Chapter 4

Wavelet multiplier associated with the Watson transform

4.1 Introduction

Among many integral transforms, the Watson transform is an important tool that contains a deep mathematical background. The theory of Watson transform was initiated by Hardy and Titchmarsh [64] in 1948 and they studied many interesting properties of Watson transform. In 1949, Bochner and Chandrashekharan [7], discussed the properties of Watson transform and solved the functional equation. In 1960, Goldberg considered the Watson transform on groups in his research paper [24] and found many interesting results. In 1976, Braaksma and Schuitman [8] observed some classes and related integral equation for generalized function associated with the Watson transform.

The Watson transform is an exciting tool and significant in the sense that it is a generalization of the Hankel transform, G-transform, H-transform and other integral transforms. This theory is widely discussed in the book of Schuitman [60], and Titchmarsh [64]. As per Schuitman [60], the Laplace transform, the Fourier transform and the Hankel transform are examples of the Watson transforms. The complete mathematical relation of the Watson transform with the Hankel transform are given in [33, p. 36].

Using the theory of the Hankel transform, Pandey et al. [40], Upadhyay et al. [43] and many others discussed the boundedness of pseudo-differential operators associated with the Bessel operator. Ghobber [22], and Mejjaoli [36] found the properties of wavelet multipliers with the help of the Hankel transform technique.

Motivated from [16, 22, 26, 34, 36], our main objective of this chapter is to investigate the Watson wavelet multiplier associated with the unitary representation and discuss its boundedness on L^p space. Using the Watson transform theory, we find Hilbert-Schmidt class, compactness of Watson wavelet multiplier. Some applications and relationship of Sobolev-type spaces with Watson wavelet multiplier are also given.

From the above discussions, our present chapter is organized by the following way:

Section 4.1 is introductory, it gives brief history and motivation about this chapter. In Section 4.2, for $1 \leq p \leq \infty$ it is shown that the Watson wavelet multiplier $P_{\sigma,f}$ is bounded linear operator for a suitable choice of the admissible wavelet f in $L^1(0,\infty) \cap L^\infty(0,\infty)$ and symbols $\sigma \in L^1(0,\infty)$. It is also shown that the Watson wavelet multiplier $P_{\sigma,f}$ are bounded linear operator on $L^p(0,\infty)$, $r \leq p \leq r'$ associated with the symbol $\sigma \in L^r(0,\infty)$ for $1 \leq r \leq 2$ and admissible wavelet $f \in L^1(0,\infty) \cap L^2(0,\infty) \cap L^\infty(0,\infty)$. In Section 4.3, we have shown that the Watson wavelet multiplier $P_{\sigma,f} : L^2(0,\infty) \to L^2(0,\infty)$ is a Hilbert-Schmidt operator and observed its compactness property for the suitable choice of admissible wavelet associated with the unitary representation. In Section 4.4, the Landau-Pollak-Slepian operator associated with the unitary representation is introduced using Watson transform technique. With the Watson wavelet multiplier, Section 4.5 studied Sobolev-type spaces and their various properties. In Section 4.6, the trace class of Watson wavelet multiplier is obtained.

The space $T(\lambda,\mu)$ consists of all functions $\phi \in \mathbb{C}^\infty(0,\infty)$ with the property that

$$\beta_n(\phi) = \sup_{\substack{t>0 \\ \rho=0,1,2....n \\ \lambda_n \le c \le \mu_n}} |t^{c+p}\phi^p(t)| < \infty \qquad \forall n \in N_0 \tag{4.1.1}$$

where $\lambda, \mu \in R^* = R \cup \{-\infty, \infty\}$, $\lambda < \mu$.

A complex-valued continuous function σ defined on $I = (0,\infty)$ is a symbol belongs to the class T_m if and only if there exists a constant $C > 0$ such that

$$|\sigma(y)| \le C(1+y)^m, \tag{4.1.2}$$

where m is a fixed real number.

Then the pseudo-differential operator $A(x,D)$ associated with the symbol σ is defined by

$$A(x,D)f(x) = \int_0^\infty k(xy)\sigma(y)(Wf)(y)dy, \qquad f \in T(\lambda,\mu), \tag{4.1.3}$$

where Wf is the Watson transform of f and k is the kernel of the Watson transform given in (1.4.3).

Let $\sigma \in L^\infty(0,\infty)$. Then we define a linear operator $A(x,D) : L^2(0,\infty) \to L^2(0,\infty)$ by

$$A(x,D)u = W^{-1}(\sigma W u), \quad u \in L^2(0,\infty), \tag{4.1.4}$$

where Wu is the Watson transform of u.

With the help of Wong [34], let $\pi : \mathbb{R}^+ \to U(L^2(\mathbb{R}^+))$ be the unitary representation of the multiplicative group $\mathbb{R}^+ = (0,\infty)$ on $L^2(0,\infty)$ is defined by

$$(\pi_\xi u)(x) = k(x\xi)u(x) \qquad x,\xi \in (0,\infty), \tag{4.1.5}$$

for all functions u in $L^2(0,\infty)$, where $U(L^2(0,\infty))$ is the group of all unitary operators on $L^2(0,\infty)$.

Let $\sigma \in L^1(0,\infty)\cap L^\infty(0,\infty)$ and $f \in L^2(0,\infty)\cap L^\infty(0,\infty)$. Then for $u,v \in T(\lambda,\mu)$, we give the definition of localization operator $P_{\sigma,f}$ with the help of the unitary representation (4.1.5)

$$\langle P_{\sigma,f}u,v\rangle = \int_0^\infty \sigma(\xi)\langle u,\pi_\xi f\rangle\langle \pi_\xi f,v\rangle d\xi, \quad \xi \in (0,\infty), \tag{4.1.6}$$

where $\langle , \rangle$ represents

$$\langle f,g\rangle = \int_0^\infty f(y)\bar{g}(y)dy \quad \text{for all } f,g \in L^2(0,\infty). \tag{4.1.7}$$

Remark: $P_{\sigma,f}$ initially defined on $T(\lambda,\mu)$, can be extended to a bounded linear operator on $L^2(0,\infty)$. From [47] it can be easily shown that $P_{\sigma,f}$ is a continuous linear operator from $T(\lambda,\mu)$ to $T(\lambda,\mu)$.

