7 Spatially Homogeneous Cosmological Models in $f(R,T)$ Gravity Theory

7.1 Introduction

General relativistic cosmological models provide a framework for the investigation of evolution of the universe. Present cosmology is based on the Friedmann-Robertson-walker(FRW) model. In this model, the universe is completely homogeneous and isotropic which is in good agreement with the observational data about the large scale structure of the universe. The adequacy of a FRW model for describing the present state of the universe is no basis for expecting that it is equally suitable for describing the early stages of evolution of the universe. There are theoretical arguments (Misner (1968), Chimento (2004)) and recent experimental data of the cosmic microwave background radiation which support the existence of an anisotropic phase that approaches an isotropic one (Land and Maguejo (2005)). This stipulates search for anisotropic cosmologically acceptable models of the universe at least in its early stages of evolution.

The content of this chapter has been published in **Electronic Journal of Theoretical Physics 12(32), 58-68 (2014).**

It is believed that the early universe evolved through some phase transitions thereby yielding a vacuum energy density which is at present is at least 118 orders of magnitudes smaller than in plank time (Weinberg (1989)). Such a discrepancy between theoretical expectations and empirical observations constitutes a fundamental problem in the interface uniting astrophysics, particles physics and cosmology is the cosmological constant problem. The recent cosmological observations obtained by type Ia supernova (Riess et al. (1998),(1999), Perlmutter et al. (1999), Tonry et al. (2003)), large scale structure (Tegmark et al. (2004), Seljak et al. (2005), Percival et al. (2007), Kamatsu et al. (2009)), baryon oscillation (Eisenstein et al. (2005)) and weak lens (Jain and Tayler (2003)) have suggested that the expansion of the universe is accelerating. These observations seem to change the entire picture of our matter filled universe. It has been observed that a fluid known as dark energy with large negative pressure is responsible for this acceleration. Many dark energy models have been proposed to explain the cosmic accelerated expansion (Copeland et al. (2006)). The cosmological constant Λ, responsible for cosmic accelerated expansion, is the simplest candidate of dark energy (Sahni and Starobinsky (2000), Padmnabhan (2003)).

In recent years, there has been a lot of interest in alternative theories of gravitation (Brans and Dicke (1961), Canuto et al. (1977), Saez and Ballester (1985)). In view of the late time acceleration of the universe and the existence of the dark matter and dark energy, very recently, modified theories of gravity have been developed. Noteworthy amongst them are $f(R)$ theory of gravity formulated by Nojiri and Odintsov (2003) and $f(R,T)$ theory of gravity proposed by Harko et al. (2011).

Harko et al.(2011) developed a generalized $f(R,T)$ gravity theory where the gravitational Lagrangian is given in terms of any arbitrary function of the Ricci scalar R and the trace T of the energy-momentum tensor T_{ij} and obtained field equations in metric formalism from Einstein- Hilbert type variational principle. They have presented several models corresponding to three explicit forms of the function $f(R,T)$.

In this chapter, we investigate a general Bianchi space-time model filled with a perfect fluid in $f(R,T)$ gravity theory. Exact solutions of the field equations are obtained explicitly by choosing the average factor $a(t)=\sqrt{t^n e^t}$, where n is a positive constant. The chapter is organized as follows: In Sec.(7.2), we present the space-time metric and the field equations for a perfect fluid distribution in $f(R,T)$ gravity theory for the particular form of $f(R,T)=R+2\lambda T$, where λ is a constant. We obtain a new class of exact solutions of the field equations in Sec.(7.3). In Sec.(7.4), we discuss some physical and dynamical properties of the model. In Sec.(7.5), we study the stability of the solution by invoking a cosmological perturbative approach. Finally, conclusions are summarized in the last Sec.(7.6).

7.2 The Metric and Field Equations

We consider the diagonal form of the metric of general class of Bianchi type cosmological models given by

$$ds^2 = dt^2 - A^2 dx^2 - B^2 e^{-2x} dy^2 - C^2 e^{-2mx} dz^2. \tag{7.1}$$

The metric (7.1) corresponds to a Bianchi type-III model for $m = 0$, type-V model for $m = 1$, type-VI_0 model for $m = -1$ and type- VI_h model for all other $m = h-1$.

The field equations in $f(R,T)$ theory of gravity for the function $f(R,T) = R + 2f(T)$, when the matter source is perfect fluid, is given by Eq.(6.2).

