authors studied the properties of pseudo-differential operators and localization operators using the Fourier transform tool. The localization operator was introduced by Daubechies [14, 15] in connection with the time frequency analysis. The boundedness of localization operators has been studied by He and Wong in [26]. Du and Wong [16] investigated the trace class of localization operators. Later on, an L^p-boundedness of localization operators associated with the left regular representation of a locally compact and Hausdroff group was investigated by Wong [78].

Concerning the Hankel transform technique, an L^p_μ-boundedness of the localization operator associated with the integral representation, was investgated by Upadhyay [66]. More recently, in the year 2020, the boundedness and compactness of the localization operators, which involve the Hankel transform technique were discussed by Baccar et al. [3] and Mejjaoli et al. [37].

The complete descriptions about the Watson transform were discussed in the previous chapters. In 1960, Watson transform on groups was given in the paper of Goldberg [24]. He defined Watson transform on locally compact group in $L^2(G)$-space and studied many properties. The characterizations of Watson transform and Watson convolution were presented by Pathak and Tiwari [47]. In [47], the authors studied the mapping properties of pseudo-differential operators involving Watson transform and Watson convolution. The continuous wavelet transform and Bessel-wavelet transform are generated by convolution and Hankel convolution with the help of the Fourier transform and Hankel transform technique. In similar way, the Watson wavelet transform can be constructed with the help of Watson convolution see [67], which plays an important role to find the properties of $L^p_{m_\nu}$-boundedness of localization operators in the present chapter.

Motivated essentially by the aforementioned investigations by Goldberg [24], Upadhyay [66], Wong [27, 78] and other results, the main objective of this chapter is to obtain the properties of localization operator and wavelet multipliers involving the Watson transform associated with the integral representation.

In the present chapter, contents of the work are presented below:
Section 5.1, gives the basic concept, formulae, and useful properties of the Watson transform. In Section 5.2, the properties of the localization operator associated with the integral representation are investigated by taking the Watson transform technique. Section 5.3 is devoted to the $L^p_{m_\nu}$-boundedness of localization operators associated with the Watson transform. In Section 5.4, the relation between wavelet multipliers and localization operators associated with the integral representation are obtained, and in last Section 5.5, the trace class and Schatten-von Neumann property associated with wavelet multipliers are found.

5.2 Properties of the localization operator

In this section, by using the theory of the Watson transform, we consider several properties of localization operators associated with the integral representation.

Let G be a locally compact Hausdorff group on which the left Haar measure is denoted by m_ν.
Suppose also that $\pi : G \to B(L^p_{m_\nu}(G))$, $1 \leq p < \infty$ is the integral representation of G on $L^p_{m_\nu}(G)$, which is defined by

$$(\pi_g f)(h) = \int_G f(t)w(h,g,t)dm_\nu(t), \quad g,h \in G \quad \text{for} \quad f \in L^p_{m_\nu}(G), \tag{5.2.1}$$

where w is the basic function defined in (1.4.7).

Theorem 5.2.1. *Let* $f \in \bigcap_{1\le p<\infty} L^p_{m_\nu}(G)$ *be such that* $\|f\|_{L^p_{m_\nu}(G)} = 1$. *Then*

$$\langle f, \pi_g f\rangle = \left(f\#\bar{f}\right)(g)$$

and

$$C_f = \int\limits_G |\langle f, \pi_g f\rangle|^2 dm_\nu(t) \le \|f\|^2_{L^1_{m_\nu}(G)}.$$

Proof. From (5.2.1), we have

$$\begin{aligned}
\langle f, \pi_g f\rangle =& \Big\langle f, \int\limits_G f(t)w(x,g,t)dm_\nu(t)\Big\rangle \\
=& \int\limits_G f(x)\overline{\Big(\int\limits_G f(t)w(x,g,t)dm_\nu(t)\Big)}dm_\nu(x) \\
=& \int\limits_G f(x)\bar{f}(x,g)dm_\nu(x) \\
=&(f\#\bar{f})(g). \qquad (5.2.2)
\end{aligned}$$

Using (5.2.2), we get

$$C_f = \int\limits_G |\langle f, \pi_g f\rangle|^2 dm_\nu(g) = \int_G |(f\#\bar{f})(g)|^2 dm_\nu(g).$$

Thus, by Parseval formula (1.4.12) for the Watson transform, we find that

$$C_f = \int\limits_G |\langle f, \pi_g f\rangle|^2 dm_\nu(g) = \int\limits_G |W(f\#\bar{f})(\xi)|^2 dm_\nu(\xi).$$