Lemma 4.1.1. *For u,v, $f \in L^2(0,\infty)$, we obtain the following relations:*

$$\langle u,\pi_\xi f\rangle = W(u\bar{f})(\xi) \tag{4.1.8}$$

and

$$\langle \pi_\xi f,v\rangle = W(f\bar{v})(\xi). \tag{4.1.9}$$

Proof. Since

$$\langle u, \pi_\xi f\rangle = \int_0^\infty u(x)\overline{(\pi_\xi f)(x)}dx.$$

By (4.1.5), we get

$$\begin{aligned}\langle u, \pi_\xi f\rangle &= \int_0^\infty u(x)\overline{k(x\xi)f(x)}dx \\ &= W(u\bar{f})(\xi).\end{aligned}$$

In a similar way, we prove the following:

$$\begin{aligned}\langle \pi_\xi f, v\rangle &= \int_0^\infty (\pi_\xi f)(y)\overline{v}(y)dy \\ &= \int_0^\infty k(y\xi)f(y)\overline{v}(y)dy \\ &= W(f\overline{v})(\xi).\end{aligned}$$

□

Lemma 4.1.2. *Let* $f \in L^2(0,\infty) \cap L^\infty(0,\infty)$*, and* $\|f\|_2 = 1$*. Then for functions* $u, v \in T(\lambda,\mu)$*, we have*

$$\int_0^\infty \langle u, \pi_\xi f\rangle\langle \pi_\xi f, v\rangle d\xi = \langle u\bar{f}, f\overline{v}\rangle. \tag{4.1.10}$$

Proof. Using Lemma 4.1.1, we have

$$\int_0^\infty \langle u, \pi_\xi f\rangle\langle \pi_\xi f, v\rangle d\xi = \int_0^\infty W(u\bar{f})(\xi)W(f\overline{v})(\xi)d\xi.$$

By (4.1.7) and the Parseval relation of the Watson transform (1.4.12), the above yields

$$\int_0^\infty \langle u, \pi_\xi f\rangle \langle \pi_\xi f, v\rangle d\xi = \langle u\bar{f}, f\bar{v}\rangle.$$

□

Lemma 4.1.3. *Let $f \in L^\infty(0,\infty)$ and $u \in L^1(0,\infty)$, then*

$$\|W(uf)\|_\infty \leq C\|u\|_1\|f\|_\infty. \tag{4.1.11}$$

Proof. Using (1.4.1), we have

$$\begin{aligned} |W(uf)(x)| &= |\int_0^\infty k(x\xi)(uf)(\xi)d\xi| \\ &\leq \int_0^\infty |k(x\xi)| \cdot |(uf)(\xi)|d\xi \\ &\leq C\int_0^\infty |u(\xi)| \cdot |f(\xi)|d\xi \\ &\leq Csup|f(\xi)|\int_0^\infty |u(\xi)|d\xi \\ &\leq C\|f\|_\infty\|u\|_1. \end{aligned}$$

□

Theorem 4.1.4. *Let $\sigma \in L^\infty(0,\infty)$, $f \in L^2(0,\infty) \cap L^\infty(0,\infty)$, and $u, v \in T(\lambda,\mu)$, such that $\|f\|_2 = 1$. Then we have*

$$\langle P_{\sigma,f}u, v\rangle = \langle fA(x,D)\bar{f}u, v\rangle. \tag{4.1.12}$$

Proof. Using (4.1.6), we have

$$\langle P_{\sigma,f}u, v\rangle = \int_0^\infty \sigma(\xi)\langle u, \pi_\xi f\rangle\langle \pi_\xi f, v\rangle d\xi, \quad \xi \in (0,\infty).$$

By Lemma 4.1.1

$$\langle P_{\sigma,f}u, v\rangle = \int_0^\infty \sigma(\xi)W(u\bar{f})(\xi)W(f\overline{v})(\xi)d\xi.$$

From (1.4.1), we get

$$\begin{aligned}\langle P_{\sigma,f}u, v\rangle &= \int_0^\infty \sigma(\xi)W(u\bar{f})(\xi)\left(\int_0^\infty k(x\xi)(f\overline{v})(x)dx\right)d\xi \\ &= \int_0^\infty \left(\int_0^\infty k(x\xi)\sigma(\xi)W(u\bar{f})(\xi)d\xi\right)(f\overline{v})(x)dx \\ &= \int_0^\infty f(x)W^{-1}\left(\sigma(\xi)W(u\bar{f})\right)(x)\overline{v}(x)dx.\end{aligned}$$

By the inner product definition (4.1.7), we obtain

$$\begin{aligned}\langle P_{\sigma,f}u, v\rangle =&\langle fW^{-1}(\sigma(\xi)W(u\bar{f})), v\rangle \\ =&\langle (fA(x,D)\bar{f})u, v\rangle.\end{aligned}$$

□

Remark: From Theorem 4.1.4, it is proved that a bounded linear operator $P_{\sigma,f} : L^2(0,\infty) \to L^2(0,\infty)$ and $fA(x,D)\bar{f} : L^2(0,\infty) \to L^2(0,\infty)$ are unitarily equivalent, hence we denote $fA(x,D)\bar{f}$ as $P_{\sigma,f}$. Here f plays the role of admissible wavelet in the localization operator $P_{\sigma,f}$ then the localization operator $P_{\sigma,f}$ is the Watson wavelet multiplier associated with the unitary representation $\pi : \mathbb{R}^+ \to U(L^2(\mathbb{R}^+))$ of the multiplicative group $\mathbb{R}^+ = (0,\infty)$.

A function $f \in L^2(0,\infty)$ which satisfies $\|f\|_2 = 1$ and

$$\int_0^\infty |\langle f, \pi_\xi f\rangle|^2 d\xi < \infty \tag{4.1.13}$$

is called an admissible wavelet of $\pi : \mathbb{R}^+ \to U(L^2(\mathbb{R}^+))$. Now we denote

$$\begin{aligned} C_f &= \int_0^\infty \left|\langle f, \pi_\xi f\rangle\right|^2 d\xi \\ &= \int_0^\infty \Big| \int_0^\infty f(x)(\pi_\xi f)(x) dx\Big|^2 d\xi. \end{aligned} \tag{4.1.14}$$

From (4.1.5), the above can be written as

$$\begin{aligned} C_f &= \int_0^\infty \Big| \int_0^\infty f(x)k(x\xi)f(x)dx\Big|^2 d\xi \\ &= \int_0^\infty \Big| \int_0^\infty k(x\xi)f^2(x)dx\Big|^2 d\xi. \end{aligned}$$

In view of (1.4.1), we get

$$C_f = \int_0^\infty \left|W(f^2)(\xi)\right|^2 d\xi.$$

Now by the Parseval formula of the Watson transform (1.4.12), we have

$$C_f = \|f\|_2^4. \tag{4.1.15}$$

4.2 Boundedness of wavelet multipliers

This section discuss the boundedness of the Watson wavelet multiplier $P_{\sigma,f}$ on $L^p(0,\infty)$, for $1 \le p \le \infty$, for a suitable choice of the admissible wavelet associated with the unitary representation.