Choosing comoving coordinates, for particular choice of the function $f(T) = \lambda T$, where λ is a constant, the field equations (6.2), for the metric (7.1), can be explicitly written as:

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} - \frac{m^2+m+1}{A^2} = \lambda p - (8\pi + 3\lambda)\rho, \tag{7.2}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{m}{A^2} = (8\pi + 3\lambda)p - \lambda\rho, \tag{7.3}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{m^2}{A^2} = (8\pi + 3\lambda)p - \lambda\rho, \tag{7.4}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{1}{A^2} = (8\pi + 3\lambda)p - \lambda\rho, \tag{7.5}$$

$$(m+1)\frac{\dot{A}}{A} - \frac{\dot{B}}{B} - m\frac{\dot{C}}{C} = 0. \tag{7.6}$$

Equations (7.2)-(7.6) are nonlinear differential equations with five unknowns A, B, C, ρ and p.

The average scale factor $a(t)$ is defined by

$$a(t) = (ABC)^{\frac{1}{3}}. \tag{7.7}$$

Now, we take the following ansatz for the average scale factor as

$$a(t) = \sqrt{t^n e^t}, \tag{7.8}$$

n being a positive constant. Pradhan and Amirhashchi (2011) and Saha et al. (2012) examined this form of the scale factor to study accelerating dark energy models in

Bianchi type$-V$ space-time and a two-fluid scenario for dark energy models in an FRW universe respectively. Pradhan (2013) assumed this form of $a(t)$ to discuss some features of Bianchi type-VI_0 models in the presence of a perfect fluid that has an anisotropic equation of state parameter in general relativity. This choice of average scale factor yields a time- dependent deceleration parameter such that before the DE era, the corresponding solution gives the inflation and radiation/matter dominated era, with subsequent transition from deceleration to acceleration. For $n = 0$, this choice of scale factor gives an exponential law of variation for the scale factor. The choice (7.8) of the average scale factor is physically acceptable. From Eqs.(1.34) and (7.8), the time-dependent $q(t)$ is obtained as

$$q(t) = \frac{2n}{(n+t)^2} - 1. \tag{7.9}$$

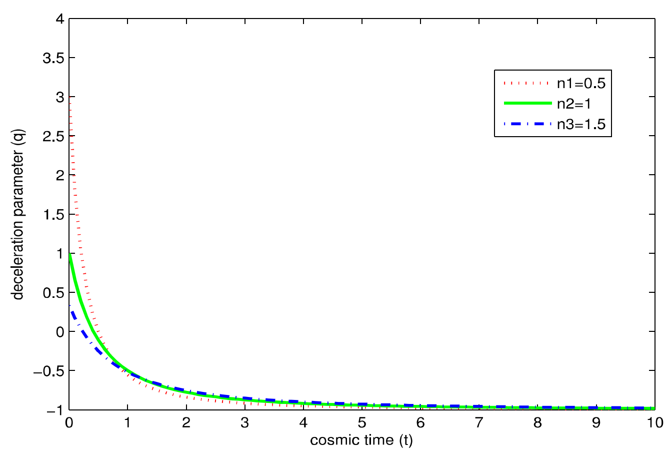

Figure 7.1: The plot of deceleration parameter q vs. cosmic time t

From Eq.(7.9), it is clear that $q > 0$ for $t < \sqrt{2n} - n$ and $q < 0$ for $t > \sqrt{2n} - n$. For $0 < n < 2$, the model is evolving from deceleration phase to acceleration phase. Recent observations of SNe Ia have shown that the present universe is accelerating and q lies in the range $-1 < q < 0$. Thus, the model has accelerated expansion at present epoch which in consistent with recent observations of Type Ia supernova. Figure (7.1) depicts the deceleration parameter q versus time which gives the behavior of q from decelerating to accelerating phase for different values of n.

7.3 Cosmological Solutions

We now obtain physically realistic cosmological models to describe the decelerating and accelerating phases of the universe. We assume that $a_3 = V^b$ where b be any constant number. Then, from Eqs.(7.6), (7.7) and (7.8), we obtain the explicit solutions of A, B and C as follows:

$$A = (t^n e^t)^{\frac{3(1+mb-b)}{2(m+2)}}, \tag{7.10}$$

$$B = (t^n e^t)^{\frac{3(1+m-b-2mb)}{2(m+2)}}, \tag{7.11}$$

$$C = (t^n e^t)^{\frac{3b}{2}}. \tag{7.12}$$

Thus, the metric (7.1) can be written in the form

$$ds^2 = dt^2 - (t^n e^t)^{\frac{3(1+mb-b)}{(m+2)}} dx^2 - (t^n e^t)^{\frac{3(1+m-b-2mb)}{(m+2)}} e^{-2x} dy^2 - (t^n e^t)^{3b} e^{-2mx} dz^2. \tag{7.13}$$

In the next section, we discuss the physical and kinematical behaviors of the model (7.13).

