Chapter 6

Watson wavelet transform: Convolution product and two wavelet multipliers

6.1 Introduction

The wavelet transform is of great importance for the analysis of non-stationary signals and provides the information of time-frequency representation at a time. Many researchers exploited the theory of the wavelet transform and explored their research works in the areas of mathematical sciences and engineering. The concept of the wavelet transform is heavily depended on the theory of convolution and each integral transform has its own convolution with its rich calculus. Using the theory of convolutions of different integral transforms, many problems of the wavelet transform have been solved by many mathematicians. Using the convolution theory of the Fourier transform, Wong [27] discussed the boundedness of wavelet multipliers and signals.

In 2000, Du and Wong [17] obtained the traces of localization operator on a separable complex Hilbert space and got many important results. In 2001, Du and Wong [18] observed the trace formula for wavelet multipliers as a bounded linear operator in the trace class from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ and were able to compute the trace of n-dimensional Landau-Pollak-Slepian operators. Wong et al. [77] studied the trace formula for two-wavelet multipliers. In 2002, Wong [78] proved an L^p-boundedness result for the localization operator associated to left regular representations of locally compact and Hausdorff groups and gave an application of the wavelet multiplier. In 2003, Wong et al. [79] found the resolution of the identity formula for a localization operator with two admissible wavelets on a separable and complex Hilbert space and the traces of these operators. Wong et al. [34], examined the boundedness and compactness of the localization operator on various functional spaces in terms of the wavelet multiplier. In the same year, Pinsky [52] developed the Heuristic treatment of the wavelet transform and its inversion formula by exploiting the theory of the Fourier transform.

For the consideration of the Hankel transform, the theory of Hankel convolution is introduced by Haimo [25], Hirschmann [28] and many others. Using this theory, Pathak and Dixit [44] introduced the Bessel wavelet transform and studied many properties. Motivated from the results of [44, 49], many authors extensively studied the characterizations of Bessel wavelet transform on certain functional spaces and applied this theory in Sobolev spaces and other problems of mathematics. Taking the Hankel transform theory, Pathak et al. [51] considered the Bessel wavelet convolution product and its properties. Upadhyay et al. [68] found the relation between the Bessel wavelet convolution product and Hankel convolution.

Motivated from the above results and concepts, our main objective in this chapter is to study the relation between the Watson wavelet convolution product and Watson convolution by exploiting the theory of the Watson transform. Later on, we shall also find the relation between the Watson wavelet convolution product and Watson two wavelet multipliers involving the Watson transform. Using the same technique the relation between the trace class of Watson two wavelet multipliers and Watson wavelet convolution product will be established.

The present chapter is organized by the following way:
Section 6.1 is introductory, which gives the brief history and motivations of the entire research work. In Section 6.2, the Watson wavelet convolution product is formally defined, after that we found the relation between the Watson wavelet convolution product and Watson convolution and its various properties. In Section 6.3, we use the results of Pinsky [52] and found the Heuristic treatment of the Watson wavelet transform and its inversion formula by exploiting the theory of the Watson transform. In Section 6.4, taking Watson wavelet transform, two-wavelet multipliers are introduced and their various properties studied. The relation between the Watson wavelet convolution product and two-wavelet multipliers are obtained.

With the help of [67], we give the definition of the Watson wavelet for $\psi \in L^p(0,\infty)$

$$\psi_{b,a}(x) = D_a \tau_b \psi(x) = D_a \psi(b,x) = \frac{1}{a}\psi\Big(\frac{b}{a},\frac{x}{a}\Big). \tag{6.1.1}$$

From (1.4.10), we have

$$\psi_{b,a}(x) = \frac{1}{a}\int_0^\infty w\Big(\frac{b}{a},\frac{x}{a},z\Big)\psi(z)dz, \tag{6.1.2}$$

where $b \geq 0$ and $a > 0$.

Using (6.1.2), the Watson wavelet transform is defined by

$$\begin{aligned} W(b,a) = (W_\psi \phi)(b,a) &= \langle \phi(t), \psi_{b,a}(t) \rangle && (6.1.3) \\ &= \int_0^\infty \phi(t) \overline{\psi_{b,a}(t)} dt \\ &= \int_0^\infty \phi(t) \Big(\frac{1}{a} \int_0^\infty w\left(\frac{b}{a}, \frac{t}{a}, z\right) \overline{\psi(z)} dz \Big) dt \\ &= \frac{1}{a} \int_0^\infty \int_0^\infty \phi(t) w\left(\frac{b}{a}, \frac{t}{a}, z\right) \overline{\psi(z)} dz dt, && (6.1.4) \end{aligned}$$

provided the integral is convergent.

With the help of [67, p. 644], we see that above integral is convergent for $\phi \in L^p(0,\infty)$ and $\psi \in L^q(0,\infty)$

$$|(W_\psi \phi)(b,a)| \leq ||\phi||_p ||\bar{\psi}_a||_q < \infty.$$

If ϕ and ψ are in $L^1(0,\infty) \cap L^2(0,\infty)$, then the following Parseval formula holds:

$$\int_0^\infty (W\phi)(t)(W\psi)(t) dt = \int_0^\infty \phi(x)\psi(x) dx. \tag{6.1.5}$$

From [67, p. 643], the continuous Watson wavelet transform of a function $\phi, \psi \in L^1(0,\infty) \cap L^2(0,\infty)$

$$(W_\psi \phi)(b,a) = \int_0^\infty k(wb)(W\phi)(w)\overline{(W\psi)(aw)} dw. \tag{6.1.6}$$

For $f \in L^2(0,\infty)$ and $g \in L^2(0,\infty)$, we state the Parseval formula of the Watson wavelet transform as

$$\int_0^\infty \int_0^\infty \frac{(W_\psi f)(b,a)\overline{(W_\psi g)(b,a)} da db}{a} = C_\psi \langle f, g \rangle, \tag{6.1.7}$$

where C_ψ satisfy the admissible condition for $\psi \in L^2(0,\infty)$, which is given below

$$C_\psi = \int_0^\infty \frac{|(W\psi)(\omega)|^2}{\omega} d\omega. \tag{6.1.8}$$

Remark: The Watson transform is generalization of Hankel transform. Exploiting the theory of the Hankel transform, Bessel wavelet convolution product was studied in the paper [44, 67, 68].

6.2 Watson wavelet convolution product

In this section, the Watson wavelet convolution product is introduced and its associated results are obtained by exploiting the theory of the Watson transform. We also find the relation between Watson wavelet convolution product and Watson convolution.