The Watson wavelet multiplier $P_{\sigma,f}$ for $\sigma \in L^1(0,\infty)$ and $f \in L^1(0,\infty) \cap L^\infty(0,\infty)$ is defined by

$$(P_{\sigma,f}u)(x) = \frac{1}{C_f}\int_0^\infty \sigma(\xi)\langle u, \pi_\xi f\rangle(\pi_\xi f)(x)d\xi, \quad x,\xi \in (0,\infty), \tag{4.2.1}$$

for all $u \in L^1(0,\infty)$ and $\langle , \rangle$ denotes the

$$\langle f, g\rangle = \int_0^\infty f(x)\bar{g}(x)dx, \quad \text{where } f, g \text{ are measurable function.}$$

Theorem 4.2.1. *Let $\sigma \in L^1(0,\infty)$ and $f \in L^1(0,\infty) \cap L^\infty(0,\infty)$. Then the Watson wavelet multiplier $P_{\sigma,f} : L^1(0,\infty) \to L^1(0,\infty)$ is a bounded linear operator and*

$$\|P_{\sigma,f}\|_{B(L^1(0,\infty))} \leq \frac{1}{C_f}K\|\sigma\|_1\|f\|_1\|f\|_\infty, \tag{4.2.2}$$

where $\| \ \|_{B(L^1(0,\infty))}$ is the norm in the Banach space $B(L^1(0,\infty))$ of all bounded linear operators from $L^1(0,\infty)$ into $L^1(0,\infty)$.

Proof. Using (4.2.1), we have

$$(P_{\sigma,f}u)(x) = \frac{1}{C_f}\int_0^\infty \sigma(\xi)\langle u, \pi_\xi f\rangle(\pi_\xi f)(x)d\xi.$$

Now, we have

$$\begin{aligned}\|P_{\sigma,f}u\|_1 &= \frac{1}{C_f}\int_0^\infty \left|\int_0^\infty \sigma(\xi)\langle u, \pi_\xi f\rangle(\pi_\xi f)(x)d\xi\right|dx \\ &\leq \frac{1}{C_f}\int_0^\infty \left(\int_0^\infty |\sigma(\xi)|\cdot|\langle u, \pi_\xi f\rangle|\cdot|(\pi_\xi f)(x)|d\xi\right)dx.\end{aligned}$$

In view of (4.1.5), the above yields

$$\|P_{\sigma,f}u\|_1 \le \frac{1}{C_f}\int_0^\infty \left(\int_0^\infty |\sigma(\xi)|\cdot|\langle u,\pi_\xi f\rangle|\cdot|k(x\xi)f(x)|d\xi\right)dx.$$

Taking Lemma 4.1.1, we obtain

$$\|P_{\sigma,f}u\|_1 \le \frac{1}{C_f}\int_0^\infty \left(\int_0^\infty |\sigma(\xi)|\cdot|W(uf)(\xi)|\cdot|k(x\xi)f(x)|d\xi\right)dx.$$

Since $k(x\xi)$ is a kernel of Watson transform so it is bounded by some constant K on $I=(0,\infty)$.

$$\begin{aligned}\|P_{\sigma,f}u\|_1 &\le \frac{1}{C_f}K\sup_{\xi\in(0,\infty)}|W(uf)(\xi)|\int_0^\infty\left(\int_0^\infty|\sigma(\xi)|d\xi\right)|f(x)|dx\\ &\le \frac{1}{C_f}K\|W(uf)\|_\infty\|\sigma\|_1\|f\|_1.\end{aligned}$$

From Lemma 4.1.3, the above yields

$$\|P_{\sigma,f}u\|_1 \le \frac{1}{C_f}K\|u\|_1\|f\|_\infty\|\sigma\|_1\|f\|_1. \tag{4.2.3}$$

Hence, the required result is

$$\|P_{\sigma,f}\|_{B(L^1(0,\infty))} \le \frac{1}{C_f}K\|f\|_\infty\|\sigma\|_1\|f\|_1.$$

□

Theorem 4.2.2. *Let $\sigma\in L^1(0,\infty)$, and $f\in L^1(0,\infty)\cap L^\infty(0,\infty)$ then $P_{\sigma,f}: L^\infty(0,\infty)\to L^\infty(0,\infty)$ is a bounded linear operator such that*

$$\|P_{\sigma,f}\|_{B(L^\infty(0,\infty))} \le \frac{1}{C_f}K\|\sigma\|_1\|f\|_1\|f\|_\infty. \tag{4.2.4}$$

Proof. From (4.2.1), we have

$$(P_{\sigma,f}u)(x) = \frac{1}{C_f}\int_0^\infty \sigma(\xi)\langle u, \pi_\xi f\rangle(\pi_\xi f)(x)d\xi.$$

Then

$$|P_{\sigma,f}u| = |\frac{1}{C_f}\int_0^\infty \sigma(\xi)\langle u, \pi_\xi f\rangle(\pi_\xi f)(x)d\xi|.$$

From (4.1.5), we get

$$\begin{aligned}|P_{\sigma,f}u| &\leq \frac{1}{C_f}\left(\int_0^\infty |\sigma(\xi)|\cdot|\langle u, \pi_\xi f\rangle|\cdot|k(x\xi)f(x)|d\xi\right)\\ &\leq \frac{1}{C_f}K\left(\int_0^\infty |\sigma(\xi)|\cdot|\langle u, \pi_\xi f\rangle|\cdot|f(x)|d\xi\right).\end{aligned}$$

By Lemma 4.1.1

$$|P_{\sigma,f}u| \leq \frac{1}{C_f}K\left(\int_0^\infty |\sigma(\xi)|\cdot|W(uf)(\xi)|\cdot|f(x)|d\xi\right).$$

Therefore, we have

$$\begin{aligned}\|P_{\sigma,f}u\|_\infty &\leq \frac{1}{C_f}K\sup_{x\in(0,\infty)}|f(x)|\left(\int_0^\infty |\sigma(\xi)||W(uf)(\xi)|d\xi\right)\\ &\leq \frac{1}{C_f}K\|f\|_\infty\|W(uf)\|_\infty\|\sigma\|_1.\end{aligned}$$

Using Lemma 4.1.3, we get

$$\|P_{\sigma,f}u\|_\infty \leq \frac{1}{C_f}K\|f\|_\infty\|u\|_\infty\|f\|_1\|\sigma\|_1.$$

Thus

$$\|P_{\sigma,f}\|_{B(L^\infty(0,\infty))} \leq \frac{1}{C_f}K\|f\|_1\|\sigma\|_1\|f\|_\infty.$$

□

Theorem 4.2.3. *Let $f \in L^2(0,\infty) \cap L^\infty(0,\infty)$ and $\sigma \in L^2(0,\infty)$, then $P_{\sigma,f}$: $L^2(0,\infty) \to L^2(0,\infty)$ is a bounded linear operator and can be find by the following norm*

$$\|P_{\sigma,f}\|_{B(L^2(0,\infty))} \leq \frac{1}{C_f} K \|\sigma\|_2 \|f\|_2 \|f\|_\infty. \tag{4.2.5}$$

Proof. From (4.2.1), we have

$$(P_{\sigma,f}u)(x) = \frac{1}{C_f} \int_0^\infty \sigma(\xi) \langle u, \pi_\xi f \rangle (\pi_\xi f)(x) d\xi.$$