7.4 Physical and Geometrical Behaviors of the Model

The Hubble parameters H_x, H_y and H_z have values given by

$$H_x = \frac{3(1-b+mb)}{2(m+2)}\left(1+\frac{n}{t}\right), \tag{7.14}$$

$$H_y = \frac{3(1+m-b-2mb)}{2(m+2)}\left(1+\frac{n}{t}\right), \tag{7.15}$$

$$H_z = \frac{3b}{2}\left(1+\frac{n}{t}\right). \tag{7.16}$$

The average Hubble's parameter H has the value given by

$$H = \frac{1}{2}\left(1+\frac{n}{t}\right). \tag{7.17}$$

The dynamical scalars σ, θ and anisotropy parameter A_m are given by

$$\sigma^2 = \frac{3[18m^2b^2+18mb^2-12m^2b-12mb+2m^2+18b^2+2m-12b+2]}{8(m+2)^2}\left(1+\frac{n}{t}\right)^2, \tag{7.18}$$

$$\theta = \frac{3}{2}\left(1+\frac{n}{t}\right), \tag{7.19}$$

$$A_m = \frac{2[9m^2b^2+9mb^2-6m^2b-6mb+m^2+9b^2+m-6b+1]}{(m+2)^2}. \tag{7.20}$$

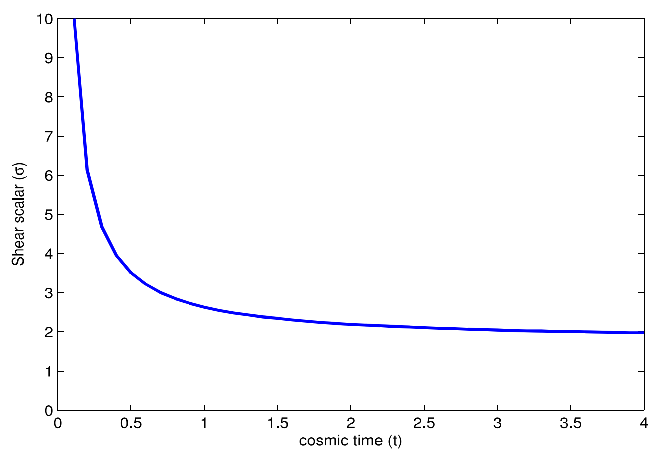

Figure 7.2: The plot of shear scalar σ vs. cosmic time t for m=1.5, n= 0.5, b=1.5

The energy density and isotropic pressure of the model are given as

$$\rho = \frac{1}{8(\lambda^2+6\pi\lambda+8\pi^2)}\left[\frac{3}{4t^2(m+2)^2}\{3(n+t)^2(\lambda A_1-8\pi A_2-3\lambda A_2) +2n\lambda(m^2b+mb-m^2-3m-2b-2)\} +\{(8\pi+3\lambda)(m^2+m+1)-m\lambda\}(e^t t^n)^{\frac{-3(1+mb-b)}{(m+2)}}\right], \tag{7.21}$$

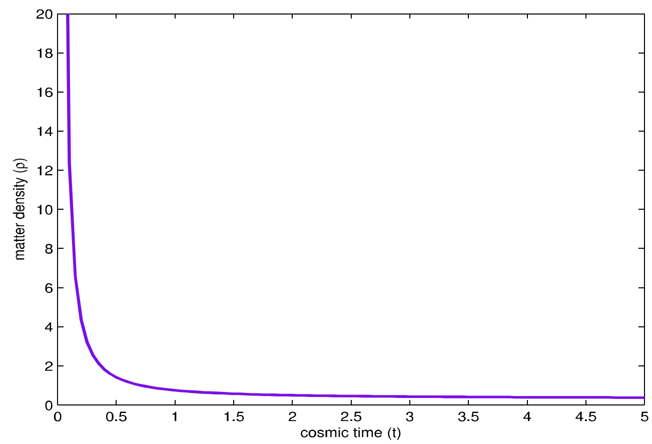

Figure 7.3: The plot of matter densityρ vs. cosmic time t for m=1.5, b=1.5, n=0.5 and $\lambda = 0.05$