Using (1.4.14), we have

$$\begin{aligned}C_f = \int\limits_G |\langle f, \pi_g f\rangle|^2 dm_\nu(g) &= \int\limits_G |(Wf)(\xi)(W\bar{f})(\xi)|^2 dm_\nu(\xi)\\ &\leq \int\limits_G |(Wf)(\xi)|^2 \cdot |(W\bar{f})(\xi)|^2 dm_\nu(\xi)\\ &\leq \int\limits_G |(Wf)(\xi)|^2 \cdot \|f\|^2_{L^1_{m_\nu}(G)} dm_\nu(\xi)\\ &\leq \|f\|^2_{L^1_{m_\nu}(G)}, \qquad \Big(\text{since} \int\limits_G |(Wf)(\xi)|^2 dm_\nu(\xi) = 1\Big).\end{aligned}$$

Therefore, the above expression yields

$$C_f = \int\limits_G |\langle f, \pi_g f\rangle|^2 dm_\nu(g) \leq \|f\|^2_{L^1_{m_\nu}(G)}.$$

□

Theorem 5.2.2. *Let f be an admissible wavelet for the square integrable representation $\pi : G \to B(L^2_{m_\nu}(G))$. Then*

$$\langle u, v\rangle = \frac{1}{C_f} \int\limits_G \langle u, \pi_g f\rangle \langle \pi_g f, v\rangle dm_\nu(\xi). \tag{5.2.3}$$

where

$$C_f = \int_G \left|(f\#f)(\xi)\right|^2 dm_\nu(\xi) < \infty.$$

Proof. First of all, by using (5.2.2), we get

$$\frac{1}{C_f} \int\limits_G \langle u, \pi_g f\rangle \langle \pi_g f, v\rangle dm_\nu(\xi) = \frac{\int\limits_G (u\#\bar{f})(\xi)(f\#\bar{v})(\xi) dm_\nu(\xi)}{\int\limits_G \left|(f\#f)(\xi)\right|^2 dm_\nu(\xi)}. \tag{5.2.4}$$

By Parseval formula of Watson transform (1.4.12), we get

$$\frac{1}{C_f}\int_G \langle u,\pi_g f\rangle\langle \pi_g f, v\rangle dm_\nu(\xi) = \frac{\int_G W(u\#\bar{f})(g)W(f\#\bar{v})(g)dm_\nu(g)}{\int_G \left|W(f\#f)(g)\right|^2 dm_\nu(g)}.$$

Using (1.4.14), we find that

$$\begin{aligned}\frac{1}{C_f}\int_G \langle u,\pi_g f\rangle\langle \pi_g f, v\rangle dm_\nu(\xi) =& \frac{\int_G (Wu)(g)\left|(Wf)(g)\right|^2(W\bar{v})(g)dm_\nu(g)}{\int_G \left|(Wf)(g)\right|^2\left|(Wf)(g)\right|^2 dm_\nu(g)}\\ =&\int_G \frac{(Wu)(g)\left|(Wf)(g)\right|^2(W\bar{v})(g)}{|(Wf)(g)|^2|(Wf)(g)|^2}dm_\nu(g)\\ =&\int_G \frac{(Wu)(g)(W\bar{v})(g)}{|(Wf)(g)|^2}dm_\nu(g)\\ =&\frac{\int_G (Wu)(g)(W\bar{v})(g)dm_\nu(g)}{\int_G |(Wf)(g)|^2 dm_\nu(g)}\\ =&\int_G (Wu)(g)(W\bar{v})(g)dm_\nu(g), \quad \Big(\int_G |(Wf)(g)|^2 dm_\nu(g)=1\Big).\end{aligned}$$

Again by Parseval formula of Watson transform (1.4.12), we get

$$\begin{aligned}\frac{1}{C_f}\int_G \langle u,\pi_g f\rangle\langle \pi_g f, v\rangle dm_\nu(\xi) =& \int_G u(\xi)\overline{v}(\xi)dm_\nu(\xi)\\ =&\langle u,v\rangle.\end{aligned}$$

□

Definition 5.2.3. Let $F\in L^1_{m_\nu}(G)\cap L^\infty_{m_\nu}(G)$. Then the localization operator $L_{F,f}$: $L^p_{m_\nu}(G)\to L^p_{m_\nu}(G)$ with symbol F and the admissible wavelet f are defined by

$$\langle L_{F,f}u,v\rangle = \frac{1}{C_f}\int_G F(g)\langle u,\pi_g f\rangle\langle \pi_g f, v\rangle dm_\nu(g), \tag{5.2.5}$$

for all $u \in L^p_{m_\nu}(G)$ and $v \in L^{p'}_{m_\nu}(G)$, where $\langle , \rangle$ defined as

$$\langle x, y\rangle = \int_G x(g)\overline{y(g)}dm_\nu(g) \quad \text{for} \quad 1 \leq p < \infty. \tag{5.2.6}$$

Theorem 5.2.4. *Let $F \in L^1_{m_\nu}(G)$. Then for $1 \leq p < \infty$, the localization operator $L_{F,f} : L^p_{m_\nu}(G) \to L^p_{m_\nu}(G)$ is a bounded linear operator and satisfies the following norm inequality*

$$\|L_{F,f}\|_{B(L^p_{m_\nu}(G))} \leq \frac{1}{C_f}\|F\|_{L^1_{m_\nu}(G)}||f||^2_{L^{p'}_{m_\nu}(G)}$$

for all $u \in L^p_{m_\nu}(G)$ and $v \in L^{p'}_{m_\nu}(G)$.

