6 Anisotropic Bianchi Type-III Cosmological Models in $f(R,T)$ Gravity Theory

6.1 Introduction

Over the last decade, the most significant progress in astrophysics and cosmology is the observational evidence that the present universe is undergoing a phase of accelerated expansion. This late time accelerated expansion of the universe has been confirmed by high redshift supernovae experiments (Riess et al.(1998), Perlmutter et al.(1999) and Bennett et al.(2003). Also, observations such as cosmic background radiation (Spergel et al. 2003) and large scale structure (Tegmark (2004)) provide an indirect evidence for late time accelerated expansion of the universe. Currently there are two different approaches to address the cosmic acceleration issue. One approach is to introduce various scalar fields of matter in Einstein gravity such as quintessence, phantom fields, tachyon field, Chaplygin gas etc and also cosmic fluids with anisotropic equation of state (Akarsu et al.(2010)). The other approach is based on modification of the Einstein-Hilbert action to get alternative theories

The contents of this chapter have been Published in **Prespacetime Journal 5(8), 753-757 (2014) and Indian J Physics 87(12), 1283-1287 (2013).**

of gravity such as $f(R)$ gravity (Nojori and Odintsov (2007)), $f(T)$ gravity (Ferraro and Fiorini (2007)), Gauss-Bonnet gravity (Carroll et al.(2005)). Harko et al.(2011) have introduced another extension of GR, known as $f(R,T)$ modified theory of gravity, where the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar R and the trace T of the energy-momentum tensor. The authors suggested that the coupling of matter and geometry leads to a model which depends on a source term representing the variation of the energy-momentum tensor with respect to the metric. This theory has been recently introduced as modifications of Einstein's theory possessing some interesting solutions which are relevant in cosmology and astrophysics. Adhav (2012) have investigated LRS Bianchi type-I model in $f(R,T)$ theory of gravity. Houndjo et al.(2013) investigated $f(R,T)$ gravity models to reproduce the four known finite-time future singularities. Sharif and Zubair (2012) considered two forms of the energy-momentum tensor of dark components and demonstrated that the equilibrium description of thermodynamics can not be achieved at the apparent horizon of Friedmann-Robertson-Walker (FRW) universe in $f(R,T)$ gravity. Alvarenga et al. (2013) tested some $f(R,T)$ gravity models through energy conditions. Pasqua et al. (2013) studied a particular model $f(R,T) = \mu R + \nu T$ which describes a quintessence-like behavior and exhibits transition from decelerated to accelerated phase.

FRW models, being spatially homogeneous and isotropic in nature, are best fit for the representation of the large scale structure of the present universe. however, it is believed that the early universe may not have been exactly uniform. Thus, the models with anisotropic background are the most suitable to describe the early stages of the universe. Bianchi type models are among the simplest models with anisotropic background. Lorenz-Petzold (1982) have studied exact Bianchi type-III solutions in the presence of electromagnetic field. Shri Ram (1989) has presented some analytic solutions to Einstein's field equations with perfect fluid in Bianchi type III space-time. Singh et al.(1991) studied some Bianchi type-III cosmological models in Saez Ballester theory of gravitation. Shri Ram and Singh (1992) investigated cosmological models of Bianchi type-III in presence of stiff matter in Lyra geometry. Tikekar and Patel (1992) presented Bianchi type-III cosmological models of massive string in the absence and presence of magnetic field. Singh and Shri Ram (1997) developed a technique to generate new exact Bianchi type-III cosmological solutions of massive string in the presence of magnetic field. Upadhaya (2008) explored some magnetized Bianchi type-III massive string cosmological model in general relativity. Adhav et al.(2009) obtained an exact solution the vacuum Brans-Dicke field equations for the metric tensor of spatially homogeneous anisotropic Bianchi type-III model. Shamir (2011) discussed the plane symmetric vacuum Bianchi type-III cosmology in $f(R)$ gravity. Reddy et al.(2013) have obtained a dark energy model with the equation of state parameter in $f(R,T)$ gravity in Bianchi type-III space time in presence of a perfect fluid source.

