of the solution of the Riemann problem for non-linear systems. The solution of the Riemann problem (1.3) is composed of $n+1$ constant states separated by n waves corresponding to the different characteristic fields. The structure of the solution in the $x-t$ plane is depicted in figure 1.1.

1.1.4 Dusty Gas

The study of a two-phase flow of gas and dust particles has been of great interest because of many applications to different engineering problems. Gas flows, which carry an appreciable amount of solid particles, may exhibit significant relaxation effects as a result of particles being unable to follow rapid changes of the velocity and temperature of the gas. When the mass concentration of the particles is comparable with that of the gas, the flow properties become significantly different from that of a pure gas. Here, we consider a mixture of a perfect gas and a large number of small dust particles of uniform spherical shape.

Dusty gas is considered to be mixture of gas and small solid dust particles where these dust particles attain less than five percent of total volume [2]. At very high speed of fluid, these small solid particles behave as a pseudo fluid [3].We consider the mixture as the mixture of two fluids: one is gas and the other is the pseudo fluid of solid particles. The solid particles are spheres of identical mass m_{sp}, radius r_{sp} and specific heat c_{sp}. We consider an element of mixture of gas and solid particles (dusty gas) with total mass $M = M_g + M_{sp}$ and with total volume $V = V_g + V_{sp}$, where subscript g refers to the value for the gas and subscript sp refers to that of the solid particles. The volume of solid particles in mixture is obtained as:

$$V_{sp} = n_{sp}.V.\tau_{sp},$$

where τ_{sp} and n_{sp} is the volume of a solid particle and the number of solid dust particles per unit volume of dusty gas respectively. The mass of solid particles in the volume V of the mixture is written as:

$$M_{sp} = n_{sp}.V.m_{sp}.$$

The species density of the solid particles is defined as:

$$\rho_{sp} = \frac{M_{sp}}{V_{sp}} = \frac{m_{sp}}{\tau_{sp}}.$$

Also, The partial density of the pseudo-fluid of solid particles is defined as:

$$\overline{\rho}_{sp} = \frac{M_{sp}}{V_{sp}} = n_{sp}.m_{sp} = Z\rho_{sp} = n_{sp}.\rho_{sp}.\tau_{sp},$$

where Z represents the fraction of volume of solid particles in the mixture.

Further, volume fraction of solid particles is given as:

$$Z = \frac{V_{sp}}{V} = n_{sp}.\tau_{sp}.$$

The species density of the gas or the fluid is defined as:

$$\rho_g = \frac{M_g}{V_g}.$$

Similarly, the partial density of a gas is defined as:

$$\overline{\rho}_g = \frac{M_g}{V} = (1 - Z)\rho_g.$$

Let us consider the thermodynamic equilibrium condition such as:

$$T_{sp} = T_g = T.$$

The density of the mixture is obtained as

$$\rho = Z\rho_{sp} + (1 - Z)\rho_g = \overline{\rho}_{sp} + \overline{\rho}_g.$$

The mass concentration of the pseudo fluid of the solid particles is obtained as:

$$k_p = \frac{\overline{\rho}_{sp}}{\rho} = \frac{Z\rho_{sp}}{\rho}.$$

The pressure of the mixture is written as:

$$p = p_{sp} + p_g.$$

The total pressure of the mixture is p which is obtained from the perfect gas law as:

$$p = R\rho_g T_g$$

with the help of above analysis, the pressure of the mixture as a whole is obtained as:

$$p_m = p = R\rho_g T_g = R\left(\frac{\rho_m - Z\rho_{sp}}{1-Z}\right)T_g = R\rho_m\left(\frac{1-k_{sp}}{1-Z}\right)T.$$

Therefore, $p_m = \frac{\rho_m R_m T}{1-Z}$ where, $R_m = (1-k_p)R$. Here, R may be considered as an effective gas constant of the mixture and subscript m refers to the value of the gas constant in the mixture as a whole.

1.1.5 Chaplygin Gas

Sergey Alexeyevich Chaplygin was a Russian and Soviet physicist, mathematician, and mechanical engineer. He is known for mathematical formulas such as Chaplygin's equation and for a hypothetical substance in cosmology called Chaplygin gas, named after him. Chaplygin's theories were greatly inspired by N. Ye. Zhukovsky, who founded the Central Institute of Aerodynamics. Chaplygin was elected to the Russian Academy of Sciences in 1924. The lunar crater Chaplygin and town Chaplygin are named in his honor.
An interesting model of dark energy is based on Chaplygin gas equation of state. we propose the following EoS,

$$p(\rho) = \sum_{j=1}^{m} A_j \rho^j - \frac{B}{\rho^\alpha},$$

where A_j and B are positive constants. p is pressure , ρ is the density. This is called extended Chaplygin gas equation of state. It reduces to MCG equation of state