For finding the properties of Watson wavelet convolution product, we formally define

$$W_\psi(f \otimes g)(b,a) = (W_\psi f)(b,a)(W_\psi g)(b,a). \tag{6.2.1}$$

Theorem 6.2.1. *Let $f, g \in L^1(0,\infty) \cap L^2(0,\infty)$ and $(W\psi)(\omega) \neq 0$. Then the Watson wavelet convolution product can be written in the following form:*

$$(f \otimes g)(z) = \int_0^\infty (\tau_{z,a} f)(y) g(y) dy,$$

where

$$(\tau_{z,a} f)(y) = \int_0^\infty f(x) W_a(x,y,z) dx,$$

$$W_a(x,y,z)=\int_0^\infty\int_0^\infty k(y\xi)k(xt)(W\psi)(at)(W\psi)(a\xi)L_a(t,\xi,z)dtd\xi,$$

$$L_a(t,\xi,z)=\int_0^\infty k(yt)k(y\xi)Q_a(y,z)dy,$$

and

$$Q_a(y,z)=\int_0^\infty \frac{k(\omega z)k(\omega y)}{(W\psi)(a\omega)}d\omega.$$

Proof. From (6.1.6), we have

$$W[(W_\psi\phi)(b,a)](\omega)=\overline{(W\psi)}(a\omega)(W\phi)(\omega). \tag{6.2.2}$$

Putting $\phi=f\otimes g$, in the above expression (6.2.2)

$$W[W_\psi(f\otimes g)(b,a)](\omega)=\overline{(W\psi)}(a\omega)(W(f\otimes g))(\omega).$$

Thus

$$\overline{(W\psi)}(a\omega)(W(f\otimes g))(\omega)=W[W_\psi(f\otimes g)(b,a)](\omega).$$

Using (6.2.1), we have

$$\overline{(W\psi)}(a\omega)(W(f\otimes g))(\omega)=W[(W_\psi f)(b,a)(W_\psi g)(b,a)](\omega).$$

In view of (6.1.6), we get

$$\overline{(W\psi)}(a\omega)W(f\otimes g)(\omega)$$
$$=W\{W^{-1}[(Wf)(\omega)\overline{(W\psi)}(a\omega)](b)W^{-1}[(Wg)(\omega)\overline{(W\psi)}(a\omega)](b)\}. \tag{6.2.3}$$

By the definition of the Watson convolution, we get

$$\begin{aligned}
&\overline{(W\psi)}(a\omega)W(f\otimes g)(\omega)\\
&\qquad = W(W^{-1}\{(Wf)(\omega)(\overline{W\psi}(a\omega))\#(Wg)(\omega)(\overline{W\psi}(a\omega))\}(b))\\
&\qquad = (Wf)(\omega)(\overline{W\psi})(a\omega)\#(Wg)(\omega)(\overline{W\psi})(a\omega).
\end{aligned}$$

Let $F_a = (Wf)(\omega)\overline{(W\psi)}(a\omega)$ and $G_a = (Wg)(\omega)\overline{(W\psi)}(a\omega)$. Then we have

$$\begin{aligned}
\overline{(W\psi)}(a\omega)W(f\otimes g)(\omega) &= (F_a\#G_a)(\omega)\\
&= \int_0^\infty\int_0^\infty F_a(\eta)G_a(\xi)w(\omega,\xi,\eta)d\xi d\eta.
\end{aligned}$$

From (1.4.7), we write

$$\begin{aligned}
(W\psi)(a\omega)W(f\otimes g)(\omega) &= (F_a\#G_a)(\omega)\\
&= \int_0^\infty\int_0^\infty F_a(\eta)G_a(\xi)\left(\int_0^\infty k(\omega y)k(\xi y)k(\eta y)dy\right)d\xi d\eta.\\
&= \int_0^\infty\left(\int_0^\infty F_a(\eta)k(\eta y)d\eta\right)\left(\int_0^\infty G_a(\xi)k(\xi y)d\xi\right)k(\omega y)dy.\\
&= \int_0^\infty (WF_a)(y)(WG_a)(y)k(\omega y)dy.
\end{aligned}$$

Therefore

$$W(f\otimes g)(\omega) = \int_0^\infty \frac{1}{(W\psi)(a\omega)}(WF_a)(y)(WG_a)(y)k(\omega y)dy.$$

By the inversion formula of the Watson transform (1.4.2), we get

$$\begin{aligned}(f \otimes g)(z) &= \int_0^\infty \frac{k(\omega z)}{(W\psi)(a\omega)} \left(\int_0^\infty (WF_a)(y)(WG_a)(y)k(\omega y)dy \right) d\omega \\ &= \int_0^\infty (WF_a)(y)(WG_a)(y) \left(\int_0^\infty \frac{k(\omega z)k(\omega y)}{(W\psi)(a\omega)} d\omega \right) dy \\ &= \int_0^\infty (WF_a)(y)(WG_a)(y)Q_a(y,z)dy, \end{aligned} \tag{6.2.4}$$

where

$$Q_a(y,z) = \int_0^\infty \frac{k(\omega z)k(\omega y)}{(W\psi)(a\omega)} d\omega \quad \text{and} \quad (W\psi)(a\omega) \neq 0.$$

Therefore, by using value of F_a and G_a in (6.2.4), we find that

$$\begin{aligned}&(f \otimes g)(z) \\ &= \int_0^\infty \left(\int_0^\infty k(yt)(W\psi)(at)(Wf)(t)dt \right) \left(\int_0^\infty k(y\xi)(W\psi)(a\xi)(Wg)(\xi)d\xi \right) \\ &\qquad \times Q_a(y,z)dy \\ &= \int_0^\infty \int_0^\infty (W\psi)(at)(W\psi)(a\xi)(Wf)(t)(Wg)(\xi) \left(\int_0^\infty k(yt)k(y\xi)Q_a(y,z)dy \right) \\ &\qquad \times dtd\xi \\ &= \int_0^\infty \int_0^\infty (W\psi)(at)(W\psi)(a\xi)(Wf)(t)(Wg)(\xi)L_a(t,\xi,z)dtd\xi, \end{aligned}$$

where

$$L_a(t,\xi,z) = \int_0^\infty k(yt)k(y\xi)Q_a(y,z)dy.$$

Thus, we have

$$
\begin{aligned}
(f\otimes g)(z) &= \int_0^\infty\int_0^\infty (W\psi)(at)(W\psi)(a\xi)\left(\int_0^\infty k(xt)f(x)dx\right)\left(\int_0^\infty k(y\xi)g(y)dy\right)\\
&\qquad \times L_a(t,\xi,z)dtd\xi\\
&= \int_0^\infty\int_0^\infty f(x)g(y)\Big(\int_0^\infty\int_0^\infty k(y\xi)k(xt)(W\psi)(at)(W\psi)(a\xi)L_a(t,\xi,z)\\
&\qquad \times dtd\xi\Big)dxdy\\
&= \int_0^\infty\int_0^\infty f(x)g(y)W_a(x,y,z)dxdy, \qquad (6.2.5)
\end{aligned}
$$

where

$$
W_a(x,y,z) = \int_0^\infty\int_0^\infty k(y\xi)k(xt)(W\psi)(at)(W\psi)(a\xi)L_a(t,\xi,z)dtd\xi.
$$

Therefore, (6.2.5) yields the required result

$$
\begin{aligned}
(f\otimes g)(z) &= \int_0^\infty\int_0^\infty f(x)g(y)W_a(x,y,z)dxdy\\
&= \int_0^\infty (\tau_{z,a}f)(y)g(y)dy.
\end{aligned}
$$