In this chapter, we study spatially homogeneous and anisotropic Bianchi type-III cosmological models with perfect fluid within the framework of $f(R,T)$ theory of gravity. In Sec.(6.2), we present the metric and field equations. We obtain exact solutions to the field equations by two methods in Sec.(6.3). Finally, some concluding remarks are given in Sec.(6.4).

6.2 The Metric and Field Equations

We consider the spatially homogeneous and anisotropic Bianchi type-III metric in the form

$$ds^2 = dt^2 - A^2(t)dx^2 - e^{-2mx}B^2(t)dy^2 - C^2(t)dz^2 \tag{6.1}$$

where A, B and C are cosmic scale factors and m is a positive constant.

The field equations in $f(R,T)$ theory of gravity for function $f(R,T) = R + 2f(T)$, when the matter source is a perfect fluid, are given by (Harko et al.(2011))

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi T_{ij} + 2f'(T)T_{ij} + [2pf'(T) + f(T)]g_{ij} \tag{6.2}$$

For a perfect fluid source energy momentum tensor T_{ij} is given by (1.26).

In comoving coordinates, the field equations (6.2) with particular choice of the function $f(T) = \lambda T$, where λ is a constant, for the metric (6.1) are obtained as follows:

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = (8\pi + 3\lambda)p - \lambda\rho, \tag{6.3}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = (8\pi + 3\lambda)p - \lambda\rho, \tag{6.4}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{m^2}{A^2} = (8\pi + 3\lambda)p - \lambda\rho, \tag{6.5}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{m^2}{A^2} = -(8\pi + 3\lambda)\rho + \lambda p, \tag{6.6}$$

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = 0. \tag{6.7}$$

Integration of Eq.(6.7) provides $B = c_1 A$, where c_1 is a constant of integration. Without loss of generality, we take $c_1 = 1$, so that

$$B = A. \tag{6.8}$$

Using Eq.(6.8), the field equations (6.3)-(6.6) reduce to

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}=(8\pi+3\lambda)p-\lambda\rho, \tag{6.9}$$

$$\frac{2\ddot{A}}{A}+\frac{\dot{A}^2}{A}-\frac{m^2}{A^2}=(8\pi+3\lambda)p-\lambda\rho, \tag{6.10}$$

$$\frac{\dot{A}^2}{A^2}+\frac{2\dot{A}\dot{C}}{AC}-\frac{m^2}{A^2}=-(8\pi+3\lambda)\rho+\lambda p. \tag{6.11}$$

For the metric (6.1) dynamical parameters are given by

$$V=a^3=A^2C. \tag{6.12}$$

The expansion scalar θ and shear scalar σ^2 are given by

$$\theta=3H=\frac{2\dot{A}}{A}+\frac{\dot{C}}{C}, \tag{6.13}$$

$$\sigma^2=\frac{1}{3}\left(\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)^2. \tag{6.14}$$

For any physically relevant cosmological model, the Hubble parameter and the deceleration parameter are the most important observational quantities. Berman (1983) and Berman and Gomide (1988) proposed a law of variation of Hubble parameter in FRW model that yields a constant value of the deceleration parameter leading to viable forms of the scale factor, one of power law form and other of exponential form. Recently, several author (Singh and Kumar(2006) , Singh et al (2008), Singh (2009a, 2009b), Singh and Beesham (2010), Sharif and Zubair (2012a, 2012b)) have generalized these assumptions in anisotropic model and have taken the constant value of deceleration parameter and derived decelerating/accelerating models of expanding universe.

6.3 Solutions of Field Equations

In this section, we obtain exact solutions of the Eqs.(6.9)-(6.11) for physically realistic cosmological models.