$$p = \frac{1}{8(\lambda^2 + 6\pi\lambda + 8\pi^2)} \Bigg[\frac{3}{4t^2(m+2)^2} \{3(n+t)^2(8\pi A_1 + 3\lambda A_1 - \lambda A_2) + (16n\pi + 6n\lambda)(m^2 b + mb - m^2 - 3m - 2b - 2)\} + \{m(8\pi + 2\lambda) - (m^2+1)\lambda\}(e^t t^n)^{\frac{-3(1+mb-b)}{(m+2)}} \Bigg] \quad (7.22)$$

where $A_1 = 3b^2(m^2+m+1) - 3b(m^2+m) + (m^2+2m+1)$ and $A_2 = -3b^2(m^2 + m+1) + 2b(m^2+m+1) + (1+m)$ are constants.

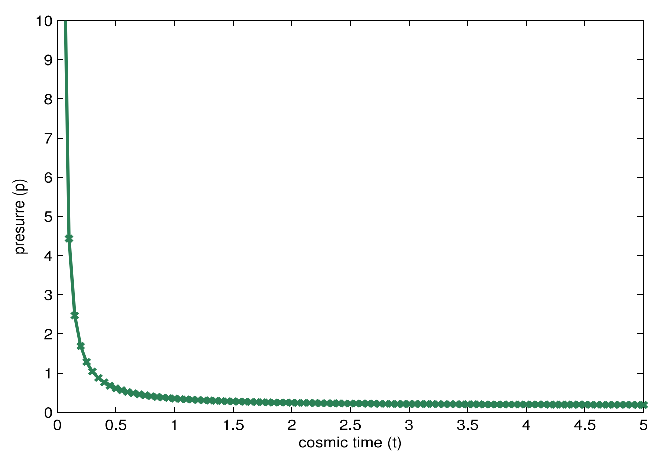

Figure 7.4: The plot of pressure p vs. cosmic time t for m=1.5, b=1.5, n=0.5 and λ=0.05

The scalar curvature R for the model is given by

$$R = \frac{9(n+t)^2}{2t^2(m+2)^2}\{(1+mb-b)^2 + (1+m-b-2mb)^2 + b^2(m+2)^2 + (1+mb-b)$$

$$(1+m-b-2mb) + b(m+2)(2-mb-2b+m)\} - \frac{3n}{t^2} - (m^2+m+1)(t^n e^t)^{\frac{-3(1+mb-b)}{(m+2)}} \quad (7.23)$$

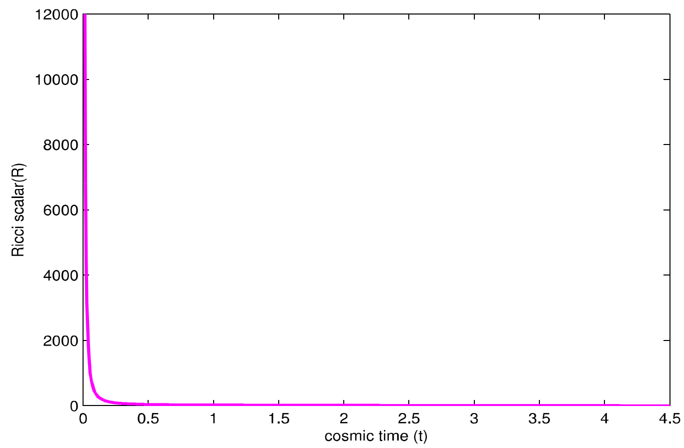

Figure 7.5: The plot of Ricci scaler R vs. cosmic time t for m=1.5, b=1.5, n=0.5 and λ=0.05

From the above results, we observe that the spatial volume is zero at $t = 0$ and it increases with the increase of t. The expansion scalar is infinite at $t = 0$. These show that the universe starts evolving with zero volume at $t = 0$ and expands with cosmic time t. All the three directional Hubble's parameters and the average Hubble parameter diverge at $t = 0$. These indicate that the model has a point-type singularity at $t = 0$. From Figures (7.2), (7.3) and (7.4), we observe that the physical parameters σ, ρ and p diverge at $t = 0$. As $t \to \infty$, the scale factors and volume become infinite where ρ and p approach to zero and expansion scalar θ and shear scalar σ obtain constant value. Since the anisotropy parameter A_m is constant throughout the passage of time, the model is anisotropic for all time. We also see from fig.(7.5), scaler curvature R is positive throughout the whole evolution of the universe and $R \to 0$ as $t \to \infty$ and $R \to \infty$ when $t \to 0$ showing initial singularity at $t = 0$. It is interesting to note that for the model with $n = 0$, we get $q = -1$, indicating that the universe is accelerating.