□

Theorem 6.2.2. *Let $f, g \in L^1(0,\infty)\cap L^2(0,\infty)$. Then the Watson convolution product can be written in the following form:*

$$
\begin{aligned}
&E_\psi\cdot[W(f\otimes g)](\omega)\\
&\quad = \int_0^\infty\int_0^\infty (Wf)(\eta)(Wg)(\xi)w(\omega,\eta,\xi)\left(\int_0^\infty \overline{(W\psi)}(a\eta)\overline{(W\psi)}(a\xi)\frac{da}{a}\right)d\eta d\xi,
\end{aligned}
\qquad (6.2.6)
$$

where

$$E_\psi = \int_0^\infty \frac{\overline{(W\psi)}(a\omega)}{a} da.$$

Proof. From (6.1.6), we have

$$W[(W_\psi \phi)(b,a)](\omega) = \overline{(W\psi)}(a\omega)(W\phi)(\omega). \tag{6.2.7}$$

Putting $\phi = f \otimes g$, in the above expression

$$W[W_\psi(f \otimes g)(b,a)](\omega) = \overline{(W\psi)}(a\omega)(W(f \otimes g))(\omega). \tag{6.2.8}$$

Thus

$$\overline{(W\psi)}(a\omega)(W(f \otimes g))(\omega) = W[W_\psi(f \otimes g)(b,a)](\omega). \tag{6.2.9}$$

Using (6.2.1), we have

$$\overline{(W\psi)}(a\omega)(W(f \otimes g))(\omega) = W[(W_\psi f)(b,a)(W_\psi g)(b,a)](\omega). \tag{6.2.10}$$

In view of (6.1.6), we get

$$\begin{aligned}&\overline{(W\psi)}(a\omega)W(f \otimes g)(\omega)\\ &\quad = W\{W^{-1}[(Wf)(\omega)\overline{(W\psi)}(a\omega)](b)W^{-1}[(Wg)(\omega)\overline{(W\psi)}(a\omega)](b)\}. \end{aligned} \tag{6.2.11}$$

By the definition of Watson convolution, we get

$$\begin{aligned}&\overline{(W\psi)}(a\omega)W(f \otimes g)(\omega)\\ &\qquad = W(W^{-1}\{(Wf)(\omega)(\overline{W\psi}(a\omega))\#(Wg)(\omega)(\overline{W\psi}(a\omega))\}(b))\\ &\qquad = (Wf)(\omega)(\overline{W\psi})(a\omega)\#(Wg)(\omega)(\overline{W\psi})(a\omega).\end{aligned}$$

Let $F_a = (Wf)(\omega)\overline{(W\psi)}(a\omega)$ and $G_a = (Wg)(\omega)\overline{(W\psi)}(a\omega)$. Then we have

$$\overline{(W\psi)}(a\omega)W(f \otimes g)(\omega) = (F_a \# G_a)(\omega) \tag{6.2.12}$$

$$= \int_0^\infty \int_0^\infty F_a(\eta)G_a(\xi)w(\omega,\xi,\eta)d\xi d\eta. \tag{6.2.13}$$

Putting the value of F_a and G_a in (6.2.13), the R.H.S of the above expression will be

$$(F_a \# G_a)(\omega) = \int_0^\infty \int_0^\infty (Wf)(\omega)\overline{(W\psi)}(a\eta)(Wg)(\omega)\overline{(W\psi)}(a\xi)w(\omega,\xi,\eta)d\xi d\eta. \tag{6.2.14}$$

Therefore using (6.2.12) and (6.2.14)

$$\int_0^\infty W(f \otimes g)(\omega)\frac{\overline{(W\psi)}(a\omega)}{a}da$$
$$= \int_0^\infty \Big(\int_0^\infty \int_0^\infty (Wf)(\omega)\overline{(W\psi)}(a\eta)(Wg)(\omega)\overline{(W\psi)}(a\xi)w(\omega,\xi,\eta)d\xi d\eta\Big) \times \frac{da}{a}.$$

Thus, we can write

$$W(f \otimes g)(\omega)\int_0^\infty \frac{\overline{(W\psi)}(a\omega)}{a}da$$
$$= \int_0^\infty \Big(\int_0^\infty \int_0^\infty (Wf)(\omega)(Wg)(\omega)w(\omega,\xi,\eta)(\overline{W\psi})(a\eta)(\overline{W\psi})(a\xi)d\xi d\eta\Big) \times \frac{da}{a}.$$

This implies

$$E_\psi \cdot W(f \otimes g)(\omega)$$
$$= \int_0^\infty \left(\int_0^\infty \int_0^\infty (Wf)(\omega)(Wg)(\omega) w(\omega, \xi, \eta) (\overline{W\psi})(a\eta)((\overline{W\psi})(a\xi) d\xi d\eta \right) \frac{da}{a},$$

where

$$E_\psi = \int_0^\infty \frac{\overline{(W\psi)}(a\omega)}{a} da. \tag{6.2.15}$$

□

Theorem 6.2.3. *Let $\psi \in L^2(0, \infty)$ be the basis wavelet and satisfies the admissibility condition*

$$C_\psi = \int_0^\infty \frac{|(W\psi)(a\omega)|^2}{a} da. \tag{6.2.16}$$

Then

$$\int_0^\infty \frac{|(W\psi)(a\omega)(W\psi)(a\eta)|}{a} da \leq C_\psi. \tag{6.2.17}$$

Proof. Let

$$\int_0^\infty \frac{|W\psi(a\omega) W\psi(a\eta)|}{a} da = \int_0^\infty \frac{|W\psi(a\omega) W\psi(a\eta)|}{a^{1/2} a^{1/2}} da.$$

By applying the Holder's inequality, we get

$$\int_0^\infty \frac{|(W\psi)(a\omega)(W\psi)(a\eta)|}{a} da \leq \int_0^\infty \left(\frac{|(W\psi)(a\omega)|^2}{(a^{1/2})^2} da \right)^{1/2} \times \int_0^\infty \left(\frac{|(W\psi)(a\eta)|^2}{(a^{1/2})^2} da \right)^{1/2}.$$

Therefore, we have

$$\int_0^\infty \frac{|(W\psi)(a\omega)(W\psi)(a\eta)|}{a} da \leq C_\psi^{1/2} C_\psi^{1/2}$$
$$= C_\psi.$$

□

Theorem 6.2.4. *Let $f, g \in L^2(0,\infty)$, then we have*

$$W(f \otimes g)(\omega) = C'_\psi (Wf \# Wg)(\omega). \tag{6.2.18}$$

Proof. Let

$$\int_0^\infty \overline{(W\psi)}(a\eta)\overline{(W\psi)}(a\xi)\frac{da}{a} = \int_0^\infty \left(\frac{\overline{(W\psi)}(a\eta)}{a^{1/2}} da^{1/2}\right)\left(\frac{\overline{(W\psi)}(a\xi)}{a^{1/2}} da^{1/2}\right).$$

If we put $a\eta = u$ and $a\xi = v$, then we get

$$\int_0^\infty \overline{(W\psi)}(a\eta)\overline{(W\psi)}(a\xi)\frac{da}{a} = \int_0^\infty \left(\frac{\overline{(W\psi)}(u)}{(u/\eta)^{1/2}}\frac{du^{1/2}}{\eta^{1/2}}\right)\left(\frac{\overline{(W\psi)}(v)}{(v/\xi)^{1/2}}\frac{dv^{1/2}}{\xi^{1/2}}\right).$$