Proof. From (5.2.5), we have

$$\begin{aligned}
|\langle L_{f,F}u, v\rangle| &\leq \frac{1}{C_f}\int_G |F(g)| \cdot |\langle u, \pi_g f\rangle| \cdot |\langle \pi_g f, v\rangle| dm_\nu(g) \\
&\leq \frac{1}{C_f}\int_G |F(g)| \cdot |(u\#f)(g)| \cdot |(f\#v)(g)| dm_\nu(g) \\
&\leq \frac{1}{C_f}||(u\#f)||_{L^\infty_{m_\nu}(G)}||(f\#v)||_{L^\infty_{m_\nu}(G)}\int_G |F(g)|dm_\nu(g) \\
&\leq \frac{1}{C_f}||u||_{L^p_{m_\nu}(G)}||f||_{L^{p'}_{m_\nu}(G)}||v||_{L^p_{m_\nu}(G)}||f||_{L^{p'}_{m_\nu}(G)}||F(g)||_{L^1_{m_\nu}(G)},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Therefore, we yields the required result

$$|\langle L_{F,f}u, v\rangle| \leq \frac{1}{C_f}||u||_{L^p_{m_\nu}}||f||_{L^{p'}_{m_\nu}}||v||_{L^p_{m_\nu}}||f||_{L^{p'}_{m_\nu}}||F(g)||_{L^1_{m_\nu}}, \tag{5.2.7}$$

for all $u \in L^p_{m_\nu}(G)$ and $v \in L^{p'}_{m_\nu}(G)$. □

Theorem 5.2.5. *For $1 \leq p < \infty$, the localization operator $L_{F,f} : L^p_{m_\nu}(G) \to L^p_{m_\nu}(G)$ is a bounded linear operator and satisfies the norm inequality*

$$\|L_{F,f}\|_{B(L^p_{m_\nu}(G))} \leq \frac{1}{C_f}\|F\|_{L^\infty_{m_\nu}(G)}||f||^2_{L^1_{m_\nu}(G)}$$

for $F \in L^\infty_{m_\nu}(G)$ and $f \in L^1_{m_\nu}(G)$.

Proof. (5.2.5), yields

$$|\langle L_{F,f}u, v\rangle| \leq \frac{1}{C_f}\int_G |F(g)| \cdot |\langle u, \pi_g f\rangle| \cdot |\langle \pi_g f, v\rangle| dm_\nu(g). \tag{5.2.8}$$

For all $u \in L^p_{m_\nu}(G)$ and $v \in L^{p'}_{m_\nu}(G)$. Then last equation (5.2.8), together with (5.2.1)

$$\begin{aligned} |\langle L_{F,f}u, v\rangle| &\leq \frac{1}{C_f} sup|F(g)| \int_G |\langle u, \pi_g f\rangle| \cdot |\langle \pi_g f, v\rangle| dm_\nu(g) \\ &\leq \frac{1}{C_f} sup|F(g)| \int_G |(u\#f)(g)(f\#v)(g)| dm_\nu(g). \end{aligned}$$

Since $F \in L^\infty_{m_\nu}(G)$, and by using Holder inequality, we have

$$\begin{aligned} |\langle L_{F,f}u, v\rangle| &\leq \frac{1}{C_f} sup|F(g)| \cdot ||u\#f||_{L^p_{m_\nu}} \cdot ||f\#v||_{L^{p'}_{m_\nu}} \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1 \\ &\leq \frac{1}{C_f}||F||_{L^\infty_{m_\nu}}||f||_{L^1_{m_\nu}}||u||_{L^p_{m_\nu}}||f||_{L^1_{m_\nu}}||v||_{L^{p'}_{m_\nu}} \\ &\leq \frac{1}{C_f}||F||_{L^\infty_{m_\nu}}||f||^2_{L^1_{m_\nu}}||u||_{L^p_{m_\nu}}||v||_{L^{p'}_{m_\nu}} \end{aligned} \tag{5.2.9}$$

for all $u \in L^p_{m_\nu}(G)$ and $v \in L^{p'}_{m_\nu}(G)$. □

5.3 $L^p_{m_\nu}$-boundedness of localization operators

In this section, we derive an $L^p_{m_\nu}(G)$-boundedness of localization operator involving the Watson transform. Exploiting the Risez-Thorin theorem from [78, p. 2916], we restate the following result.