Hence

$$\begin{aligned} \|P_{\sigma,f}u\|_2^2 &= \int_0^\infty |(P_{\sigma,f}u)(x)|^2 dx \\ &= \int_0^\infty |\frac{1}{C_f} \int_0^\infty \sigma(\xi) \langle u, \pi_\xi f \rangle (\pi_\xi f)(x) d\xi|^2 dx. \end{aligned}$$

By Lemma 4.1.1 and (4.1.5), we get

$$\begin{aligned} \|P_{\sigma,f}u\|_2^2 &= \int_0^\infty |\frac{1}{C_f} \int_0^\infty \sigma(\xi) W(uf)(\xi) \left(k(x\xi) f(x)\right) d\xi|^2 dx \\ &\leq \int_0^\infty \left(\frac{1}{C_f} \int_0^\infty |\sigma(\xi)| \cdot |W(uf)(\xi)| \cdot |k(x\xi)| \cdot |f(x)| d\xi \right)^2 dx \\ &\leq K' \int_0^\infty \left(\frac{1}{C_f} \int_0^\infty |\sigma(\xi)| \cdot |W(uf)(\xi)| \cdot |f(x)| d\xi \right)^2 dx. \end{aligned}$$

Using Fubini's theorem, we have

$$\begin{aligned}
\|P_{\sigma,f}u\|_2^2 &\leq \frac{1}{C_f^2}K' \int_0^\infty |f(x)|^2 \left(\int_0^\infty |\sigma(\xi)| \cdot |W(uf)(\xi)|d\xi\right)^2 dx \\
&\leq \frac{1}{C_f^2}K' \int_0^\infty |f(x)|^2 dx \left(\int_0^\infty |\sigma(\xi)| \cdot |W(uf)(\xi)|d\xi\right)^2 \\
&= \frac{1}{C_f^2}K'\|f\|_2^2 \left(\int_0^\infty |\sigma(\xi)| \cdot |W(uf)(\xi)|d\xi\right)^2.
\end{aligned}$$

By Holder's inequality, we get

$$\|P_{\sigma,f}u\|_2 \leq \frac{1}{C_f}K\|f\|_2\|\sigma\|_2\|W(uf)\|_2.$$

Take Parseval relation of the Watson transform (1.4.12), the above can be written as

$$\begin{aligned}
\|P_{\sigma,f}u\|_2 &= \frac{1}{C_f}K\|f\|_2\|\sigma\|_2\|uf\|_2 \\
&\leq \frac{1}{C_f}K\|f\|_2\|\sigma\|_2\|u\|_2\|f\|_\infty.
\end{aligned}$$

Therefore, we obtain

$$\|P_{\sigma,f}\|_{B(L^2(0,\infty))} \leq \frac{1}{C_f}K\|f\|_2\|\sigma\|_2\|f\|_\infty.$$

□

We can now state and prove a theorem on Lp-boundedness of Watson wavelet multiplier with the help of Risez-Thorin theorem from [78, p. 2916].

Theorem 4.2.4. *Let (X,μ) be a measure space and (Y,ν) be a σ-finite measure space. Let T be a linear transformation with domain D consisting of all μ-simple functions f on X such that $\mu\{x \in X : f(x) \neq 0\} < \infty$ and such that range of T is* *contained in the set of all* ν*-measurable functions on* Y*. Suppose that* $\alpha_1, \alpha_2, \beta_1, \beta_2$ *are real numbers in [0, 1] and there exist positive constant* M_1 *and* M_2 *such that*

$$||Tf||_{L^{1/\beta_j}(Y)} \leq M_j ||f||_{L^{1/\alpha_j}(X)}. \qquad f \in D \quad j = 1, 2 \tag{4.2.6}$$

Then for $0 < \theta < 1$

$$\alpha = (1-\theta)\alpha_1 + \theta\alpha_2 \tag{4.2.7}$$

$$\beta = (1-\theta)\beta_1 + \theta\beta_2. \tag{4.2.8}$$

We have

$$||Tf||_{L^{1/\beta_j}(Y)} \leq M_1^{1-\theta} M_2^{\theta} ||f||_{L^{1/\alpha_j}(X)}. \tag{4.2.9}$$

Theorem 4.2.5. *Let* $f \in L^1(0,\infty) \cap L^2(0,\infty) \cap L^\infty(0,\infty)$ *such that* $\|f\|_2 = 1$*. Let* $\sigma \in L^r(0,\infty)$ *for* $1 \leq r \leq 2$*, then there exists a unique bounded linear operator* $P_{\sigma,f} : L^p(0,\infty) \to L^p(0,\infty)$ *for all* $p \in [r,\ r']$ *where* r *is conjugate index of* r'*such that*

$$\|P_{\sigma,f}\|_{B(L^p(0,\infty))} \leq K \|\sigma\|_1^{1-2/r'} \|\sigma\|_2^{2/r'} \|f\|_1^{1-2/r'} \|f\|_2^{2/r'} \|f\|_\infty. \tag{4.2.10}$$

Proof. By interpolation of the Theorem 4.2.1 and Theorem 4.2.2, we get a unique bounded linear operator $P_{\sigma,f} : L^p(0,\infty) \to L^p(0,\infty)$ for $1 < p < \infty$.
For taking the suitable choice of $\alpha_1 = 1$, $\alpha_2 = 1/2$, $\frac{1}{\beta_1} = p$ and $\frac{1}{\beta_2} = p$.
By (4.2.7) and (4.2.8) we find that $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2 = 1 - \frac{\theta}{2}$ and $\beta = (1 - \theta)(1/p) + \theta(1/p)$.
Now suppose $\alpha = 1/r$ this implies $1/r = 1 - \theta/2$ and $1 - 1/r' = 1 - \theta/2$ or $\theta = 2/r'$.
This implies

$$\begin{aligned} \|P_{\sigma,f}\|_{B(L^p(0,\infty))} &\leq K(\|\sigma\|_1 \|f\|_1 \|f\|_\infty)^{1-\theta} (\|f\|_2 \|\sigma\|_2 \|f\|_\infty)^{\theta} \\ &= K(\|\sigma\|_1 \|f\|_1 \|f\|_\infty)^{1-2/r'} (\|f\|_2 \|\sigma\|_2 \|f\|_\infty)^{2/r'}. \end{aligned}$$

Therefore, we get the required result

$$\|P_{\sigma,f}\|_{B(L^p(0,\infty))} \leq K\|\sigma\|_1^{1-2/r'}\|\sigma\|_2^{2/r'}\|f\|_1^{1-2/r'}\|f\|_2^{2/r'}\|f\|_\infty.$$

□

4.3 Hilbert-Schmidt operator and compactness

In this section, we investigate the Watson wavelet multiplier $P_{\sigma,f}$ associated with unitary representation in the Hilbert-Schmidt class and obtain its compactness property.