Taking $u = v$, we have

$$\begin{aligned}\int_0^\infty \overline{(W\psi)}(a\eta)\overline{(W\psi)}(a\xi)\frac{da}{a} &= \int_0^\infty \left(\frac{\overline{(W\psi)}(u)}{(u)^{1/2}} du^{1/2}\right)\left(\frac{\overline{(W\psi)}(u)}{(u)^{1/2}} du^{1/2}\right) \\ &= \int_0^\infty \frac{[\overline{(W\psi)}(u)]^2}{u} du \\ &= D_\psi. \end{aligned} \tag{6.2.19}$$

Using (6.2.19) in (6.2.6), we obtained

$$\begin{aligned} &E_\psi \cdot [W(f \otimes g)](\omega) \\ &= \int_0^\infty \int_0^\infty (Wf)(\eta)(Wg)(\xi)w(\omega,\eta,\xi)\left(\int_0^\infty \overline{(W\psi)}(a\eta)\overline{(W\psi)}(a\xi)\frac{da}{a}\right) d\eta d\xi \\ &= D_\psi \int_0^\infty \int_0^\infty (Wf)(\eta)(Wg)(\xi)w(\omega,\eta,\xi) d\eta d\xi \\ &= D_\psi((Wf)\#(Wg))(\omega). \end{aligned}$$

Thus

$$W(f \otimes g)(\omega) = \frac{D_\psi}{E_\psi}(Wf\#Wg)(\omega)$$
$$= C'_\psi(Wf\#Wg)(\omega),$$

where $C'_\psi = \frac{D_\psi}{E_\psi}$. □

Lemma 6.2.5. *Let $f, g \in L^1(0,\infty) \cap L^2(0,\infty)$ and $Wf, Wg \in L^1(0,\infty) \cap L^2(0,\infty)$, then we have the following relation*

$$(f \otimes g)(w) = C'_\psi(fg)(w) \qquad a.e.$$

Proof. From above Theorem 6.2.4, we have

$$W(f \otimes g)(w) = C'_\psi(Wf\#Wg)(w). \tag{6.2.20}$$

Taking inverse Watson transform on both sides in (6.2.20), we get

$$W^{-1}(W(f \otimes g)(w)) = W^{-1}(C'_\psi(Wf\#Wg)(w))$$
$$= C'_\psi W^{-1}((Wf\#Wg)(w))$$
$$= C'_\psi W^{-1}(Wf)(w')W^{-1}(Wg)(w').$$

Therefore,

$$(f \otimes g)(w) = C'_\psi(fg)(w) \qquad a.e. \tag{6.2.21}$$

□

Theorem 6.2.6. *Let $f \in L^p(0,\infty)$, $g \in L^{p'}(0,\infty)$ then for $1 \leq p, p' < \infty$ and $\psi \in L^q(0,\infty) \cap L^{q'}(0,\infty)$ such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$*

$$|W_\psi(f \otimes g)(b,a)| \leq ||f||_p ||\psi||_{p'} ||g||_q ||\psi||_{q'} \tag{6.2.22}$$

Proof. Using (6.2.1), we have

$$W_\psi(f \otimes g)(b,a) = (W_\psi f)(b,a)(W_\psi g)(b,a).$$

Using (6.1.6), we get

$$W_\psi(f \otimes g)(b,a) = W^{-1}\left((Wf)(\omega)\overline{(W\psi)(a\omega)}\right)(b) W^{-1}\left((Wg)(\omega)\overline{(W\psi)(a\omega)}\right)(b).$$

Applying convolution formula (1.4.14)

$$\begin{aligned} W_\psi(f \otimes g)(b,a) &= W^{-1}\left(W(f\#\psi)(b)\right)(b) W^{-1}\left(W(g\#\psi)(b)\right) \\ &= |(f\#\psi)(b)||(g\#\psi)(b)| \\ &\leq ||f||_p ||\psi||_{p'} ||g||_q ||\psi||_{q'}. \end{aligned}$$

□

Theorem 6.2.7. *Let $f \in L^2(0,\infty)$, $g \in L^2(0,\infty)$ then*

$$\int_0^\infty \int_0^\infty W_\psi(f \otimes g)(b,a) \frac{dadb}{a} \leq ||f||_2 ||g||_2. \tag{6.2.23}$$

Proof. Using (6.2.1), we have

$$\int_0^\infty \int_0^\infty W_\psi(f \otimes g)(b,a) \frac{dadb}{a} = \int_0^\infty \int_0^\infty (W_\psi f)(b,a)(W_\psi g)(b,a) \frac{dadb}{a}.$$

Using (6.1.7), we obtained

$$\int_0^\infty \int_0^\infty W_\psi(f \otimes g)(b,a)\frac{dadb}{a} = C_\psi |\langle f, g\rangle|.$$

By using Cauchy-Schwartz inequality, we get the required result

$$\int_0^\infty \int_0^\infty W_\psi(f \otimes g)(b,a)\frac{dadb}{a} \leq C_\psi ||f||_2||g||_2.$$

□

Theorem 6.2.8. *Let $f, g \in L^1(0,\infty) \cap L^2(0,\infty)$, then we have*

$$||W(f \otimes g)||_2 \leq C'_\psi ||f||_1||g||_2. \tag{6.2.24}$$

Proof. Using Theorem 6.2.4, we have

$$\begin{aligned} ||W(f \otimes g)||_2 &= C'_\psi ||Wf \# Wg||_2 \\ &\leq C'_\psi ||Wf||_1||Wg||_2 \\ &\leq C'_\psi ||f||_1||g||_2. \end{aligned}$$

□

Theorem 6.2.9. *Let $k_n(\omega) = (Wg_n)(\omega)$ for $n \in \mathbb{N}$ and $\phi(\omega) = (Wf)(\omega)$ satisfy the following conditions:*

1. $k_n(\omega) \geq 0,\ 0 < \omega < \infty$,
2. $\int_0^\infty k_n(\omega)d\omega = 1, \omega = 0, 1, 2.....$
3. $\lim_{n\to\infty} \int_\delta^\infty k_n(\omega)d\omega = 0$, *for each* $\delta > 0$.
4. $\phi(\omega) \in L^\infty(0,\infty)$.

5. ϕ is continuous at ω_0.

Then

$\lim_{n\to\infty} W(f \otimes g_n)(\omega_0) = C'_\psi (Wf)(\omega_0)$, *where C'_ψ defined in theorem (6.2.4).*

Proof. From (6.2.18), we write

$$\begin{aligned} W(f \otimes g_n)(\omega_0) &= C'_\psi (Wf \# Wg_n)(\omega_0) \\ &= C'_\psi (\phi \# k_n)(\omega_0). \end{aligned} \tag{6.2.25}$$

Let

$$\begin{aligned} I &= (\phi \# k_n)(\omega_0) - \phi(\omega_0) \\ &= \int_0^\infty \int_0^\infty [\phi(\omega) - \phi(\omega_0)] k_n(x) w(\omega_0, \omega, x) d\omega dx. \end{aligned} \tag{6.2.26}$$

Since ϕ is continuous at ω_0, then for given $\epsilon > 0$, there is a $\delta > 0$ so small that $|\phi(\omega) - \phi(\omega_0)| < \epsilon$ for $|\omega - \omega_0| < \delta$.