[4] for $m = 1$, and can recover barotropic fluid with quadratic equation of state by setting $m = 2$. Also, higher m may recover higher order barotropic fluid which is indeed our motivation to suggest extended Chaplygin gas. The case of $A_j = 0$ recovers generalized Chaplygin gas equation of state, and $A_j = 0$ together $\alpha = 1$ recovers the original Chaplygin gas equation of state. Chaplygin gas, which occurs in certain theories of cosmology, is a hypothetical substance that satisfies an exotic equation of state in the form [5]

$$p(\rho) = \frac{-B}{\rho^{\alpha}}.$$

The substance is named after Sergey Chaplygin. In some models, generalized Chaplygin gas is considered, where α is a parameter, which can take on values $0 < \alpha < -1$. In exotic background fluid, the Chaplygin gas equation plays an important role to describe the accelerated expansion of the universe and evolution of the perturbations of energy density. It also describes the dark energy and dark matter in the united form. The Chaplygin gas equation is used to describe the concept of dark energy and dark matter related mathematical problems.

1.2 Motivation

Any mathematical model of a continuum is given by a system of partial differential equations (PDEs). In continuum mechanics, the conservation laws of mass, momentum and energy form a common starting point and each medium is then characterized by its constitutive laws. The conservation laws and constitutive equations for the field variables, under quite natural assumptions, reduce to field equations, i.e. partial differential equations, which in general, are nonlinear and nonhomogeneous due to complicated physical phenomena. The system of non-linear PDEs is classified into elliptic, parabolic and hyperbolic. Out of these, the hyperbolic system of conservation laws is one of the most important class of non-linear PDEs. Euler's equations are the most common examples of hyperbolic PDEs. The Euler equations arise from the compressible Navier-Stokes equations by neglecting the viscosity and heat conduction. The theoretical foundation of gasdynamics is formed by the application of the basic conservation laws of mechanics and the second law of thermodynamics to a moving volume of a compressible gas.

The most outstanding new phenomenon of the nonlinear theory is the appearance of shock waves, which are abrupt jumps in pressure, density, and velocity: the blast waves of explosions and the sonic booms of high speed aircraft. But the whole intricate machinery of nonlinear hyperbolic equations had to be developed for their prediction, and a full understanding is required to analyze the viscous effects and some aspects of kinetic theory.

The detailed study of mathematical properties and applications of nonlinear wave propagation problems in the context of hyperbolic systems of PDEs may be found in the books and monographs by Courant and Friedrichs [6], Jeffrey [7], Whitham [8], Zheng [9], Bressan [10], Dafermos [11], Sharma [12], Smoller [13], Holden and Risebro [14] etc. Solving a Cauchy problem for hyperbolic conservation laws, we find two important challenges. The first one is that the solution may not be continuous after a finite time and the second one is how to choose an admissible solution from several weak solutions. In order to obtain a unique solution to the Cauchy problem, terms such as admissible wave, stable solution, physically relevant solution and so on are introduced. The complete theoretical basis of conservation laws and the admissibility of the weak solution are given by Lax [15].

Since, a lot of interesting work has been contributed to the one-dimensional Riemann problem for various hyperbolic systems of conservation laws; indeed, the Riemann problem has been playing an important role in all three areas of theory, applications and computation. It is fully recognized that the Riemann problem is the most fundamental problem in the entire field of nonlinear hyperbolic conservation laws and Riemann's work is his most important contribution in the area of mathematical physics. Thus, it serves as a touchstone for numerical schemes due to the explicit structure of Riemann solution. For nonlinear hyperbolic system of conservation laws we refer to Jeffrey [7]; Ruggeri and Simic [16]; Liu [17]; Godlewski [18]; Lax [15] and references cited therein. It is well known that for nonlinear hyperbolic systems of conservation laws, what the most important feature is that no matter how smooth flux functions and even in the case of sufficiently smooth and small initial data, discontinuities may occur in the solution. The theory of nonlinear hyperbolic system of conservation laws in one space dimension usually assume that the system to be strictly hyperbolic with genuinely nonlinear or linearly degenerate characteristic fields. Moreover, general results on the existence of entropy weak solutions to these systems are established only for initial values with small total variation, see Lax [15] and Glimm [19]. However, it is recognized that most of the physical systems do not fit into the standard theory of hyperbolic systems of conservation laws. A natural question is whether these results remain true for a nonstrictly hyperbolic system, or even for strictly hyperbolic systems for initial data with large total variation. Keyfitz and Kranzer [20] considered the Riemann problem for one strictly hyperbolic system whose characteristic fields are genuinely nonlinear. They found that for large initial data, the Riemann problem may admit no classical wave solution (which consists of contact discontinuities, shock, and/or rarefaction waves).