Theorem 4.3.1. *Let* $f \in L^2(0,\infty) \cap L^\infty(0,\infty)$, *Then* $P_{\sigma,f} : L^2(0,\infty) \to L^2(0,\infty)$ *is a Hilbert-schmidt operator such that*

$$\|P_{\sigma,f}\|_{S_2}^2 \leq \|\sigma\|_2\|\sigma\|_1\|f\|_2^2\|\bar{f}\|_\infty^2.$$

for all $\sigma \in L^1(0,\infty) \cap L^2(0,\infty)$.

Proof. Let $\sigma \in L^1(0,\infty) \cap L^2(0,\infty)$ and $\{f_k : k \in \mathbb{N}\}$ be an orthonormal basis in $L^2(0,\infty)$. Then

$$\sum_k \|P_{\sigma,f} f_k\|_2^2 = \sum_k \langle P_{\sigma,f} f_k, P_{\sigma,f} f_k\rangle. \tag{4.3.1}$$

Using (4.1.6), we have

$$\langle P_{\sigma,f} u, v\rangle = \int_0^\infty \sigma(\zeta)\langle u, \pi_\zeta f\rangle\langle \pi_\zeta f, v\rangle d\zeta. \tag{4.3.2}$$

Therefore from (4.3.1), we obtain

$$\sum_k \|P_{\sigma,f} f_k\|_2^2 = \sum_k \int_0^\infty \sigma(\zeta)\langle f_k, \pi_\zeta f\rangle\langle \pi_\zeta f, P_{\sigma,f} f_k\rangle d\zeta$$
$$= \sum_k \int_0^\infty \sigma(\zeta)\langle f_k, \pi_\zeta f\rangle\langle P_{\bar{\sigma},f}\pi_\zeta f, f_k\rangle d\zeta.$$

From [34, p. 1014] and [54, p. 409], we get

$$\sum_k \|P_{\sigma,f} f_k\|_2^2 = \int_0^\infty \sigma(\zeta)\langle P_{\bar{\sigma},f}\pi_\zeta f, \pi_\zeta f\rangle d\zeta.$$

Exploiting (4.3.2), we have

$$\sum_k \|P_{\sigma,f} f_k\|_2^2 = \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\langle \pi_\zeta f, \pi_\eta f\rangle\langle \pi_\eta f, \pi_\zeta f\rangle d\eta\Big) d\zeta. \tag{4.3.3}$$

From (4.3.3) and Fubini theorem, we get

$$\sum_k \|P_{\sigma,f} f_k\|_2^2 = \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)|\langle \pi_\zeta f, \pi_\eta f\rangle|^2 d\eta\Big) d\zeta.$$

In view of (4.1.5), we have

$$\sum_k \|P_{\sigma,f} f_k\|_2^2 = \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\left(\left|\int_0^\infty k(x\zeta)f(x)k(x\eta)\bar{f}(x)dx\right|^2\right) d\eta\Big) d\zeta$$
$$= \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\left(\left|\int_0^\infty k(x\zeta)k(x\eta)f(x)\bar{f}(x)dx\right|^2\right) d\eta\Big) d\zeta.$$

From (1.4.8), we obtain

$$\sum_k \|P_{\sigma,f} f_k\|_2^2 = \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\Big(\Big|\int_0^\infty \Big(\int_0^\infty k(x\xi)w(\zeta,\eta,\xi)d\xi\Big) f(x)\bar{f}(x)dx\Big|^2\Big) \times d\eta\Big)d\zeta$$
$$= \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\Big(\Big|\int_0^\infty \Big(\int_0^\infty k(x\xi)f(x)\bar{f}(x)dx\Big)w(\zeta,\eta,\xi)d\xi\Big|^2\Big) \times d\eta\Big)d\zeta.$$

Taking (1.4.1), the above expression yields

$$\sum_k \|P_{\sigma,f} f_k\|_2^2 = \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\Big(\Big|\int_0^\infty W(f(x)\bar{f}(x))(\xi)w(\zeta,\eta,\xi)d\xi\Big|^2\Big) \times d\eta\Big)d\zeta.$$

Thus, by (1.4.10) we find

$$\sum_k \|P_{\sigma,f} f_k\|_2^2 = \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\big(|W(f(x)\bar{f}(x))(\zeta,\eta)|^2\big)d\eta\Big)d\zeta.$$

If we take $\big(W(f(x)\bar{f}(x))(\zeta,\eta)\big)^2 = G'(\zeta,\eta)$, then above expression yields

$$\sum_k \|P_{\sigma,f} f_k\|_2^2 = \int_0^\infty \sigma(\zeta)(\int_0^\infty \bar{\sigma}(\eta)|G'(\zeta,\eta)|d\eta)d\zeta.$$

Using (1.4.11), we have

$$\begin{aligned}\sum_k \|P_{\sigma,f} f_k\|_2^2 &= \int_0^\infty \sigma(\zeta)(\sigma\#|G'|)(\zeta)d\zeta \\ &\leq \int_0^\infty |\sigma(\zeta)(\sigma\#|G'|)(\zeta)|d\zeta \\ &\leq \|\sigma\|_2\|(\sigma\#|G'|)\|_2 \\ &\leq \|\sigma\|_2\|\sigma\|_1\||G'|\|_2 \\ &\leq \|\sigma\|_2\|\sigma\|_1\|\big|W(f(x)\bar{f}(x))\big|^2\|_2 \\ &\leq \|\sigma\|_2\|\sigma\|_1\|W(f(x)\bar{f}(x))\|_2^2.\end{aligned}$$

Using Parseval formula of Watson transform (1.4.12), we have

$$\begin{aligned}\sum_k \|P_{\sigma,f} f_k\|_2^2 &\leq \|\sigma\|_2\|\sigma\|_1\|f(x)\bar{f}(x)\|_2^2 \\ &\leq \|\sigma\|_2\|\sigma\|_1\|f\|_2^2\|\bar{f}\|_\infty^2.\end{aligned}$$

This implies $P_{\sigma,f} : L^2(0,\infty) \to L^2(0,\infty)$ is a Hilbert-Schmidt operator. □

Theorem 4.3.2. *Let $\sigma \in L^1(0,\infty)$, and $f \in L^1(0,\infty) \cap L^\infty(0,\infty)$. Then $P_{\sigma,f}$: $L^1(0,\infty) \to L^1(0,\infty)$ is a compact operator.*

Proof. From Theorem 4.2.1, the bounded linear operator $P_{\sigma,f} : L^1(0,\infty) \to L^1(0,\infty)$ satisfies

$$\|P_{\sigma,f}\|_{B(L^1(0,\infty))} \leq \frac{K}{C_f}\|\sigma\|_1\|f\|_1\|f\|_\infty. \tag{4.3.4}$$

Consider $\{u_j\}$ be a sequence of functions in $L^1(0,\infty)$ such that $\|u_j\| \leq 1$, then (4.3.4) gives

$$\|P_{\sigma,f}u_j\|_1 \leq \frac{K}{C_f}\|u_j\|_1\|\sigma\|_1\|f\|_1\|f\|_\infty.$$

For $\|u_j\| \leq 1$, the above yields

$$\|P_{\sigma,f}u_j\|_1 \leq \frac{K}{C_f}\|\sigma\|_1\|f\|_1\|f\|_\infty \quad \text{for} \quad j = 1, 2, 3\ldots \tag{4.3.5}$$

Hence the above shows that $\{P_{\sigma,f}u_j\}$ is uniformly bounded in $L^1(0, \infty)$.