Theorem 5.3.1. *Let (X, μ) be a measure space and (Y, ν) a σ-finite measure space. Let T be a linear transformation with domain D consisting of all μ-simple functions f on X such that $\mu\{x \in X : f(x) \neq 0\} < \infty$ and such that range of T is contained in the set of all ν-measurable functions on Y. Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real numbers in $[0, 1]$ and there exist positive constants M_1 and M_2 such that*

$$||Tf||_{L^{1/\beta_j}(Y)} \leq M_j ||f||_{L^{1/\alpha_j}(X)}. \quad f \in D \quad j = 1, 2 \tag{5.3.1}$$

Then for $0 < \theta < 1$

$$\alpha = (1 - \theta)\alpha_1 + \theta\alpha_2 \tag{5.3.2}$$

$$\beta = (1 - \theta)\beta_1 + \theta\beta_2. \tag{5.3.3}$$

We have

$$||Tf||_{L^{1/\beta_j}(Y)} \leq M_1^{1-\theta} M_2^{\theta} ||f||_{L^{1/\alpha_j}(X)}. \tag{5.3.4}$$

Theorem 5.3.2. *Let $F \in L^r_{m_\nu}(G), 1 \leq r < \infty$ and $f \in \bigcap_{1 \leq p < \infty} L^p_{m_\nu}(G)$. Then there exists a unique localization operator $L_{F,f} : L^p_{m_\nu}(G) \to L^p_{m_\nu}(G)$ such that*

$$\langle L_{F,f} u, v\rangle = \frac{1}{C_f} \int\limits_G F(g) \langle u, \pi_g f\rangle \langle \pi_g f, v\rangle dm_\nu(g). \tag{5.3.5}$$

Moreover,

$$\|L_{F,f}\|_{B(L^p_{m_\nu}(G))} \leq \frac{1}{C_f} \|f\|^{\frac{1}{r}}_{L^p_{m_\nu}(G)} \|f\|^{\frac{1}{r}}_{L^{p'}_{m_\nu}(G)} \|f\|^{\frac{2}{r'}}_{L^1_{m_\nu}(G)} \|F\|_{L^r_{m_\nu}(G)}.$$

Proof. Let $T_u f = L_{F,f}u$, for $u \in D$, where T_u be the linear transformation with domain D consisting of all simple functions on G with property that

$$\nu\{g \in G : F(g) \neq 0\} < \infty.$$

Then from Theorem 5.2.4 and Theorem 5.2.5, we get

$$\begin{aligned} \|T_u F\|_{(L^p_{m_\nu}(G))} &= \|L_{F,f}u\|_{(L^p_{m_\nu}(G))} \\ &\leq \frac{1}{C_f}\|f\|_{L^p_{m_\nu}(G)}\|f\|_{L^{p'}_{m_\nu}(G)}\|u\|_{L^p_{m_\nu}(G)}\|F\|_{L^1_{m_\nu}(G)}. \end{aligned} \tag{5.3.6}$$

and

$$\begin{aligned} \|T_u F\|_{(L^p_{m_\nu}(G))} &= \|L_{F,f}u\|_{(L^p_{m_\nu}(G))} \\ &\leq \frac{1}{C_f}\|F\|_{L^\infty_{m_\nu}(G)}\|f\|^2_{L^1_{m_\nu}(G)}\|u\|_{L^p_{m_\nu}(G)}, f \in D. \end{aligned} \tag{5.3.7}$$

With a view to applying the Reisz-Thorin interpolation theorem, we let

$$\alpha_1 = 1, \alpha_2 = 0 \quad \text{and} \quad \beta_1 = \beta_2 = \tfrac{1}{p}.$$

Let $\alpha = \frac{1}{r}$ then $\theta = \frac{1}{r'}$ where r' is the conjugate index of r. Hence $\alpha = \frac{1}{r}, \beta = \frac{1}{p'}$. From (5.3.6) and (5.3.7), we obtain

$$\begin{aligned} \|L_{F,f}u\|_{(L^p_{m_\nu}(G))} &= \|T_u F\|_{(L^p_{m_\nu}(G))} \\ &\leq \frac{1}{C_f}\|f\|^{\frac{1}{r}}_{L^p_{m_\nu}(G)}\|f\|^{\frac{1}{r}}_{L^{p'}_{m_\nu}(G)}\|f\|^{\frac{2}{r'}}_{L^1_{m_\nu}(G)} \\ &\quad \times \|F\|_{L^r_{m_\nu}(G)}\|u\|_{L^p_{m_\nu}(G)}, \end{aligned} \tag{5.3.8}$$

for all $u \in D$. Since D is dense in $L^p_{m_\nu}(G)$. So the proof is completed.