Since the 70s of 20th century, there are nonclassical solutions in contrast to Lax's and Glimm's results such as measure solutions, singular shocks, etc. Especially, the discovery of a new type of nonlinear nonclassical wave, called delta shock wave, has attracted the mathematicians and physicists widespread attention. A delta shock wave is a generalization of an ordinary shock wave. Informally speaking, it is a kind of discontinuity, on which at least one of the state variables may develop an extreme concentration in the form of a weighted Dirac delta function with the discontinuity line as its support. It is more compressive than an ordinary shock wave in the sense that more characteristics enter the discontinuity line. Physically, the delta shock waves are interpreted as the process of formation of the galaxies in the universe or the process of concentration of particles.

1.3 Literature Review

The phenomenon of shock waves is mainly associated with aerospace engineering and in particular with the supersonic flight. The development of this particular branch of physics began, in 1746 when a mathematician Robins determined velocity of the bullet by ballistic pendulum and noticed a growth in aerodynamic drag as velocity tends to the sound speed. However, in the 19th century, the phenomenon of the shock wave was still a mystery to many researchers. In 1759, without mentioning the word shock waves, Euler talked about the "size of disturbance" of a sound wave meaning its amplitude. Nevertheless, his assumption that velocity would diminish with increasing amplitude was incorrect. In 1808, Poisson [21] was the first researcher to solve the Euler equation for the one-dimensional unsteady fluid-flow and got the exact solutions. In 1823, Poisson created a milestone in non-linear wave theory by constructing isentropic gas law for the sound wave with infinitesimal amplitude. In 1848, Stokes [22] used the term "surface of discontinuity". He further extended the theory of acoustic wave, having a finite amplitude, by considering the problem of wave steepening. Stokes was not sure about the possibility of discontinuous motion, in this confusion he used isentropic relation which decides the role of dissipation and energy conservation in shock formation, instead of the correct energy equation. He derived the conservation laws for mass and momentum which are used very frequently in modern days. In 1889, Hugoniot [23] independently derived the correct jump conditions for the shock waves. The formulated theory of Rankine and Hugoniot is, even at present also, the basic model for the propagation of shock waves.

The study of Riemann problem started with the work " theory of waves of finite amplitude" by great mathematician G. F. B. Riemann (1859), which was not limited to a single progressive wave and suited to calculate the propagation of planar waves of finite amplitude proceeding in both directions. In 1860, Riemann [1] introduced the Riemann problem for a system of conservation laws in gas dynamics. Lax determined the solution of the Riemann problem for the condition when the initial data of the problem consists of two constant states U_*^1 and U_*^2, where U_*^1 and U_*^2 are respectively the vector of conserved variables to the left and right of $x = 0$ such that $\|U_*^1 - U_*^2\|$ is appropriately small and left and right constant states are divided by jump discontinuity at $x = 0$. In the consideration of Euler equations, the Riemann problem consists of the shock tube problem and for detailed discussion of shock tube problem and other physical problems in form of conservation laws of gasdynamics, the readers are recommended to study the book by Courant and Friedrichs [6].

The exact solution to the Riemann problem is of great significance. For instance, it constitutes the basic building block for the construction of solutions to general initial value problems using the well known random choice method proposed by Glimm [19]. The exact solutions of the Riemann problem for hyperbolic system of conservation laws are proposed by many authors for detailed methodologies, the reader is referred to the book by [24, 25, 26, 27]. Also, Chorin in [28, 29] proposed the new approach to obtain the exact solution to the Riemann problem. Another improvement to the Godunov's first Riemann solver was presented in [30].

For an illuminating treatment on Riemann problem, we also refer to an article by Liu [17] and the books of Dafermos [11], Bressan [10] and LeVeque [31]. The special solution of Euler equations in which one of the Riemann invariants remains constant throughout the flow field is called a simple wave. In simple wave solutions, waves break and the solution has to be complemented by the introduction of shock waves. When the shock strength is small and even moderate, jumps in entropy and the Riemann invariants are surprisingly small, see Whitham [8]; this, indeed, formed the basis for an approximate theory for shocks of weak or moderate strength developed by Friedrichs [32], where the actual shock conditions are replaced by the transition through a corresponding simple compression wave.