Let

$$I_1 = \int_0^\delta \int_0^\infty [\phi(\omega) - \phi(\omega_0)] k_n(x) w(\omega_0, \omega, x) d\omega dx \tag{6.2.27}$$

and

$$I_2 = \int_\delta^\infty \int_0^\infty [\phi(\omega) - \phi(\omega_0)] k_n(x) w(\omega_0, \omega, x) d\omega dx. \tag{6.2.28}$$

Now

$$\begin{aligned} |I_2| &\le \int_\delta^\infty \int_0^\infty |\phi(\omega) - \phi(\omega_0)| k_n(x) w(\omega_0, \omega, x) d\omega dx \\ &\le 2||\phi||_\infty \int_\delta^\infty \left(\int_0^\infty w(\omega_0, \omega, x) d\omega \right) k_n(x) dx \\ &= 2||\phi||_\infty \int_\delta^\infty k_n(x) dx, \qquad \text{since} \quad \left(\int_0^\infty w(\omega_0, \omega, x) d\omega = 1 \right). \end{aligned}$$

Taking limit $n \to \infty$ in the last expression and using (3.), we get $\lim_{n\to\infty} I_2 = 0$.

Now we have

$$\begin{aligned}
|I_1| &\leq \int_0^\delta \int_0^\infty |\phi(\omega) - \phi(\omega_0)| k_n(x) w(\omega_0, \omega, x) d\omega dx \\
&\leq \epsilon \int_0^\delta \int_0^\infty k_n(x) w(\omega_0, \omega, x) d\omega dx \\
&\leq \epsilon \int_0^\delta \left(\int_0^\infty w(\omega_0, \omega, x) d\omega \right) k_n(x) dx \\
&\leq \epsilon \int_0^\delta k_n(x) dx \leq \epsilon.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} ||I|| \leq \epsilon$. Since ϵ is arbitrary, we have $\lim_{n\to\infty} I = 0$.

From (6.2.25), we have

$$\begin{aligned}
\lim_{n\to\infty} W(f \otimes g_n)(\omega_0) &= \lim_{n\to\infty} C'_\psi (\phi \# k_n)(\omega_0) \\
&= C'_\psi \phi(\omega_0) \\
&= C'_\psi (Wf)(\omega_0).
\end{aligned}$$

□

6.3 Heuristic treatment of the Watson wavelet transform

In this section, we discussed the heuristic treatment of the Watson wavelet transform and investigated inversion formula of Watson wavelet transform.

Theorem 6.3.1. *Let* $(W_\psi f)(b, a)$ *be the Watson wavelet transform and* $(W_\psi^* f)(b, a)$ *be the adjoint Watson wavelet transform on a function* $f \in L^2(0, \infty)$ *with respect to* *the wavelet* $\psi \in L^2(0,\infty)$*. Then*

$$f = \int_0^\infty W_\psi^* W_\psi f \frac{da}{a}, \tag{6.3.1}$$

where $f(t) = k(t\xi)$*.*

Proof. Watson wavelet transform is given by

$$(W_\psi\phi)(b,a) = \int_0^\infty f(t)\overline{\psi_{b,a}}(t)dt.$$

From (6.1.2), we find

$$(W_\psi\phi)(b,a) = \int_0^\infty f(t) \left(\frac{1}{a}\int_0^\infty w\left(\frac{b}{a},\frac{t}{a},z\right)\overline{\psi}(z)dz\right)dt. \tag{6.3.2}$$

Putting $f(t) = k(t\xi)$ in (6.3.2), we get

$$(W_\psi\phi)(b,a) = \int_0^\infty k(t\xi) \left(\frac{1}{a}\int_0^\infty w\left(\frac{b}{a},\frac{t}{a},z\right)\overline{\psi}(z)dz\right)dt. \tag{6.3.3}$$

On choosing $\frac{t}{a} = u$, we obtained

$$\begin{aligned}(W_\psi\phi)(b,a) &= \frac{1}{a}\int_0^\infty \left(\int_0^\infty k(\xi ua)w\left(\frac{b}{a},u,z\right)\overline{\psi}(z)dz\right)du \times a\\ &= \int_0^\infty \left(\int_0^\infty k(\xi ua)w\left(\frac{b}{a},u,z\right)du\right)\overline{\psi}(z)dz\\ &= \int_0^\infty k(b\xi)k(\xi za)\overline{\psi}(z)dz\\ &= k(b\xi)\overline{(W\psi)}(a\xi). \end{aligned} \tag{6.3.4}$$

Now taking adjoint operator of Watson wavelet transform and using (6.3.4), we get

$$\begin{aligned}(W_\psi^* W_\psi \phi)(t) &= \int_0^\infty (W_\psi \phi)(b,a)\psi_{b,a}(t)db \\ &= \overline{(W\psi)}(a\xi)\int_0^\infty k(b\xi)\left(\frac{1}{a}\int_0^\infty w\left(\frac{b}{a},\frac{t}{a},z\right)\overline{\psi}(z)dz\right)db \qquad (6.3.5)\\ &= \overline{(W\psi)}(a\xi)\int_0^\infty \left(\frac{1}{a}\int_0^\infty w\left(\frac{b}{a},\frac{t}{a},z\right)k(b\xi)db\right)\overline{\psi}(z)dz.\end{aligned}$$

If we substitute $\frac{b}{a} = v$, then

$$\begin{aligned}(W_\psi^* W_\psi \phi)(t) &= \overline{(W\psi)}(a\xi)\int_0^\infty \left(\frac{1}{a}\int_0^\infty w\left(v,\frac{t}{a},z\right)k(av\xi)a\times dv\right)\overline{\psi}(z)dz \\ &= \overline{(W\psi)}(a\xi)\int_0^\infty k(t\xi)k(za\xi)\overline{\psi}(z)dz \\ &= k(t\xi)\overline{(W\psi)}(a\xi)\int_0^\infty k(za\xi)\overline{\psi}(z)dz \\ &= k(t\xi)\overline{(W\psi)}(a\xi)(W\psi)(a\xi) \\ &= k(t\xi)|(W\psi)(a\xi)|^2.\end{aligned}$$

Therefore

$$\int_0^\infty (W_\psi^* W_\psi \phi)(t)\frac{da}{a} = k(t\xi)\int_0^\infty |(W\psi)(a\xi)|^2\frac{da}{a}.$$

Thus we have

$$f(t) = k(t\xi) = \frac{\int_0^\infty (W_\psi^* W_\psi \phi)(t)\frac{da}{a}}{\int_0^\infty |(W\psi)(a\xi)|^2\frac{da}{a}}.$$

By imposing the normalization $\int_0^\infty |(W\psi)(a\xi)|^2\frac{da}{a} = 1$, we obtain the Watson wavelet representation

$$f(t) = \int_0^\infty (W_\psi^* W_\psi \phi)(t)\frac{da}{a}.$$

□

Theorem 6.3.2. *Suppose that* ψ *is a continuum Watson wavelet with*

$$A_\psi = \int_0^\infty \frac{|\psi(w)|^2}{w} dw = 1.$$

Then for any $f \in L^2(0,\infty)$*, we have the inversion formula*

$$\begin{aligned} f(x) &= \int_0^\infty \int_0^\infty (W_\psi f)(b,a)\psi_{b,a}(x)\frac{dbda}{a} \\ &= \lim_{\epsilon\to 0, A,B\to\infty} \int_{\epsilon<a<A, b<B} (W_\psi f)(b,a)\psi_{b,a}(x)\frac{dbda}{a}, \end{aligned} \tag{6.3.6}$$

where $S(\epsilon, A, B)f = \int_{\epsilon<a<A,b<B}(W_\psi f)(b,a)\psi_{b,a}(x)\frac{dbda}{a}$.