Now, consider the function $f \in C_0^\infty(0, \infty)$, which is compactly supported then the function $(\pi_\xi f)(x) = k(x\xi)f(x)$, (where k is the kernel of Watson transform) is uniformly continuous.

Thus $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that for all x and y in supp(f) with $|x - y| < \delta$, we get

$$|(\pi_\xi f)(x) - (\pi_\xi f)(y)| < \epsilon. \tag{4.3.6}$$

Hence for all j=1,2,3... and with the help of (4.2.1), we have

$$\begin{aligned}
&|(P_{\sigma,f}u_j)(x) - (P_{\sigma,f}u_j)(y)| \\
&\qquad = \frac{1}{C_f}|\int_0^\infty \sigma(\xi)\langle u_j, \pi_\xi f\rangle(\pi_\xi f)(x)d\xi - \int_0^\infty \sigma(\xi)\langle u_j, \pi_\xi f\rangle(\pi_\xi f)(y)d\xi| \\
&\qquad = \frac{1}{C_f}|\int_0^\infty \sigma(\xi)\langle u_j, \pi_\xi f\rangle\left((\pi_\xi f)(x) - (\pi_\xi f)(y)\right)d\xi| \\
&\qquad \leq \frac{1}{C_f}\int_0^\infty |\sigma(\xi)| \cdot |\langle u_j, \pi_\xi f\rangle| \cdot |(\pi_\xi f)(x) - (\pi_\xi f)(y)|d\xi.
\end{aligned}$$

Using (4.3.6), we find that

$$\begin{aligned}
|(P_{\sigma,f}u_j)(x) - (P_{\sigma,f}u_j)(y)| &\leq \epsilon\frac{1}{C_f}\int_0^\infty |\sigma(\xi)| \cdot |\langle u_j, \pi_\xi f\rangle|d\xi \\
&\leq \epsilon\frac{1}{C_f}sup|\langle u_j, \pi_\xi f\rangle|\int_0^\infty |\sigma(\xi)|d\xi.
\end{aligned}$$

Exploiting (4.1.5), we get

$$\begin{aligned}|(P_{\sigma,f}u_j)(x)-(P_{\sigma,f}u_j)(y)| &\leq \epsilon\frac{1}{C_f}\|u_j\|_1\|f\|_\infty\int_0^\infty|\sigma(\xi)|d\xi\\ &\leq \epsilon\frac{1}{C_f}\|u_j\|_1\|f\|_\infty\|\sigma\|_1.\end{aligned}$$

For $\|u_j\|\leq 1$, we get

$$\leq \epsilon\frac{1}{C_f}\|f\|_\infty\|\sigma\|_1.$$

So $\{P_{\sigma,f}u_j\}_{j=1}^\infty$ is equicontinuous on $(0,\infty)$.

This implies that for every compact subset K of $(0,\infty)$, the Ascoli-Arzela theorem shows that the Watson wavelet multiplier $\{P_{\sigma,f}u_j\}_{j=1}^\infty$ has a subsequence that converges uniformly on K.

Thus by Cantor diagonal procedure we can find a subsequence $\{u_{j_k}\}$ of $\{u_j\}_{j=1}^\infty$ such that $\{P_{\sigma,f}u_{j_k}\}_{k=1}^\infty$ converges to pointwise to a function on $(0,\infty)$. Using (4.3.5) and by applying Lebesgue dominated convergence theorem, the sequence $\{P_{\sigma,f}u_{j_k}\}_{k=1}^\infty$ is converges in $L^1(0,\infty)$.

From the above conclusions, this finds that $P_{\sigma,f}:L^1(0,\infty)\to L^1(0,\infty)$ is compact. Let $\psi\in L^1(0,\infty)\cap L^\infty(0,\infty)$ and $\{\psi_j\}_{j=1}^\infty$ be a sequence of functions in $C_0^\infty(0,\infty)$ such that $\psi_j\to\psi$ in $L^1(0,\infty)$ as $j\to\infty$. By (4.3.4), we get

$$\|P_{\sigma,f,\psi_j}-P_{\sigma,f,\psi}\|_1\leq\|f\|_\infty\|\psi_j-\psi\|_1\|\sigma\|_1\to 0$$

as $j\to\infty$.

Therefore $P_{\sigma,f}:L^1(0,\infty)\to L^1(0,\infty)$ is compact provided that the support of σ is compact. □

Theorem 4.3.3. *For $1 \leq p \leq \infty$ and for any $\sigma \in L^1(0,\infty)$ and $\phi \in L^1(0,\infty) \cap L^\infty(0,\infty)$. Then $P_{\sigma,f} : L^p(0,\infty) \to L^p(0,\infty)$ is a compact operator.*

Proof. Exploiting the interpolation Theorem 4.2.4 on Theorem 4.3.2, we get the required result. □

4.4 Applications of the Watson wavelet multiplier

In this section, the Landau-Pollak Slepian operator is described using Watson transform technique. With the help of this, it is shown that the Landau-Pollak Slepian operator associated with the unitary representation $\pi : \mathbb{R}^+ \to U(L^2(\mathbb{R}^+))$ arising in signal analysis is a Watson wavelet multiplier.

Definition 4.4.1. Let $C_1 > 0$ and $C_2 > 0$. Then the linear operators $P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ and $Q_{C_2} : L^2(0,\infty) \to L^2(0,\infty)$ are defined by

$$W(P_{C_1}f)(\zeta) = \begin{cases} (Wf)(\zeta) & 0 \leq \zeta \leq C_1 \\ 0 & \zeta > C_1 \end{cases} \tag{4.4.1}$$

and

$$(Q_{C_2}f)(x) = \begin{cases} f(x) & 0 \leq x \leq C_2 \\ 0 & x > C_2 \end{cases} \tag{4.4.2}$$

for all functions f in $L^2(0,\infty)$.