For the physical acceptability of the solutions, firstly it is required that the velocity of the sound $v_s = \frac{dp}{d\rho}$ should be less than velocity of light c. As we are working in the gravitational units with unit speed of light, the velocity of sound must exist within the range $0 \leq v_s \leq 1$. Here the speed of sound is obtained as

$$v_s = \frac{P(t)}{Q(t)} \tag{7.24}$$

where

$$P(t) = 18(8\pi A_1 + 3\lambda A_1 - \lambda A_2)(n^2 + nt) + (16n\pi + 6n\lambda)(m^2 b + mb - m^2 - 3m - 2b - 2)$$

$$+12(nt^2 + t^3)(m+2)(1+mb-b)\{m(8\pi + 2\lambda) - (m^2+1)\lambda\}(t^n e^t)^{\frac{-3(1+mb-b)}{(m+2)}} \tag{7.25}$$

and

$$Q(t) = 18(\lambda A_1 - 8\pi A_2 - 3\lambda A_2)(n^2 + nt) + 12n\lambda(m^2 b + mb - m^2 - 3m - 2b - 2)$$

$$+12(nt^2+t^3)(m+2)(1+mb-b)\{m(8\pi+3\lambda)(m^2+m+1) - m\lambda\}(t^n e^t)^{\frac{-3(1+mb-b)}{(m+2)}}. \tag{7.26}$$

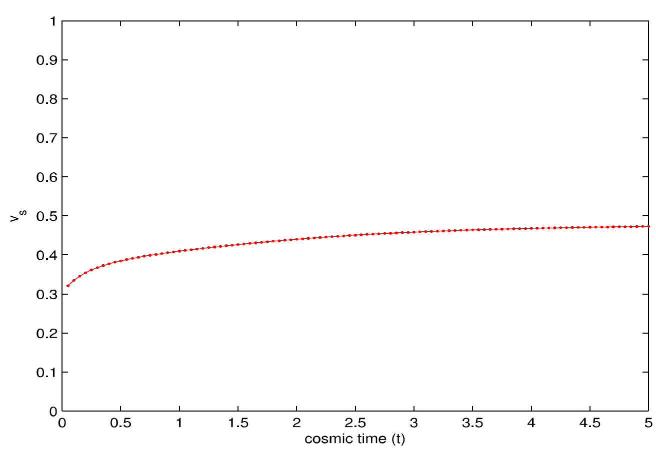

Figure 7.6: Plot of sound velocity v_s vs. t for m=1.5, b=1.5, n=0.5 and λ=0.05

It is clear from fig. (7.6) that $v_s < 1$ throughout the evolution of the universe.

Secondly, the weak energy (WEC), dominant energy (DEC) and strong energy (SEC) conditions: (i) $\rho \geq 0$, and $\rho + p \geq 0$, (ii) $\rho \geq 0$ and $\rho - p \geq 0$ and (iii) $\rho + 3p \geq 0$ and $\rho + p \geq 0$ should be satisfied identically.