Proof. Let

$$||f - S(\epsilon, A, B)f||_2 = \sup_{||g||_2=1} |\langle f - S(\epsilon, A, B)f, g\rangle|. \tag{6.3.7}$$

Applying Fubini's theorem, we have

$$\begin{aligned} \langle S(\epsilon, A, B)f, g\rangle &= \int_0^\infty \bar{g}(x)\left(\int_{\epsilon<a<A,b<B} (W_\psi f)(b,a)\psi_{b,a}(x)\frac{dbda}{a}\right)dx \\ &= \int_{\epsilon<a<A,b<B} (W_\psi f)(b,a)\left(\int_0^\infty \bar{g}(x)\psi_{b,a}(x)dx\right)\frac{dbda}{a} \\ &= \int_{\epsilon<a<A,b<B} (W_\psi f)(b,a)\overline{(W_\psi g)}(b,a)(x)\frac{dbda}{a}. \end{aligned}$$

Using Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&|\langle f - S(\epsilon, A, B)f, g\rangle| \\
&\quad = \left| \int_{(\epsilon<a<A, b<B)^c} (W_\psi f)(b,a)\overline{(W_\psi g)}(b,a)(x)\frac{dbda}{a} \right| \\
&\quad = \left(\int_{(\epsilon<a<A, b<B)^c} |(W_\psi f)(b,a)|^2 \frac{dbda}{a} \right)^{1/2} \left(\int_0^\infty \int_0^\infty |(W_\psi g)(b,a)|^2 \frac{dbda}{a} \right)^{1/2}.
\end{aligned}$$

Then by using (6.1.7), we get

$$\begin{aligned}
&|\langle f - S(\epsilon, A, B)f, g\rangle| \\
&= \left(\int_{(\epsilon<a<A, b<B)^c} |(W_\psi f)(b,a)|^2 \frac{dbda}{a} \right)^{1/2} A_\psi ||g||_2,
\end{aligned} \tag{6.3.8}$$

where $\epsilon \to 0$ and $A, B \to \infty$, the region of integration decreases to empty set. Hence the last integral tends to zero by the dominated convergence theorem. This gives that

$$||S(\epsilon, A, B)f - f||_2 \to 0.$$

□

Theorem 6.3.3. *Suppose that* ψ *is a continuum Watson wavelet with* $<\psi, \psi>_w = 1$ *and*

$$C_{\psi,s} = \int_0^\infty \frac{|(W\psi)(\xi)|}{\xi^{2s}} d\xi < \infty. \tag{6.3.9}$$

Then

$$\int_0^\infty \int_0^\infty \frac{|(W_\psi f)(b,a)|^2}{a^{2s}} dadb = C_{\psi,s} ||f||_{2,s}^2.$$

Proof. From (6.1.6), we have

$$\int_0^\infty (W_\psi f)(b,a)\overline{(W_\psi g)}(b,a)db$$
$$= \int_0^\infty W^{-1}\left((Wf)(u)\overline{(W\psi)}(au)\right)(b)\overline{W^{-1}\left((Wg)(u)\overline{(W\psi)}(au)\right)}(b)db. \tag{6.3.10}$$

Using Parseval formula of Watson wavelet transform (1.4.12)

$$\int_0^\infty\int_0^\infty \frac{(W_\psi f)(b,a)\overline{(W_\psi g)(b,a)}}{a^{2s}}dadb$$
$$= \int_0^\infty\int_0^\infty \frac{(Wf)(u)\overline{(W\psi)}(au)\overline{(Wg)(u)\overline{(W\psi)}(au)}}{a^{2s}}dadb$$
$$= \int_0^\infty\int_0^\infty (Wf)(u)\overline{(Wg)}(u)\frac{|(W\psi)(au)|^2}{a^{2s}}dadb.$$

Take $f=g$, we get

$$\int_0^\infty\int_0^\infty \frac{|(W_\psi f)(b,a)|^2}{a^{2s}}dadb = \int_0^\infty\int_0^\infty |(Wf)(u)|^2\frac{|(W\psi)(au)|^2}{a^{2s}}dadb.$$

Putting $au=\xi$ in the second term of the above expression, we get

$$\int_0^\infty\int_0^\infty \frac{|(W_\psi f)(b,a)|^2}{a^{2s}}dadb = \int_0^\infty\int_0^\infty |(Wf)(u)|^2\left(\frac{|(W\psi)(\xi)|^2}{\xi^{2s}}d\xi\right)u^{2s}du$$
$$= C_{\psi,s}\int_0^\infty |(Wf)(u)|^2u^{2s}du$$
$$= C_{\psi,s}||f||^2_{2,s}.$$

□

6.4 Two wavelet multipliers

With the help of [62, 77], two wavelet multipliers are introduced and expressed it in term of Watson wavelet convolution product by exploiting the theory of the Watson transform.

From [62], let $\pi : \mathbb{R}^+ \to U(L^2(0,\infty))$ be the unitary representation of the multiplicative group $\mathbb{R}^+$ on $L^2(0,\infty)$ is defined by

$$(\pi_\xi u)(x) = k(x\xi)u(x) \qquad x, \xi \in (0,\infty), \tag{6.4.1}$$

for all functions u in $L^2(0,\infty)$, where $U(L^2(0,\infty))$ is the group of all unitary operators on $L^2(0,\infty)$.

Then the Watson two wavelet multipliers associated with the unitary representation is given as

$$(P_{\sigma,\phi,\psi}u)(x) = \int_0^\infty \sigma(\xi)\langle u, \pi_\xi \phi\rangle(\pi_\xi\psi)(x)d\xi. \tag{6.4.2}$$

Theorem 6.4.1. *If $\sigma(\xi) \in L^1(0,\infty)$ and $\phi, \psi \in L^1(0,\infty) \cap L^\infty(0,\infty)$. Then the Watson two wavelet multipliers can be expressed in the following form:*

$$(P_{\sigma,\phi,\psi}u)(x) = \frac{1}{C'_\psi}\psi(x)W^{-1}[\sigma(\xi)W(u \otimes \phi)(\xi)](x), \quad u \in L^1(0,\infty). \tag{6.4.3}$$

Proof. Using (6.4.2), we have

$$\begin{aligned}(P_{\sigma,\phi,\psi}u)(x) &= \int_0^\infty \sigma(\xi)\langle u, \pi_\xi\phi\rangle(\pi_\xi\psi)(x)d\xi \\ &= \int_0^\infty \sigma(\xi)\left(\int_0^\infty u(\eta)(\pi_\xi\phi)(\eta)d\eta\right)(\pi_\xi\psi)(x)d\xi.\end{aligned}$$

