CERTIFICATE

It is certified that the work contained in this thesis titled "**Study of Certain Problems on Nonlinear wave Propagation involving Hyperbolic System of PDEs**" by **Rahul Kumar Chaturvedi** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy and SOTA for the award of Ph.D. degree.

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DECLARATION BY THE CANDIDATE

I, **Rahul Kumar Chaturvedi**, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of **Dr. L. P. Singh** from **July, 2017** to **March, 2022** at the **Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi**. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

Date:

Place: Varanasi **(Rahul Kumar Chaturvedi)**

CERTIFICATE BY THE SUPERVISOR

It is certified that the above statement made by the student is correct to the best of my/our knowledge.

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Dedicated

to

My Beloved Family

and

Supervisor

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Rahul Kumar Chaturvedi

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PREFACE

It is well known that Gas dynamics is a branch of compressible fluid dynamics. It evolved in the end of 19^{th} century to understand high speed fluid flow phenomenon. A wave can be thought as a propagating feature of disturbance. It is defined as any notable feature that is propagated from one medium to another or within the medium with a recognizable speed. It can be any characteristic of the disturbance, such as the formation of trough and crest or sudden change etc., in some physical quantity, provided that it can be clearly noticed and its position at any time can be found. The characteristic feature may contort, be magnified, and change its velocity provided it is still recognizable. Certain types of wave can be formulated mathematically in terms of hyperbolic partial differential equations.

In 1860, Riemann studied fluid dynamics through a shock tube. He introduced the Riemann problem for a system of conservation laws in gas dynamics which is a specific initial value problem composed of a conservation equation together with piecewise constant initial data which has a single discontinuity in the domain of interest. The Riemann problem is very useful for the understanding of equations like Euler conservation equations because all properties, such as shocks and rarefaction waves, appear as characteristics in the solution. It also gives an exact solution to some complex nonlinear equations. A shock wave is a surface of discontinuity across which the flow properties experience a sudden jump. Across a rarefaction wave the flow properties are continuous. The velocity and pressure are continuous across a contact wave but density, temperature, entropy etc. experience a sudden change. Shock waves are most challenging phenomenon occurring in non linear wave motion; they can develop and propagate, even if the initial data are continuous. The reason is that non linear partial differential equations do not admit continuous solutions.

The present thesis, embodies the results of researches carried out by me at the Department of Mathematical sciences, Indian Institute of Technology (BHU), Varanasi, during the period July 2017 to February 2022 under the supervision of Prof. L. P. Singh. The present work deals with some problems associated with the solutions of the Riemann problem for quasilinear one-dimensional conservative hyperbolic system which occur in many physical phenomena having practical importance in real life. Our aim is to solve, those homogeneous and non-homogeneous hyperbolic systems where classical and non-classical situations arise, using various approaches like flux approximation method, Differential constraints method, vanishing pressure limit method and Characteristic method for hyperbolic system. We are motivated to solve the problem for non-homogeneous hyperbolic system which is modified into homogeneous hyperbolic system of conservation laws to study the solution of Riemann problem with constant initial data by introducing new variable for the velocity. Also, this thesis concerns with the solutions of the Riemann problem with constant and non-constant initial data for different hyperbolic systems. We introduce the notions of rarefaction waves, shock waves, contact discontinuities and delta shock waves, which play an essential role in the explicit construction of the solution of the Riemann problem. Then, we discuss the local existence and uniqueness of the solution of Riemann problem for a system in the sense that the initial states are sufficiently close. It is also proved that this is true for the dusty gas dynamic equations. We consider the strictly hyperbolic system of conservation laws which describes the background flow carrying dust particles and whose Riemann solution contains classical elementary waves as well as delta shock wave in certain situation. The whole thesis is divided into six chapters as fallows:

Chapter - 1 is introductory and gives a general idea of when and how a discontinuity appears. The mathematical theory and their fundamental properties have also been briefly discussed. The physical properties of hyperbolic systems, equation of state, dusty gas, reacting gas and methods which are used throughout the thesis are briefly reviewed.

Chapter - 2 concerns with the study of the exact solution of the generalized Riemann problem for the 2×2 hyperbolic $p-$system with linear damping by using Differential constraint method. This method is used to develop the consistency conditions and constraint equations for the considered non-homogeneous hyperbolic system and obtained the exact solution of the system of governing equations. Further, the generalized Riemann problem of non-homogeneous hyperbolic $p-$system which involves non-constant discontinuous initial data is solved.

In **Chapter - 3**, the structure of the Riemann solutions for compressible hyperbolic system of PDEs with logarithmic equation of state, so called Logotropic model, in the presence of a Coulomb-type friction is analyzed. The system considered in this chapter is hyperbolic in nature and the characteristic fields associated with the characteristics are genuinely non-linear. The classical wave solutions of the Riemann problem for the Logotropic model are structured explicitly for all cases. It is shown that the Riemann solutions for the Logotropic model with a Coulomb-type friction term composed of the rarefaction wave and shock wave. Also, It is found that the Coulomb-type friction term, appearing in the governing equations, influences the Riemann solution for the system.

In **Chapter - 4**, the solution of the Riemann problem for hyperbolic system of PDEs with modified Chaplygin gas (MCG) equation of state in the presence of constant external force is studied. The analysis leads to the fact that in some special circumstances delta shock appears in the solution of the Riemann problem. Further, the Rankine-Hugoniot relations for delta shock wave which are utilized to determine the strength, position and propagation speed of the delta shocks have been derived. Delta shock wave solution to the Riemann problem for the non-homogeneous hyperbolic system with modified Chaplygin gas equation of state is obtained. It is found that the external force term, appearing in the governing equations, influences the Riemann solution for the system.

In **Chapter - 5**, the concentration and cavitation phenomenon in the solution of the Riemann problem to the pressure-less isentropic Euler equations for the dusty gas model is investigated by using two parameter flux approximation. The similar solution of the Riemann problem for dusty gas model is obtained. The formation of delta - shock and vacuum state in the flow field is discussed. Also, it is shown that the solution, containing two shock waves, of the Riemann problem to the isentropic Euler equations for dusty gas converges to the delta - shock wave solution of the transport equations and the solution, containing two rarefaction waves, of the Riemann problem converges to the vacuum state solution of the transport equations.

In **Chapter - 6**, the evolutionary process of shock wave along the characteristic path under the effect of dust particles in a polytropic reacting gasdynamics is investigated. Using the characteristics of the governing quasilinear hyperbolic system as a reference coordinate system, we transform the governing equations and obtain the solution of it. It is shown that a linear solution in the characteristic plane can exhibit a non-linear behaviour in the physical plane. It is shown how the presence of dust in reacting gas affects the growth and decay of the compressive and expansive waves. The transport equation leading to the evolution of shock wave is determined which provides the relations for the shock formation. The comparative study of the effect of reacting gas parameter and dust particles on the flow patterns and distortion of shock wave for planar, cylindrically symmetric and spherically symmetric flows is also performed.

Lastly, in **Chapter - 7**, the work done in the thesis is summarized. Major contributions made in the thesis are briefly discussed followed by a discussion on the future scope.