6.3.1 Solution by Special Law of Variation of Hubble Parameter

Since $f(R,T)$ gravity theory is concerned about the accelerated phase of the universe, we consider q as a negative constant. Then solution of Eq.(1.34) can be written as

$$a = (\alpha t + \beta)^{\frac{1}{(1+q)}} \tag{6.15}$$

where $\alpha \neq 0$ and β are constants of integration. This equation implies the condition of accelerated expansion as $1 + q > 0$ In order to solve the non-linear differential equations (6.9)-(6.11), Reddy et al. (2012) have assumed that the expansion scalar in the model is proportional to shear scalar, which implies $A = C^n, n \neq 1$. Instead of assuming the above condition, here we assume the relationship between the scale factor C and the spatial volume V of the form $C = V^b$ (Chaubey and Shukla (2012)), where b is a constant. Using this form of C in Eq.(6.12) and Eq.(6.15), we obtain the solutions for A and C as follows:

$$A = (\alpha t + \beta)^{\frac{3(1-b)}{2(1+q)}}, \tag{6.16}$$

$$C = (\alpha t + \beta)^{\frac{3b}{(1+q)}}. \tag{6.17}$$

Hence, by a suitable choice of the coordinates and constants, the metric of the solutions (6.16) and (6.17) can be written in the form

$$ds^2 = dt^2 - t^{\frac{3(1-b)}{(1+q)}}\{dx^2 + e^{-2mx}dy^2\} - t^{\frac{6b}{(1+q)}}dz^2. \tag{6.18}$$

For the model (6.18), the kinematical parameters are obtained as follows:

$$\text{Volume} : V = t^{\frac{3}{(1+q)}}, \tag{6.19}$$

$$\text{Expansion scalar} : \theta = 3H = \frac{3}{(1+q)t}, \tag{6.20}$$

$$\text{Shear scalar} : \sigma^2 = \frac{3}{4}\left(\frac{1-3b}{1+q}\right)^2 \frac{1}{t^2}, \tag{6.21}$$

$$\text{Anisotropy parameter} : A_m = \frac{1}{2}(1-3b)^2. \tag{6.22}$$

The energy density and pressure for the model (6.18) are obtained as

$$\rho = \frac{1}{\lambda^2 - (8\pi + 3\lambda)^2}\left[\frac{3(1-b)}{4(1+q)^2t^2}\{24\pi + 4\lambda + 72\pi b + 36\lambda b + 4\lambda q\} - \frac{(8\pi + 2\lambda)m^2}{t^{\frac{3(1-b)}{(1+q)}}}\right], \tag{6.23}$$

$$p = \frac{1}{(8\pi + 3\lambda)^2 - \lambda^2}\left[\frac{3(1-b)}{4(1+q)^2t^2}\{40\pi - 72\pi b - 32\pi q + 12\lambda - 12\lambda q - 24\lambda b\} - \frac{(8\pi + 2\lambda)m^2}{t^{\frac{3(1-b)}{(1+q)}}}\right] \tag{6.24}$$

For a physically realistic model we take $b < 1$. We observe that the spatial volume V is zero at $t = 0$. Therefore, the model starts evolving with a big-bang type singularity at $t = 0$. At this epoch θ, H, σ, ρ and p all have infinite values. These parameters are decreasing functions of time. As $t \to \infty$, the physical and kinematical parameters all tend to zero while scalar volume increases with time, which shows the late time acceleration of the universe. Since $\lim_{t\to\infty} \frac{\sigma}{\theta} \neq 0$, the model does not approach isotropy throughout the evolution of the universe.

6.3.2 Solutions by Generating Techniques

In this section, we derive algorithms for generating new solutions of field equations given by Eqs.(6.9)-(6.11) for Bianchi type-III space time in $f(R,T)$ gravity theory by using the procedure similar to that of Hajj-Boutros (1986), Ram (1989), Singh and Ram (1995).