From (6.4.1), we get

$$
\begin{aligned}
(P_{\sigma,\phi,\psi}u)(x) &= \int_0^\infty \sigma(\xi)\left(\int_0^\infty k(\eta\xi)\phi(\eta)u(\eta)d\eta\right)k(x\xi)\psi(x)d\xi \\
&= \int_0^\infty \sigma(\xi)W(u\phi)(\xi)k(x\xi)\psi(x)d\xi \\
&= \psi(x)\int_0^\infty k(x\xi)\sigma(\xi)W(u\phi)(\xi)d\xi \\
&= \psi(x)W^{-1}\left[\sigma(\xi)W(u\phi)(\xi)\right](x).
\end{aligned}
$$

With help of Lemma 6.2.5, we have

$$
(P_{\sigma,\phi,\psi}u)(x) = \frac{1}{C'_\psi}\left(\psi(x)W^{-1}\left[\sigma(\xi)W(u\otimes\phi)(\xi)\right]\right)(x).
$$

□

Theorem 6.4.2. *Let $\sigma \in L^1(0,\infty)$, and $\phi,\psi \in L^1(0,\infty)\cap L^2(0,\infty)$. Then the two wavelet multipliers $P_{\sigma,\phi,\psi} : L^1(0,\infty) \to L^1(0,\infty)$ is*

$$
||P_{\sigma,\phi,\psi}||_{B(L^1(0,\infty))} \leq \frac{1}{C'_\psi}||\sigma||_\infty||\phi||_2||\bar{\psi}||_2.
$$

Proof. From (6.4.3), we have

$$
||P_{\sigma,\phi,\psi}u||_1 = ||\psi(x)W^{-1}\left[\sigma(\xi)W(\bar{\phi}u)(\xi)\right](x)||_1.
$$

Using Lemma 6.2.5, we get

$$
||P_{\sigma,\phi,\psi}u||_1 = \frac{1}{C'_\psi}||\psi(x)W^{-1}\left[\sigma(\xi)W(\bar{\phi}\otimes u)(\xi)\right](x)||_1.
$$

Applying Holder's inequality, we obtained

$$||P_{\sigma,\phi,\psi}u||_1 \leq \frac{1}{C'_\psi}||\psi||_2||W^{-1}\left[\sigma(\xi)W(\bar{\phi}\otimes u)(\xi)\right]||_2.$$

By applying the Parseval relation (1.4.12) of the Watson transform, we have

$$\begin{aligned}||P_{\sigma,\phi,\psi}u||_1 &\leq \frac{1}{C'_\psi}||\psi||_2||\sigma(\xi)W(\bar{\phi}\otimes u)(\xi)||_2\\ &\leq \frac{1}{C'_\psi}||\psi||_2||\sigma||_\infty||W(\bar{\phi}\otimes u)||_2.\end{aligned}$$

Again applying Parseval formula (1.4.12), we get

$$||P_{\sigma,\phi,\psi}u||_1 \leq \frac{1}{C'_\psi}||\psi||_2||\sigma||_\infty||\bar{\phi}\otimes u||_2.$$

Using Theorem 6.2.8, we get the required result

$$||P_{\sigma,\phi,\psi}||_{B(L^1(0,\infty))} \leq \frac{1}{C'_\psi}||\sigma||_\infty||\psi||_2||\bar{\phi}||_2.$$

□

Theorem 6.4.3. *Let $\sigma \in L^1(0,\infty)$, and $\phi,\psi \in L^1(0,\infty)\cap L^\infty(0,\infty)$. Then the two wavelet multipliers $P_{\sigma,\phi,\psi} : L^2(0,\infty) \to L^2(0,\infty)$ is estimated by*

$$||P_{\sigma,\phi,\psi}||_{B(L^2(0,\infty))} \leq \frac{1}{C'_\psi}||\phi||_\infty||\sigma||_\infty||\bar{\psi}||_1.$$

Proof. From (6.4.3), we have

$$||P_{\sigma,\phi,\psi}u||_2 = ||\psi(x)W^{-1}\left[\sigma(\xi)W(\bar{\phi}u)(\xi)\right](x)||_2.$$

Using Lemma 6.2.5, we get

$$\begin{aligned} ||P_{\sigma,\phi,\psi}u||_2 &= \frac{1}{C'_\psi}||\psi(x)W^{-1}\left[\sigma(\xi)W(\bar{\phi}\otimes u)(\xi)\right](x)||_2 \\ &\leq \frac{1}{C'_\psi}||\psi||_\infty||W^{-1}\left[\sigma(\xi)W(\bar{\phi}\otimes u)(\xi)\right](x)||_2. \end{aligned}$$

By Parseval relation (1.4.12), we have

$$\begin{aligned} ||P_{\sigma,\phi,\psi}u||_2 &\leq \frac{1}{C'_\psi}||\psi||_\infty||\sigma(\xi)W(\bar{\phi}\otimes u)(\xi)||_2 \\ &\leq \frac{1}{C'_\psi}||\psi||_\infty||\sigma||_\infty||W(\bar{\phi}\otimes u)||_2. \end{aligned}$$

Again applying Parseval formula (1.4.12), we get

$$||P_{\sigma,\phi,\psi}u||_2 \leq \frac{1}{C'_\psi}||\psi||_\infty||\sigma||_\infty||\bar{\phi}\otimes u||_2.$$

Using Theorem 6.2.8, we get the required results

$$||P_{\sigma,\phi,\psi}||_{B(L^2(0,\infty))} \leq \frac{1}{C'_\psi}||\psi||_\infty||\sigma||_\infty||\bar{\phi}||_1.$$

□

From Wong [77, p. 499], we recall the following fact which are very useful to find the trace class of Watson two wavelet multipliers.

Theorem 6.4.4. *Let $\sigma \in L^1(0,\infty)$ and ϕ, ψ be any functions in $L^2(0,\infty)$ such that $||\phi||_2 = 1 = ||\psi||_2$. Then the two Watson wavelet multiplier $P_{\sigma,\phi,\psi} : L^2(0,\infty) \to L^2(0,\infty)$ is in trace class.*

$$|tr(P_{\sigma,\phi,\psi})| \leq K\int_0^\infty |\sigma(\xi)|d\xi. \tag{6.4.4}$$

Proof. Let $\{\phi_k\}$ be a sequence of orthonormal basis for $L^2(0,\infty)$. Then, we have

$$|tr(P_{\sigma,\phi,\psi}\phi_k)| = |\sum_{k=1}^{\infty}\langle P_{\sigma,\phi,\psi}\phi_k, \phi_k\rangle|.$$

From (6.4.2), we get

$$\begin{aligned}|tr(P_{\sigma,\phi,\psi}\phi_k)| &= |\sum_{k=1}^{\infty}\int_0^{\infty}\sigma(\xi)\langle\phi_k, \pi_\xi\phi\rangle\langle\pi_\xi\psi, \phi_k\rangle d\xi| \\ &\leq \sum_{k=1}^{\infty}|\int_0^{\infty}\sigma(\xi)\langle\phi_k, \pi_\xi\phi\rangle\langle\pi_\xi\psi, \phi_k\rangle d\xi|.\end{aligned}$$