Theorem 4.4.2. *$P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ and $Q_{C_2} : L^2(0,\infty) \to L^2(0,\infty)$ are self-adjoint.*

Proof. By Parseval relation of the Watson transform (1.4.12), we have

$$\begin{aligned}\langle P_{C_1}f, g\rangle_{L^2(0,\infty)} &= \langle W(P_{C_1}f), Wg\rangle_{L^2(0,\infty)}\\ &= \int_0^\infty W(P_{C_1}f)(\zeta)\overline{(Wg)}(\zeta)d\zeta.\end{aligned}$$

With the help of (4.4.1), we get

$$\begin{aligned}\langle P_{C_1}f, g\rangle_{L^2(0,\infty)} &= \int_{B_{C_1}} (Wf)(\zeta)\overline{(Wg)}(\zeta)d\zeta\\ &= \int_{B_{C_1}} (Wf)(\zeta)\overline{W(P_{C_1}g)}(\zeta)d\zeta\\ &= \int_0^\infty (Wf)(\zeta)\overline{W(P_{C_1}g)}(\zeta)d\zeta\\ &= \langle Wf, W(P_{C_1}g)\rangle.\end{aligned}$$

Exploiting (1.4.12), we have

$$\langle P_{C_1}f, g\rangle_{L^2(0,\infty)} = \langle f, P_{C_1}g\rangle_{L^2(0,\infty)}. \tag{4.4.3}$$

This implies that $P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ is a self-adjoint operator.

Similarily

$$\langle Q_{C_2}f, g\rangle_{L^2(0,\infty)} = \int_0^\infty (Q_{C_2}f)(x)\bar{g}(x)dx.$$

From (4.4.2), we have

$$\begin{aligned}\langle Q_{C_2} f, g\rangle_{L^2(0,\infty)} &= \int_{B_{C_2}} f(x)\bar{g}(x)dx \\ &= \int_{B_{C_2}} f(x)\overline{Q_{C_2} g(x)}dx \\ &= \int_0^\infty f(x)\overline{Q_{C_2} g(x)}dx \\ &= \langle f, Q_{C_2} g\rangle. \end{aligned} \tag{4.4.4}$$

This implies that $Q_{C_2} : L^2(0,\infty) \to L^2(0,\infty)$ is a self-adjoint operator. □

Theorem 4.4.3. *$P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ and $Q_{C_2} : L^2(0,\infty) \to L^2(0,\infty)$ are projection.*

Proof. Since $P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ is a self-adjoint, so we have

$$\langle P_{C_1}^2 f, g\rangle = \langle P_{C_1} f, P_{C_1} g\rangle.$$

and by Parseval relation of the Watson transform (1.4.12), we get

$$\begin{aligned}\langle P_{C_1}^2 f, g\rangle &= \langle W(P_{C_1} f), W(P_{C_1} g)\rangle \\ &= \int_0^\infty W(P_{C_1} f)(\zeta)\overline{W(P_{C_1} g)}(\zeta)d\zeta.\end{aligned}$$

Taking (4.4.1), we have

$$\begin{aligned}\langle P_{C_1}^2 f, g\rangle &= \int_{B_{C_1}} (Wf)(\zeta)\overline{(Wg)}(\zeta)d\zeta \\ &= \int_0^\infty W(P_{C_1} f)(\zeta)\overline{(Wg)}(\zeta)d\zeta \\ &= \langle W(P_{C_1} f), Wg\rangle \\ &= \langle P_{C_1} f, g\rangle\end{aligned}$$

for all $f, g \in L^2(0, \infty)$.

This implies that $P_{C_1}^2 = P_{C_1}$, hence we say that $P_{C_1} : L^2(0, \infty) \to L^2(0, \infty)$ is a projection.

Also, since $Q_{C_2} : L^2(0, \infty) \to L^2(0, \infty)$ is a self-adjoint, we have

$$\begin{aligned}\langle Q_{C_2}^2 f, g\rangle &= \langle Q_{C_2} f, Q_{C_2} g\rangle \\ &= \int_0^\infty (Q_{C_2} f)(x)\overline{(Q_{C_2} g)}(x)dx.\end{aligned}$$

Using (4.4.2), we get

$$\begin{aligned}\langle Q_{C_2}^2 f, g\rangle &= \int_{B_{C_2}} f(x)\overline{g}(x)dx \\ &= \int_0^\infty (Q_{C_2} f)(x)\overline{g}(x)dx \\ &= \langle Q_{C_2} f, g\rangle\end{aligned}$$

for all $f, g \in L^2(0, \infty)$.

This implies that $Q_{C_2}^2 = Q_{C_2}$, hence we say that $Q_{C_2} : L^2(0, \infty) \to L^2(0, \infty)$ is a projection. □

Thus $P_{C_1} Q_{C_2} P_{C_1} : L^2(0, \infty) \to L^2(0, \infty)$ is a bounded linear operator which is known as the generalized Landau-Pollak Slepian operator.

Theorem 4.4.4. *Let $f \in L^2(0, \infty)$ which is defined by*

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{\mu(B_{C_1})}} & 0 \leq x \leq C_1 \\ 0 & x > C_1 \end{cases} \tag{4.4.5}$$

where $\mu(B_{C_1})$ is the length of B_{C_1}. Let σ be the characteristic function on B_{C_2},

$$\sigma(\zeta) = \begin{cases} 1 & 0 \leq \zeta \leq C_2 \\ 0 & \zeta > C_2 \end{cases} \tag{4.4.6}$$

then the Landau Pollak Slepian operator $P_{C_1}Q_{C_2}P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ is unitarily equivalent to a scalar multiple of the Watson wavelet multiplier $fA(x,D)\bar{f}$: $L^2(0,\infty) \to L^2(0,\infty)$.

Proof. By (4.4.5), and $f \in L^2(0,\infty) \cap L^\infty(0,\infty)$ such that

$$\begin{aligned} \|f\|^2_{L^2(0,\infty)} &= \int_0^\infty |f(x)|^2 dx \\ &= \frac{1}{\mu(B_{C_1})} \int_{B_{C_1}} dx \\ &= 1. \end{aligned}$$

Theorem 4.1.4, gives

$$\langle (fA(x,D)\bar{f})u, v\rangle = \int_0^\infty \sigma(\zeta)\langle u, \pi_\zeta f\rangle \langle \pi_\zeta f, v\rangle d\zeta. \tag{4.4.7}$$

Using (4.4.5), we get

$$\begin{aligned} \langle u, \pi_\zeta f\rangle &= \int_0^\infty k(x\zeta)u(x)f(x)dx \\ &= \frac{1}{\sqrt{\mu(B_{C_1})}} \int_{B_{C_1}} k(x\zeta)u(x)dx. \end{aligned} \tag{4.4.8}$$

From definition (4.4.1), we have

$$W(P_{C_1}W^{-1}(u))(x) = \begin{cases} u(x) & 0 \leq x \leq C_1 \\ 0 & x > C_1 \end{cases} \tag{4.4.9}$$

function u in $T(\lambda, \mu)$, where W^{-1} is the inverse Watson transform of u.