From Eqs.(6.9) and (6.10), we obtain

$$\frac{\ddot{A}}{A}+\frac{\dot{A}^2}{A^2}-\frac{\ddot{C}}{C}-\frac{\dot{A}\dot{C}}{AC}-\frac{m^2}{A^2}=0. \tag{6.25}$$

To treat this equation, we introduce the functions R and S defined by

$$R=\frac{\dot{A}}{A}, \qquad S=\frac{\dot{C}}{C}. \tag{6.26}$$

Using Eq.(6.26)) into Eq. (6.25), we get

$$\dot{R}+2R^2-\dot{S}-S^2-RS-\frac{m^2}{A^2}=0. \tag{6.27}$$

Eq.(6.27) can be regarded as A Ricatti equation in S or R. If we regard Eq.(6.27) as a Ricatti equation in S, we linearize it by the change of function

$$S=S_0+\frac{1}{X} \tag{6.28}$$

where S_0 is a particular solution of Eq.(6.27), so that

$$\dot{S}_0+S_0^2+RS_0-\dot{R}-2R^2+\frac{m^2}{A^2}=0. \tag{6.29}$$

Substituting Eq.(6.28) in Eq.(6.27), then using Eq.(6.29) and integrating the resulting differential equation, we obtain

$$X=AC_0^2\left\{\int\frac{dt}{AC_0^2}+k_1\right\} \tag{6.30}$$

where k_1 is an integration constant. Finally, from Eqs.(6.27) and (6.30), we obtain the new scale factor C given by

$$C = C_0 \exp\left[\int \frac{dt}{AC_0^2\left\{\int \frac{dt}{AC_0^2} + k_1\right\}} + k_2\right] \tag{6.31}$$

where k_2 is another constant of integration. Thus, for the couple $[A, C_0]$, Eq.(6.31) allows us to obtain the new couple $[A, C]$ where A stays invariable.

If we treat Eq.(6.27) as a Ricatti equation in R, then using the same procedure as above, we obtain the formula

$$A = A_0 \exp\left[\int \frac{dt}{\frac{2A_0^4}{C}\left\{\int \frac{C}{A_0^4}dt + k_3\right\}} + k_4\right] \tag{6.32}$$

to generate a new couple $[A, C]$ starting from the known couple $[A_0, C]$ where C stays invariable, and k_3 and k_4 are integration constants.

Reddy et al. (2012) have presented a solutions of field equations given by Eqs.(6.9)-(6.11) representing Bianchi type-III cosmological model with perfect fluid source in $f(R,T)$ gravity theory with the help of a special law of variation of Hubble's parameter, proposed by Berman (1983), which yields a constant value of deceleration parameter. It may be noted that most of the well known models in general relativity and scalar-tensor theories of gravitation including inflationary models have constant deceleration parameter. Assuming q as a negative constant, since $f(R,T)$ gravity theory is about accelerated expansion of the universe, metric of their solutions with negative deceleration parameter q is given by

$$ds^2 = dt^2 - t^{\frac{6n}{(1+q)(2n+1)}}\left[dx^2 + e^{-2mx}dy^2\right] - t^{\frac{6}{(1+q)(2n+1)}}dz^2. \tag{6.33}$$

This model is physically significant for discussion on the early stages of evolution of the universe.

We now use metric (6.33) to generate new solutions of field equations by applying the generating algorithms given by Eqs. (6.31) and (6.32).

6.3.2.1 Model I

To apply Eq.(6.31), we take

$$A = t^{\frac{3n}{(1+q)(2n+1)}}, \qquad C_0 = t^{\frac{3}{(1+q)(2n+1)}}. \tag{6.34}$$

Then performing integrations in Eq.(6.31), we obtain

$$C = Kt^{\frac{(2n+1)q-(n+2)}{(1+q)(2n+1)}} \tag{6.35}$$

where K is $\log k_2$ and k_1 is set to zero. Therefore, metric of the new solution can be written as

$$ds^2 = dt^2 - t^{\frac{6n}{(1+q)(2n+1)}}[dx^2 + e^{-2mx}dy^2] - t^{\frac{2(2n+1)q-2(n+2)}{(1+q)(2n+1)}}dz^2. \tag{6.36}$$