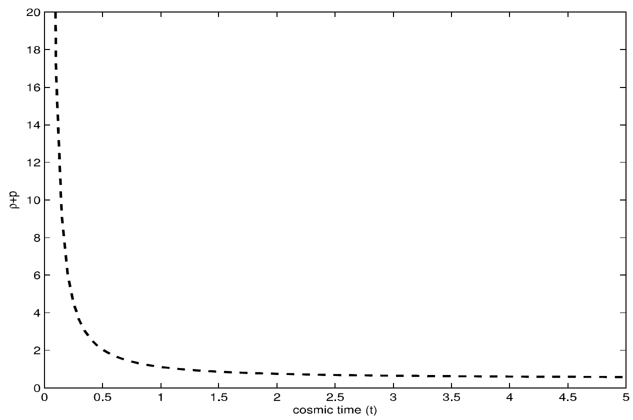

Figure 7.7: Plot of WEC vs. time t for m=1.5, b=1.5, n=0.5 and λ=0.05

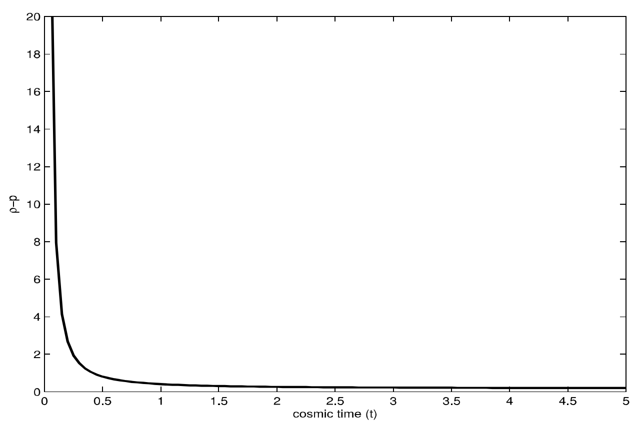

Figure 7.8: Plot of DEC vs. time t for m=1.5, b=1.5, n=0.5 and λ=0.05

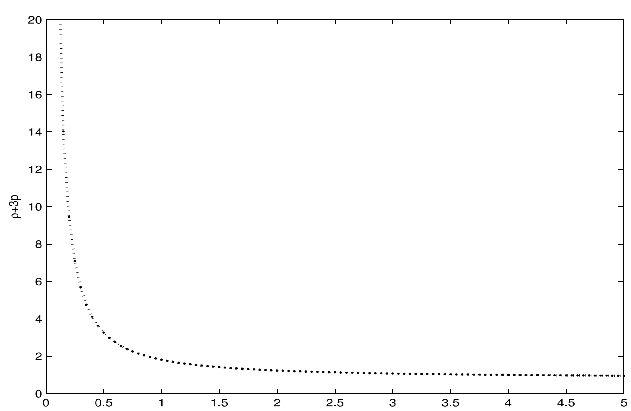

Figure 7.9: Plot of SEC vs. time t for m=1.5, b=1.5, n=0.5 and λ=0.05

From figures (7.3), (7.7) , (7.8) and (7.9), it can be seen that these energy conditions are identically satisfied throughout the evolution of universe. From the above discussion we find that our model is physically consistent with the present day observations.

7.5 Stability of the model

We now check the stability of the solution by metric perturbation method (Chen and Kao (2001)) by the same procedure as done in chapter 5, Sec.(5.5).

The background volume scalar V_B of the model (7.13) is given by

$$V_B = t^{\frac{3n}{2}} e^{\frac{3t}{2}}. \tag{7.27}$$

Substituting the value of V_B in Eq.(5.47) and integrating, we obtain

$$\delta b_i = c_i t^{\frac{-3n}{4}} e^{\frac{-3t}{4}} \quad \text{Wittaker}M\left(\frac{-3n}{4}, \frac{-3n}{4} + \frac{1}{2}, \frac{3t}{2}\right) \tag{7.28}$$

where c_i is a constant of integration. Therefore, the actual fluctuations, for each expansion factor $\delta a_i = a_{B_i}\delta b_i$, are given by

$$\delta a_i = c_i t^{\frac{-3n}{4}} e^{\frac{-3t}{4}} \quad \text{Wittaker}M\left(\frac{-3n}{4}, \frac{-3n}{4} + \frac{1}{2}, \frac{3t}{2}\right). \tag{7.29}$$

From Eq.(7.29), we observe that for $n \geq 1$, δa_i approaches zero. Consequently, the background solution is stable against the perturbation of the graviton field.

7.6 Conclusions

In this chapter, we have studied a general spatially homogeneous and anisotropic Bianchi space-time model in $f(R,T)$ theory of gravity in the presence of a perfect fluid source having initial singularity at $t = 0$. Einstein's field equations have been solved by choosing the average scale factor $a(t) = \sqrt{t^n e^t}$, which yields a time-dependent deceleration parameter. The derived model represents expanding, shearing and non-rotating universe which does not tend isotropy for all large time . We have discussed the physical and geometrical behaviors of the cosmological model. The variation of the physical parameters have been shown graphically. It is shown that model starts expanding from a decelerating phase to an accelerating phase. By cosmological perturbation method, we have shown that our model is stable. Also, the cosmological model is physically acceptable in concordance with the fulfillment of energy conditions WEC, DEC and SEC. The cosmological solution presented in this chapter may be useful for better understanding the characteristics in the evolution of the universe within the framework of $f(R, T)$ theory of gravitation.