□

5.4 Wavelet multipliers

Here, in this section, we consider the relationship between localization operators and wavelet multipliers associated with

$$\pi : \mathbb{R}^+ \to B(L^p(\mathbb{R}^+)) \quad 1 \leq p < \infty$$

is the integral representation on $L^p(\mathbb{R}^+)$, which is defined by

$$(\pi_g f)(h) = \int_0^\infty f(t)w(h, g, t)dt, \tag{5.4.1}$$

for $g, h \in (0, \infty)$ for $f \in L^p(0, \infty)$.

Theorem 5.4.1. *Let* $f \in \bigcap_{1 \leq p < \infty} L^p(0, \infty)$ *be such that* $\|f\|_{L^2(0,\infty)} = 1$. *Then, we have*

$$C_f = \int_0^\infty |\langle f, \pi_g f\rangle|^2 dg = ||f||_2^4.$$

Proof. Using (5.4.1), we get

$$C_f = \int_0^\infty \left|\left\langle f, \int_0^\infty f(z)w(x, g, z)dz\right\rangle\right|^2 dg.$$

In view of (1.4.7), we have

$$C_f = \int_0^\infty \left|\left\langle f, \int_0^\infty f(z)\left(\int_0^\infty k(xt)k(gt)k(zt)dt\right) dz\right\rangle\right|^2 dg.$$

Thus by definition of the Watson transform (1.4.1), we find that

$$C_f = \int_0^\infty \left|\left\langle f, \int_0^\infty \left(\int_0^\infty f(z)k(zt)dz\right) k(xt)k(gt)dt\right\rangle\right|^2 dg.$$

Therefore, we obtain

$$\begin{aligned} C_f &= \int_0^\infty \left| \left\langle f, \int_0^\infty k(xt)k(gt)(Wf)(t)dt \right\rangle \right|^2 dg \\ &= \int_0^\infty \left| \int_0^\infty f(x) \left(\int_0^\infty k(xt)k(gt)(Wf)(t)dt \right) dx \right|^2 dg. \end{aligned}$$

Again applying (1.4.1), we get

$$\begin{aligned} C_f &= \int_0^\infty \left| \int_0^\infty k(gt)(Wf)(t) \left(\int_0^\infty k(xt)f(x)dx \right) dt \right|^2 dg \\ &= \int_0^\infty \left| \int_0^\infty k(gt)(Wf)(t)(Wf)(t)dt \right|^2 dg \\ &= \int_0^\infty |W((Wf)^2)(g)|^2 dg \\ &= ||Wf||_2^4. \end{aligned}$$

Now, by using the Parseval formula of the Watson transform (1.4.12), we have

$$C_f = ||f||_2^4. \tag{5.4.2}$$

□

With the help of [78, p. 2913], we define wavelet multiplier involving Watson transform.

Definition 5.4.2. Let $\sigma \in L^1(0,\infty) \cap L^\infty(0,\infty)$. Then the wavelet multiplier $P_{\sigma,f} : L^p(0,\infty) \to L^p(0,\infty)$ associated with symbol σ and the admissible wavelet f is defined by

$$(P_{\sigma,f}u)(x) = \int_0^\infty \sigma(g)\langle u, \pi_g f\rangle(\pi_g f)(x)dg. \tag{5.4.3}$$

Theorem 5.4.3. *Let $\sigma \in L^r(0,\infty), 1 \le r < \infty$ and $f \in \bigcap_{1\le p<\infty} L^p(0,\infty)$. Then there exists a unique localization operator $L_{\sigma,f} : L^p(0,\infty) \to L^p(0,\infty)$ such that*

$$\langle L_{\sigma,f}u, v\rangle = \frac{1}{C_f}\int_0^\infty \sigma(g)\langle u, \pi_g f\rangle\langle \pi_g f, v\rangle dg.$$

Moreover,

$$\langle L_{\sigma,f}u, v\rangle = \|f\|^{-4}_{L^2(0,\infty)}\langle W^{-1}\left(\bar{\hat{f}}W^{-1}[\sigma(g)W(WuWf)(g)]\right), v\rangle.$$

Proof. Now for $\sigma \in L^r(0,\infty), 1 \le r < \infty$ and $u, v \in T(\lambda,\mu)$, we have

$$\big\langle L_{\sigma,f}u, v\big\rangle = \frac{1}{C_f}\int_0^\infty \sigma(g)\langle u, \pi_g f\rangle\langle \pi_g f, v\rangle dg. \tag{5.4.4}$$