The metric given by Eq.(6.36) represents a Bianchi type-III perfect fluid model in $f(R,T)$ gravity theory with following physical and kinematical prameters:
Spatial volume V of the model (6.36) is given by

$$V = t^{\frac{\{(2n+1)(1+q)+3(n-1)\}}{(1+q)(2n+1)}}. \tag{6.37}$$

We observe that spatial volume is zero at $t = 0$ if $n > 1$. Clearly the volume increases as time increases and ultimately becomes infinite at late time. Therefore for physical reality of the model (31), we must have $n > 1$. The deceleration parameter q_1 in this model is given by

$$q_1 = -1 + \frac{3(2n+1)(1+q)}{3(n-1)+(2n+1)(1+q)}. \tag{6.38}$$

The second term on right hand side is always greater than unity since it fulfills the requirements $n > 1$ and $1 + q > 0$. This model decelerates in a standard way which is not in accordance with present day scenario of accelerating universe. It may be noted that Bianchi models represent cosmos in its early stage of evolution. However, in spite of fact that the universe, in this case, decelerates in a standard way, it should accelerate in finite time due to cosmic recollapse where the universe in turns inflates "decelerates and then accelerates".

Hubble parameter H, expansion scalar θ , shear scalar σ and anisotropic parameter A_m are given by

$$\theta = 3H = \frac{(2n+1)(1+q)+3(n-1)}{(1+q)(2n+1)}\frac{1}{t}, \tag{6.39}$$

$$\sigma^2 = \frac{1}{3}\left(\frac{2-q}{1+q}\right)^2\frac{1}{t^2}, \tag{6.40}$$

$$A_m = \frac{1}{3}\left[\frac{18n^2+\{(2n+1)q-(n+2)\}^2-3\{6n+(2n+1)q-(n+2)\}^2}{\{6n+(2n+1)q-(n+2)\}^2}\right] = constant. \tag{6.41}$$

Thus, the model is expanding, shearing and anisotropic for all time.

The physical parameter, energy density ρ and pressure p have values given by

$$\rho = \frac{1}{\lambda^2-(8\pi+3\lambda)^2}\left[\frac{3n}{(1+q)^2(2n+1)^2t^2}\{(8n\pi-2n\lambda+14\lambda+32\pi) +(32n\pi+16n\lambda+16\pi+8\lambda)q\} - \frac{(8\pi+2\lambda)m^2}{t^{\frac{6n}{(1+q)(2n+1)}}}\right], \tag{6.42}$$

$$p = \frac{1}{(8\pi+3\lambda)^2-\lambda^2}\left[\frac{3n}{(1+q)^2(2n+1)^2t^2}\{(40n\pi+14n\lambda-16\pi-10\lambda) -(32n\pi+16n\lambda+16\pi+8\lambda)q\} - \frac{(8\pi+2\lambda)m^2}{t^{\frac{6n}{(1+q)(2n+1)}}}\right]. \tag{6.43}$$

We observe that this model has a big-bang type singularity at $t = 0$ since $n > 1$. The kinematical parameters H, θ and σ diverge at initial epoch while they vanish for large values of t. Energy density and pressure diverge at initial epoch and they tend to zero for large time. Also, $\lim_{t\to\infty}\frac{\sigma^2}{\theta^2}=\frac{1}{3}\frac{(2-q)^2(2n+1)^2}{(2n+1)(1+q)+3(n-1)}\neq 0$, the model does not tend to isotropy for large t .

6.3.2.2 Model II

We now apply Eq.(6.34) to generate new couple $[A, C]$ starting with $[A_0, C]$ where

$$A_0 = t^{\frac{3n}{(1+q)(2n+1)}}, \quad C = t^{\frac{3}{(1+q)(2n+1)}}. \tag{6.44}$$

Then, performing integrations of Eq. (6.34) and putting $k_3 = 0$ and $M = \log k_4$, we obtain

$$A = Mt^{\frac{(2n+1)q+4(1-n)}{2(1+q)(2n+1)}}. \tag{6.45}$$

With a suitable choice of the coordinates and $M = 1$, metric of the solution can be written in following form

$$ds^2 = dt^2 - t^{\frac{(2n+1)q+4(1-n)}{(1+q)(2n+1)}}\{dx^2 + e^{-2mx}dy^2\} - t^{\frac{6}{(1+q)(2n+1)}}dz^2. \tag{6.46}$$

For the model (6.46), spatial volume V is given by

$$V = t^{\frac{(2n+1)(q-2)+9}{(1+q)(2n+1)}} \tag{6.47}$$

which is zero at $t = 0$ if $0 < n < 1$. The volume scalar is expanding function of time and ultimately becomes infinite with late time.