By using the definition of inner product, we have

$$|tr(P_{\sigma,\phi,\psi}\phi_k)| \leq \sum_{k=1}^{\infty}\int_0^{\infty}|\sigma(\xi)\langle\phi_k, \pi_\xi\phi\rangle\overline{\langle\phi_k, \pi_\xi\psi\rangle}|d\xi$$

Using the useful result of [54, p. 402]

$$|tr(P_{\sigma,\phi,\psi}\phi_k)| \leq \int_0^{\infty}|\sigma(\xi)\langle\pi_\xi\psi, \pi_\xi\phi\rangle|d\xi.$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}|tr(P_{\sigma,\phi,\psi})| &\leq \int_0^{\infty}|\sigma(\xi)|||\pi_\xi\phi||_2||\pi_\xi\psi||_2 d\xi \\ &\leq K||\phi||_2||\psi||_2\int_0^{\infty}|\sigma(\xi)|d\xi \\ &< \infty.\end{aligned}$$

Thus, we yields the required results

$$|tr(P_{\sigma,\phi,\psi})| \leq K\int_0^{\infty}|\sigma(\xi)|d\xi.$$

□

Theorem 6.4.5. *Let $\sigma \in L^1(0,\infty)$, and ϕ and ψ be any functions in $L^2(0,\infty) \cap L^\infty(0,\infty)$ such that $||\phi||_2 = 1 = ||\psi||_2$. Then the trace class of Watson two wavelet multiplier can be expressed in term of Watson wavelet convolution product.*

$$tr(P_{\sigma,\phi,\psi}\phi_k) = \sum_{k=1}^{\infty} \frac{1}{C'_\psi} \int_0^\infty \sigma(\xi) W(\phi_k \otimes \phi)(\xi) W(\psi \otimes \phi_k)(\xi) d\xi. \tag{6.4.5}$$

Proof. Let $\{\phi_k\}$ be a sequence of orthonormal basis for $L^2(0,\infty)$. Then, by using the Fubini's theorem, we have

$$\begin{aligned} tr(P_{\sigma,\phi,\psi}\phi_k) &= \sum_{k=1}^{\infty} \langle P_{\sigma,\phi,\psi}\phi_k, \phi_k\rangle \\ &= \sum_{k=1}^{\infty} \int_0^\infty \sigma(\xi) \langle \phi_k, \pi_\xi \phi\rangle \langle \pi_\xi \psi, \phi_k\rangle d\xi \\ &= \sum_{k=1}^{\infty} \int_0^\infty \sigma(\xi) \langle \phi_k, \pi_\xi \phi\rangle \langle \pi_\xi \psi, \phi_k\rangle d\xi. \end{aligned}$$

Using the definition of the inner product, we have

$$tr(P_{\sigma,\phi,\psi}) = \sum_{k=1}^{\infty} \int_0^\infty \sigma(\xi) \langle \phi_k, \pi_\xi \phi\rangle \overline{\langle \phi_k, \pi_\xi \psi\rangle} d\xi.$$

From (6.4.1), we find that

$$tr(P_{\sigma,\phi,\psi}) = \sum_{k=1}^{\infty} \int_0^\infty \sigma(\xi) W(\phi_k \phi)(\xi) \overline{W(\phi_k \psi)}(\xi) d\xi.$$

Using Lemma 6.2.5, we express the trace class in term of the Watson wavelet convolution product

$$tr(P_{\sigma,\phi,\psi}\phi_k) = \sum_{k=1}^{\infty} \frac{1}{C'_\psi} \int_0^\infty \sigma(\xi) W(\phi_k \otimes \phi) \overline{W(\phi_k \otimes \psi)} d\xi.$$

□

Theorem 6.4.6. *Let $\sigma \in L^1(0,\infty)$, and ϕ and ψ be any functions in $L^2(0,\infty) \cap L^\infty(0,\infty)$ such that $||\phi||_2 = 1 = ||\psi||_2$. Then the trace class of Watson two wavelet multiplier is in S_1.*

$$|tr(P_{\sigma,\phi,\psi}\phi_k)| \leq \sum_{k=1}^{\infty} \frac{1}{C'_\psi} \int_0^\infty |\sigma(\xi) W(\phi_k \otimes \phi) W(\psi \otimes \phi_k)| d\xi < \infty.$$

Proof. With the help of (6.4.5), we find that

$$\begin{aligned}
|tr(P_{\sigma,\phi,\psi}\phi_k)| &\leq \sum_{k=1}^{\infty} \frac{1}{C'_\psi} sup|W(\phi_k \otimes \phi)(\xi)| \int_0^\infty |\sigma(\xi)\overline{W(\phi_k \otimes \psi)}| d\xi \\
&\leq \sum_{k=1}^{\infty} \frac{1}{C'_\psi} ||\phi_k \otimes \phi||_1 \int_0^\infty |\sigma(\xi)\overline{W(\phi_k \otimes \psi)}| d\xi \\
&\leq \sum_{k=1}^{\infty} \frac{1}{C'_\psi} ||\phi_k \otimes \phi||_1 sup|\overline{W(\phi_k \otimes \psi)}| \int_0^\infty |\sigma(\xi)| d\xi \\
&\leq \sum_{k=1}^{\infty} \frac{1}{C'_\psi} ||\phi_k \otimes \phi||_1 ||\overline{\phi_k \otimes \psi}||_1 \int_0^\infty |\sigma(\xi)| d\xi.
\end{aligned}$$

Using Lemma (6.2.5) and applying Holder inequality, we get

$$\begin{aligned}
|tr(P_{\sigma,\phi,\psi}\phi_k)| &\leq \sum_{k=1}^{\infty} \frac{1}{C'_\psi} ||\phi_k \phi||_1 ||\overline{\phi_k \psi}||_1 \int_0^\infty |\sigma(\xi)| d\xi \\
&\leq ||\phi||_2 ||\overline{\psi}||_2 \int_0^\infty |\sigma(\xi)| d\xi \\
&\leq ||\phi||_2 ||\overline{\psi}||_2 \int_0^\infty |\sigma(\xi)| d\xi \\
&\leq \int_0^\infty |\sigma(\xi)| d\xi < \infty.
\end{aligned}$$

□

6.5 Conclusions

From the works of Pathak [49], Schuitman [60] and Titchmarsh [64] the author concluded that the Watson transform contains strong mathematical background and rich calculus. The results of the Watson transform are interesting and from [33, 60] it is observed that Fourier transform, Laplace transform, and Hankel transform are examples of the Watson transform. Various mathematical relations of Watson transform with Laplace transform, Hankel transform and Fourier transform are given in [33]. This work provides an integral representation of the Watson wavelet convolution product and shows the relationship between the Watson wavelet convolution product and Watson convolution. A heuristic treatment of the inversion formula of the Watson wavelet transform is developed, and its estimation is expressed in terms of the Sobolev type space. This theory is used to derive estimations of two wavelet multipliers associated with the Watson transform. Later on, the author was able to find the connection between the Watson wavelet convolution product and the trace class of two wavelet multipliers.

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