It is well known that in the compressible fluid flow if the speed of the fluid is greater than the sound speed then shocks may form when particles collide. However, it is observed numerically that (Chang et al. [33]) for gas dynamics in the regime of small pressure: for one case, the particles seem to be more sticky and tend to concentrate at some shock location which move with the associated shock speed and for the other case, in the region of rarefaction waves, the particles seem to be far apart and tend to form cavitation in the region. Such phenomena may be regarded as a tendency towards the concentration and cavitation in terms of the density. Chen and Liu [34] rigorously justified the phenomena of concentration and cavitation for isentropic fluids by looking at the vanishing pressure limit. During the recent decades, the problems concerning the phenomena of concentration, cavitation, the formation of delta shock waves and vacuum states in solution have received much more attention ([35, 36, 37, 38, 39, 40]). The construction of delta shock wave and vacuum state for pressureless Euler systems investigated for the isothermal and nonisentropic fluids by using vanishing pressure approach [41]. Now, the vanishing pressure limit method has been widely used and the results were extended to the relativistic Euler equations by Yin and Sheng [42], for the nonlinear chromatography system by Sun [43], to the perturbed Aw-Rascle model by Shen and Sun [44], to the modified Chaplygin gas equations by Yang and Wang [45, 46], to the Aw-Rascle model of traffic flow by Liu [47] etc. It is clear that the flux approximation method is indeed a natural generalization of the vanishing pressure limit method as in [48, 49]. Also, multi-dimensional conservation laws describe the realistic situation more accurately and are more interesting. In the study of 2-dimensional hyperbolic conservation laws, the initial data with two constants on two sides of a straight line is a trivial generalization of the one-dimensional case. To get the nontrivial 2-dimensional solution for Riemann problem, a natural choice is to assume that x-axis and y-axis are all initial discontinuities and the initial value in the four quadrants are different constants. There have been many studies on 2-dimensional hyperbolic system of conservation laws from various aspects. Tan and Zhang in [50] firstly studied the four quadrant Riemann problem of 2-dimensional hyperbolic system of conservation laws, namely the initial data are four constant states in each quadrant of (x, y) plane, and they discovered that a kind of new nonlinear wave called delta shock wave was there. About the delta shock wave solution in the multi-dimensional hyperbolic conservation laws, we can also see [51, 52] and the reference therein. The Riemann problem of 2-dimensional hyperbolic conservation laws with such initial data was studied in [53, 54, 55, 56].

Furthermore, in the literature, a Generalized Riemann problem (GRP) is characterized by non-constant initial data while a Riemann problem is determined by initial constant data which are, in fact, equilibrium states of the governing system. A more hard task is to find exact solutions to a generalized Riemann problem which consists of non-constant initial data with a discontinuity at $x = 0$. During the last few decades variety of mathematical methods have been proposed to find exact solutions to PDEs. For example, similarity transformation method, perturbation method, differential constraint method etc. that lead to exact type solutions or approximate solutions. Curro et al. [57] introduced systematic reduction approach to obtain particular classes of solutions to quasilinear hyperbolic systems. The exact solution to rate-type material model by reduction procedure using differential constraint method is described by Fusco and Manganaro [58]. Curro et al. [59, 60] have obtained the exact solution to Generalized Riemann problem for traffic flow model by using the Differential constraints method. This method provides a systematic technique for determining the exact solution to system of quasilinear PDEs. In gasdynamics, this technique was introduced by Janenko [61]. Many contributions on generalized Riemann problem have been given concerning existence and uniqueness theorems. In particular, it can be proved that the solution of a generalized Riemann problem has the same behaviour as that of the classical solution of the Riemann problem in the neighborhood of the origin. For more detail on generalized Riemann problem, we also refer to readers to the articles [62, 63, 64, 65, 66, 67] .

1.4 Aims and Thesis Objectives

The aim of the present thesis is to study the certain problems associated with the solutions of the Riemann Problem for quasilinear one-dimensional conservative hyperbolic system which occur in many physical phenomena having practical importance in real life. Our aim is to solve, those homogeneous and non-homogeneous hyperbolic systems where classical and non-classical situations arise, using various approaches like flux approximation method, Differential constraints method, vanishing pressure limit method and Characteristic method for hyperbolic system. We are motivated to solve the problem for non-homogeneous hyperbolic system which is modified into homogeneous hyperbolic system of conservation laws to study the solution of Riemann problem with constant initial data by introducing new variable for the velocity. Our objective in this thesis is to address few of these issues to understand the mathematical theory of hyperbolic system of conservation laws. In order to address some of these issues, we have been motivated to work on the following objectives:

1. To study the exact solution of the generalized Riemann problem for 2×2 hyperbolic $p-$ system with linear damping with non-constant Riemann initial data by using Differential constraint method.

2. To determine the structure of solution of the Riemann Problem for the one-dimensional compressible hyperbolic system with logarithmic equation of state under the influence of friction.

3. To study the delta shock wave solution of the Riemann problem for the non-homogeneous hyperbolic system with modified Chaplygin gas equation of state.

4. To investigate the phenomenon of concentration and cavitation in the solution of the Riemann problem for the dusty gas model by using two parameter flux approximation approach.

5. To discuss the evolutionary behavior of shock wave along the characteristic path in polytropic reacting gas flow with small solid dust particles.

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