Deceleration parameter q_2 of this model is given by

$$q_2 = -1 + \frac{3(2n+1)(1+q)}{(2n+1)(1+q) - 6(n-1)} \tag{6.48}$$

where $0 < n < 1$ and $1+q > 0$. The second term on right hand side is always greater than unity since it fulfills the requirements $0 < n < 1$ and $1 + q > 0$. Therefore, this model is expanding with decelerating expansion rate.

The expressions for kinematical parameters H, θ, σ and A_m are found as

$$\theta = 3H = \frac{(2n+1)q - (4n-7)}{(1+q)(2n+1)}\frac{1}{t}, \tag{6.49}$$

$$\sigma^2 = \frac{1}{12}\left(\frac{2-q}{1+q}\right)^2 \frac{1}{t^2}, \tag{6.50}$$

$$A_m = \frac{1}{3}\left[\frac{\{(2n+1)q+4(1-n)\}^2 + 18 - 6\{(2n+1)q+4(1-n)+3\}^2}{2\{(2n+1)q+4(1-n)+3\}^2}\right] = Constant. \tag{6.51}$$

For model (6.46), the energy density and pressure are obtained as

$$\rho = \frac{1}{\lambda^2 - (8\pi+3\lambda)^2}\left[\frac{4(1-n)+(2n+1)q}{(1+q)^2(2n+1)^2t^2}\{(32\pi+10\lambda+2n\lambda-8\pi n) + q(4n\pi+2n\lambda+2\pi+\lambda)\} - \frac{(8\pi+2\lambda)m^2}{t^{\frac{4(1-n)+(2n+1)q}{(1+q)(2n+1)}}}\right], \tag{6.52}$$

$$p = \frac{1}{(8\pi+3\lambda)^2 - \lambda^2}\left[\frac{4(1-n)+(2n+1)q}{(1+q)^2(2n+1)^2t^2}\{(16\pi+2\lambda-14n\lambda-40n\pi) + q(4n\pi-\lambda-2n\lambda-2\pi)\} - \frac{(8\pi+2\lambda)m^2}{t^{\frac{4(1-n)+(2n+1)q}{(1+q)(2n+1)}}}\right]. \tag{6.53}$$

Also, $lim_{t\to\infty}\frac{\sigma^2}{\theta^2} = \frac{1}{12}\frac{(2-q)^2(2n+1)^2}{(2n+1)q-(4n-7)} \neq 0$. This model has a big-bang type singularity at $t=0$ for $0<n<1$. Kinematical and physical behaviors of the model are same as that of model I.

6.4 Conclusions

In this chapter we have studied an anisotropic Bianchi type - III cosmological model filled with a perfect fluid in $f(R,T)$ theory of gravity. In Sec.(6.3.1), we have obtained the solutions of field equations by applying the special law of variation of

Hubble parameter that yields a negative constant value of the deceleration parameter. It is observed that the model has a big-bang singularity at $t = 0$ and shows a late time accelerated expansion of the universe for large time. All the kinematical and physical parameters diverge at the initial singularity and ultimately tend to zero for large time. The anisotropy in the model of the universe is maintained throughout the evolution of the universe. This model and its properties throw a better understanding of the accelerated expansion of the universe.

In Sec.(6.3.2), we have derived new algorithms for generating new solutions of the field equations for Bianchi type III space-time filled with perfect fluid in $f(R, T)$ gravity theory. Starting with the solution due to Reddy et al. (2012), we have presented two new classes of cosmological models which have point-like singularities at initial time $t = 0$. At this initial epoch, all the physical parameters ρ, p, θ, σ and H diverge and are decreasing functions of time, which ultimately approach to zero for large time. Thus, the models essentially give empty space for large time. Since anisotropic parameter A_m is constant, anisotropy in the models is maintained throughout the passage of time.