Since

$$(\pi_g f)(x) = \int_0^\infty f(t)w(x,g,t)dt.$$

Using (1.4.7), we find that

$$\begin{aligned}(\pi_g f)(x) &= \int_0^\infty f(t)\left(\int_0^\infty k(xt')k(gt')k(tt')dt'\right)dt\\ &= \int_0^\infty \left(\int_0^\infty f(t)k(tt')dt\right)k(xt')k(gt')dt'.\end{aligned}$$

By definition of Watson transform (1.4.1), we get

$$\begin{aligned}(\pi_g f)(x) &= \int_0^\infty (Wf)(t')k(xt')k(gt')dt'\\ &= \int_0^\infty k(xt')k(gt')(Wf)(t')dt'\\ &= W^{-1}\Big(k(gt')(Wf)(t')\Big)(x). \end{aligned} \tag{5.4.5}$$

From (5.4.4) and (5.4.5), we have

$$\begin{aligned}\langle L_{\sigma,f}u, v\rangle &= \frac{1}{C_f}\int_0^\infty \sigma(g)\big\langle u, W^{-1}(k(gt')(Wf)(t'))\big\rangle \\ &\quad \times \big\langle W^{-1}(k(gt')(Wf)(t')), v\big\rangle dg \\ &= \frac{1}{C_f}\int_0^\infty \sigma(g)\big\langle (Wu)(t'), k(gt')(Wf)(t')\big\rangle \\ &\quad \times \big\langle k(gt')(Wf)(t'), (Wv)(t')\big\rangle dg.\end{aligned}$$

By (5.4.2), we obtain

$$\begin{aligned}\big\langle L_{\sigma,f}u, v\big\rangle &= \|f\|^{-4}_{L^2(0,\infty)}\int_0^\infty \sigma(g)W(WuWf)(g)W(WfWv)(g)dg \\ &= \|f\|^{-4}_{L^2(0,\infty)}\big\langle \sigma(g)W(WuWf)(g), W(WfWv)(g)\big\rangle \\ &= \|f\|^{-4}_{L^2(0,\infty)}\big\langle W^{-1}[\sigma(g)W(WuWf)(g)], WvWf\big\rangle \\ &= \|f\|^{-4}_{L^2(0,\infty)}\big\langle \bar{\hat{f}}W^{-1}[\sigma(g)W(WuWf)(g)], Wv\big\rangle \\ &= \|f\|^{-4}_{L^2(0,\infty)}\big\langle W^{-1}\left(\bar{\hat{f}}W^{-1}[\sigma(g)W(WuWf)(g)]\right), v\big\rangle. \qquad (5.4.6)\end{aligned}$$

The above relation (5.4.6) indicates that the localization operator gets converted into a wavelet multiplier and pseudo-differential operators. Thus, when p=2, (5.4.6) shows that the localization operator $L_{\sigma,f} : L^2(0,\infty) \to L^2(0,\infty)$ is unitary equivalent to Wavelet multiplier $P_{\sigma,f} : L^2(0,\infty) \to L^2(0,\infty)$. □

5.5 Application of localization operators.

In this section, we study the trace class and Schatten-von Neumann classes. We find the trace class of localization operators with the help of some useful results of the book [55, p. 211], by Reed and Simon.

Proposition 5.5.1. *Let $A : X \to X$ be a bounded linear operator in S_1 and let $\{\phi_k : k = 1, 2, 3 \ldots\}$ be any orthonormal basis for Hilbert space X. Then the series $\sum_{k=1}^{\infty}\langle A\phi_k, \phi_k\rangle$ is absolutely convergent and the sum is independent of the choice of the orthonormal basis $\{\phi_k : k = 1, 2, 3 \ldots\}$.*

Remark: In view of the above Proposition, we can define the trace $tr(A)$ of any linear operator $A : X \to X$ in S_1 by

$$tr(A) = \sum_{k=1}^{\infty}\langle A\phi_k, \phi_k\rangle. \tag{5.5.1}$$

Theorem 5.5.2. *Let $F \in L^1_{m_\nu}(G)$ and $\|f\|_2^2 = 1$. Then the trace class of localization operator is estimated by*

$$\big|tr(L_{F,f})\big| \leq A' \frac{1}{C_f}\int_G |F(g)| dm_\nu(g). \tag{5.5.2}$$

Proof. Let $\{\phi_k\}$ be a sequence of orthonormal basis for X. Then, by using formula (5.2.5) and (5.5.1), and the Fubini's theorem, we have

$$\begin{aligned}
\big|tr(L_{F,f})\big| &= \Big|\sum_{k=1}^{\infty}\langle L_{F,f}\phi_k, \phi_k\rangle\Big| \\
&= \Big|\sum_{k=1}^{\infty}\frac{1}{C_f}\int_G F(g)\langle\phi_k, \pi_g f\rangle\langle\pi_g f, \phi_k\rangle dm_\nu(g)\Big| \\
&\leq \sum_{k=1}^{\infty}\Big|\frac{1}{C_f}\int_G F(g)\langle\phi_k, \pi_g f\rangle\langle\pi_g f, \phi_k\rangle dm_\nu(g)\Big| \\
&\leq \sum_{k=1}^{\infty}\frac{1}{C_f}\int_G \big|F(g)\langle\phi_k, \pi_g f\rangle\langle\pi_g f, \phi_k\rangle\big| dm_\nu(g).
\end{aligned} \tag{5.5.3}$$

Exploiting the results of [54, p. 402], we get

$$\big|tr(L_{F,f})\big| \leq \frac{1}{C_f}\int_G |F(g)|(||\pi_g f||_2^2) dm_\nu(g). \tag{5.5.4}$$

From (5.4.1), we get

$$
\begin{aligned}
||\pi_g f||_2^2 &= \int_G |(\pi_g f)(x)|^2 dm_\nu(x) \\
&= \int_G \Big| \left(\int_G f(t) w(x,g,t) dm_\nu(t) \right) \Big|^2 dm_\nu(x).
\end{aligned}
$$

Using (1.4.7), we have

$$
\begin{aligned}
||\pi_g f||_2^2 &= \int_G \Big| \int_G f(t) \left(\int_G k(x\xi)k(g\xi)k(t\xi) dm_\nu(\xi) \right) dm_\nu(t) \Big|^2 dm_\nu(x) \\
&= \int_G \Big| \int_G \left(\int_G f(t) k(t\xi) dm_\nu(t) \right) k(x\xi)k(g\xi) dm_\nu(\xi) \Big|^2 dm_\nu(x).
\end{aligned}
$$

Moreover, by using (1.4.1), we get

$$
\begin{aligned}
||\pi_g f||_2^2 &= \int_G \Big| \left(\int_G k(g\xi)k(x\xi)(Wf)(\xi) dm_\nu(\xi) \right) \Big|^2 dm_\nu(x) \\
&= \int_G \big| W(k(g\xi)(Wf)(\xi))(x) \big|^2 dm_\nu(x).
\end{aligned}
$$

Applying Parseval relation of watson transform (1.4.12), we find that

$$
\begin{aligned}
||\pi_g f||_2^2 &\leq |k(g\xi)| \int_G |(Wf)(\xi)|^2 dm_\nu(\xi) \\
&\leq A' \int_G |(Wf)(\xi)|^2 dm_\nu(\xi) \\
&\leq A' ||f||_2^2.
\end{aligned} \tag{5.5.5}
$$

Using (5.5.4) and (5.5.5), we obtained the required results

$$
\big| tr(L_{F,f}) \big| \leq A' \frac{1}{C_f} \int_G |F(g)| dm_\nu(g).
$$

□

Proposition 5.5.3. *Let $F \in L^1(G) \cap L^2(G)$, then for m= 1,2,3....we have*

$$\big|tr(L^m_{F,f})\big| \leq K'\big(tr(L_{F,f})\big)^m. \tag{5.5.6}$$

Proof. Since $L_{F,f} : X \to X$ is in trace class S_1 and the adjoint of $L_{F,f} : X \to X$ is $L_{\bar{F},f} : X \to X$ from [76, p. 80], by using (5.5.1), we have

$$\begin{aligned}
\big|tr(L^m_{F,f})\big| &= \Big|\sum_{k=1}^{\infty}\langle L^m_{F,f}\phi_k, \phi_k\rangle\Big| \\
&= \Big|\sum_{k=1}^{\infty}\langle L_{F,f}\phi_k, L^{m-1}_{\bar{F},f}\phi_k\rangle\Big| \\
&= \Big|\sum_{k=1}^{\infty}\int_G F(g)\langle\phi_k, \pi_g f\rangle\langle\pi_g f, L^{m-1}_{\bar{F},f}\phi_k\rangle dm_\nu(g)\Big| \\
&\leq \sum_{k=1}^{\infty}\int_G \Big|F(g)\langle\phi_k, \pi_g f\rangle\langle L^{m-1}_{F,f}\pi_g f, \phi_k\rangle\Big| dm_\nu(g).
\end{aligned}$$

Therefore, by applying the Cauchy-Schwarz inequality, we get

$$\big|tr(L^m_{F,f})\big| \leq \sum_{k=1}^{\infty}\int_G |F(g)| \cdot ||\phi_k|| \cdot ||\pi_g f|| \cdot |\langle L^{m-1}_{F,f}\pi_g f, \phi_k\rangle| dm_\nu(g). \tag{5.5.7}$$

We now evaluate $\big|\big\langle L^{m-1}_{F,f}\pi_g f, \phi_k\big\rangle\big|$ for m=1,2,3...

Case 1. When m=2, we have

$$\begin{aligned}
\big|\big\langle L_{F,f}\pi_g f, \phi_k\big\rangle\big| &\leq \int_G |F(g)| \cdot |\langle\pi_g f, \pi_g f\rangle| \cdot |\langle\phi_k, \pi_g f\rangle| dm_\nu(g) \\
&\leq \int_G |F(g)| \cdot ||\pi_g f||^3 ||\phi_k|| dm_\nu(g) \\
&\leq \int_G |F(g)| \cdot ||\pi_g f||^3 ||\phi_k|| dm_\nu(g) \\
&\leq sup||\pi_g f||^3 ||\phi_k|| \int_G |F(g)| dm_\nu(g).
\end{aligned} \tag{5.5.8}$$

Case 2. When m=3, we have

$$\begin{aligned}\left|\left\langle L_{F,f}^{2}\pi_g f,\phi_k\right\rangle\right| &= \left|\left\langle L_{F,f}\pi_g f, L_{\tilde{F},f}\phi_k\right\rangle\right| \\ &\leq \int_G |F(g)|\cdot|\langle\pi_g f,\pi_g f\rangle|\cdot|\langle\pi_g f, L_{\tilde{F},f}\phi_k\rangle| dm_\nu(g) \\ &\leq \int_G |F(g)|\cdot|\langle\pi_g f,\pi_g f\rangle|\cdot|\langle L_{F,f}\pi_g f,\phi_k\rangle| dm_\nu(g).\end{aligned}$$

Using (5.5.8), we have

$$\begin{aligned}\left|\left\langle L_{F,f}^{2}\pi_g f,\phi_k\right\rangle\right| &\leq \int_G |F(g)|\cdot||\pi_g f||^2\left(\int_G |F(g)|\cdot||\pi_g f||^3||\phi_k||dm_\nu(g)\right)dm_\nu(g) \\ &\leq \int_G\left(\int_G |F(g)|\cdot|F(g)|\cdot||\pi_g f||^5||\phi_k||dm_\nu(g)\right)dm_\nu(g) \\ &\leq sup||\pi_g f||^5||\phi_k||\left(\int_G |F(g)|dm_\nu(g)\right)^2. \qquad (5.5.9)\end{aligned}$$

In a similar manner, with m replaced by m-1, we find that

$$\begin{aligned}\left|\left\langle L_{F,f}^{m-1}\pi_g f,\phi_k\right\rangle\right| &= \left|\left\langle L_{F,f}\pi_g f, L_{\tilde{F},f}^{m-2}\phi_k\right\rangle\right| \\ &\leq sup||\pi_g f||^{2m-1}||\phi_k||\left(\int_G |F(g)|dm_\nu(g)\right)^{m-1}. \qquad (5.5.10)\end{aligned}$$

Using (5.5.10),(5.5.9),(5.5.8) in (5.5.7), we get

$$\left|tr(L_{F,f}^{m})\right| \leq K'(tr(L_{F,f}))^m.$$

□

Theorem 5.5.4. *Let F be a non negative function in $L^1_{m_\nu}(G)$. Then, for p=1,2,3...*

$$||L_{F,f}||_{S_p} \leq K' tr(L_{F,f}).$$

Proof. Let $\{\psi_k\}$ be an orthonormal basis in X and let a Schatten-von Neumann class of localization operator is given by

$$\begin{aligned} ||L_{F,f}||_{S_p} &= \left(\sum_{k=1}^{\infty} \left\langle L^p_{F,f}\psi_k, \psi_k\right\rangle\right)^{1/p} \\ &= \left(tr(L^p_{F,f})\right)^{1/p}. \end{aligned}$$

Then, by Proposition [5.5.3], we get the required result

$$||L_{F,f}||_{S_p} \leq K' tr(L_{F,f}).$$

□

5.6 Conclusions

Localization operators and wavelet multipliers are known to play constructive roles in the problems of image processing and signal processing by exploiting the theory of many different integral transforms. In the present chapter, we have studied the various properties of localization operators associated with the integral representation involving the Watson transform. Among other results, we found the relation between localization operators and wavelet multipliers. From the monumental works by Schuitman [60] and Titchmarsh [64], we observe that the results involving the Watson transform are more general than the Laplace transform, the Fourier transform, the Hankel transform and other integral transforms. We are also introduced the properties of the trace class and the Schatten-von Neumann class for localization operators which are considered here.

***