Hence, (4.4.8) and (4.4.9) yields

$$\begin{aligned} \langle u, \pi_\zeta f\rangle &= \frac{1}{\sqrt{\mu(B_{C_1})}} \int_{B_{C_1}} k(x\zeta) W(P_{C_1}W^{-1}(u))(x)dx. \\ &= \frac{1}{\sqrt{\mu(B_{C_1})}} (P_{C_1}W^{-1}(u))(\zeta) \quad \text{for all} \quad u \in T(\lambda, \mu). \end{aligned} \tag{4.4.10}$$

Since P_{C_1} is a self adjoint and (4.4.6), (4.4.7) and (4.4.10) gives

$$\begin{aligned} \langle (fA(x,D)\bar{f})u, v\rangle &= \frac{1}{\mu(B_{C_1})} \int_0^\infty \sigma(\zeta)(P_{C_1}W^{-1}(u))(\zeta)\overline{(P_{C_1}W^{-1}(v))(\zeta)}d\zeta \\ &= \frac{1}{\mu(B_{C_1})} \int_{B_{C_2}} (P_{C_1}W^{-1}(u))(\zeta)\overline{(P_{C_1}W^{-1}(v))(\zeta)}d\zeta \\ &= \frac{1}{\mu(B_{C_1})} \int_0^\infty (Q_{C_2}P_{C_1}(W^{-1}u))(\zeta)\overline{(P_{C_1}(W^{-1}v))(\zeta)}d\zeta \\ &= \frac{1}{\mu(B_{C_1})} \langle (Q_{C_2}P_{C_1}(W^{-1}u)), (P_{C_1}(W^{-1}v))\rangle. \end{aligned}$$

With the help of the Theorem (4.4.2), we get

$$\begin{aligned} \langle (fA(x,D)\bar{f})u, v\rangle &= \frac{1}{\mu(B_{C_1})} \langle (P_{C_1}Q_{C_2}P_{C_1}(W^{-1}u)), (W^{-1}v)\rangle \\ &= \frac{1}{\mu(B_{C_1})} \langle WP_{C_1}Q_{C_2}P_{C_1}W^{-1}u, v\rangle \end{aligned}$$

for all $u, v \in T(\lambda, \mu)$.

Thus the Landau-Pollak Slepian operator $P_{C_1}Q_{C_2}P_{C_1}$ is unitary equivalent to $(fA(x,D)\bar{f})$ Watson wavelet multiplier. □

4.5 Watson Wavelet multiplier in Sobolev-type space

In this section, the Watson wavelet multiplier is used to construct a Sobolev-type space using the Watson transform technique.

Theorem 4.5.1. *If* $\sigma_t(\zeta) = (1+\zeta^2)^{-t/2} \in L^1(0,\infty)$, *for* $t > 0$ *and* $f \in L^1(0,\infty) \cap L^\infty(0,\infty)$. *Then the Watson wavelet multiplier*

$$(P_{\sigma_t,f}u)(x) = \int_0^\infty \sigma_t(\zeta)\langle u, \pi_\zeta f\rangle(\pi_\zeta f)(x)d\zeta, \quad \zeta \in (0,\infty) \tag{4.5.1}$$

can be expressed in the following form:

$$(P_{\sigma_t,f}u)(x) = f(x)W^{-1}[\sigma_t(\zeta)W(uf)(\zeta)](x), \quad u \in L^1(0,\infty). \tag{4.5.2}$$

Proof. Using (4.5.1), we have

$$(P_{\sigma_t,f}u)(x) = \int_0^\infty \sigma_t(\zeta)\langle u, \pi_\zeta f\rangle(\pi_\zeta f)(x)d\zeta, \quad \zeta \in (0,\infty).$$

From (4.1.5) and (4.1.7), we get

$$\begin{aligned}
(P_{\sigma_t,f}u)(x) =& \int_0^\infty \sigma_t(\zeta)\left(\int_0^\infty u(\eta)(\pi_\zeta f)(\eta)d\eta\right)(\pi_\zeta f)(x)d\zeta \\
=& \int_0^\infty \sigma_t(\zeta)\left(\int_0^\infty k(\eta\zeta)f(\eta)u(\eta)d\eta\right)k(x\zeta)f(x)d\zeta \\
=& \int_0^\infty \sigma_t(\zeta)W(uf)(\zeta)k(x\zeta)f(x)d\zeta \\
=& f(x)\int_0^\infty k(x\zeta)\sigma_t(\zeta)W(uf)(\zeta)d\zeta \\
=& f(x)W^{-1}\left[\sigma_t(\zeta)W(uf)(\zeta)\right](x).
\end{aligned}$$

□

Definition 4.5.2. If O_M denotes the linear space of all smooth functions $\theta(x)$ defined on $I = (0, \infty)$ such that for each non-negative integer ν, there is a non-negative integer n_ν for which

$$|(1 + x^2)^{-n_\nu}(x^{-1}D_x)^\nu \theta(x)| < \infty \tag{4.5.3}$$

for all $x \in I$, then $\theta(x)$ is called multiplier in $T(\lambda, \mu)$.

Theorem 4.5.3. *Let $\phi \in O_M$ and $\psi \in T(\lambda, \mu)$. Then $\phi\psi \in T(\lambda, \mu)$.*

Proof. The above theorem can be easily proved in [50]. □

Theorem 4.5.4. *Let $\phi \in T(\lambda, \mu)$. Then $W\phi \in T(1 - \mu, 1 - \lambda)$.*

Proof. The proof is obvious, see [50]. □

Theorem 4.5.5. *Let $\sigma_t \in L^1(0, \infty)$ and $\phi \in O_m$ and $\psi \in T(\lambda, \mu)$, Then we have to prove that*

$$W_t(\phi\psi)(x) = W^{-1}[\sigma_t(\zeta)W(\phi\psi)(\zeta)](x) \in T(\lambda, \mu). \tag{4.5.4}$$

Proof. Now, we have

$$\begin{aligned}|x^{c+l}D_x^l(W_t(\phi\psi)(x)| &= |x^{c+l}D_x^l W^{-1}[\sigma_t(\zeta)W(\phi\psi)(\zeta)](x)| \\ &= |x^{c+l}D_x^l \left(\int_0^\infty k(x\zeta)\sigma_t(\zeta)W(\phi\psi)(\zeta)d\zeta\right)|\end{aligned}$$

Using (1.4.3), we get

$$
\begin{aligned}
&|x^{c+l}D_x^l(W_t(\phi\psi)(x)| \\
&\quad = |x^{c+l}D_x^l\int_0^\infty\left(\int_{c-i\infty}^{c+i\infty}K(s)(x\zeta)^{-s}ds\right)\sigma_t(\zeta)W(\phi\psi)(\zeta)d\zeta| \\
&\quad = |x^{c+l}\int_0^\infty\left(\int_{c-i\infty}^{c+i\infty}D_x^lK(s)(x\zeta)^{-s}ds\right)\sigma_t(\zeta)W(\phi\psi)(\zeta)d\zeta| \\
&\quad = |x^{c+l}\int_0^\infty\left(\int_{c-i\infty}^{c+i\infty}K(s)(\zeta)^{-s}(-s)(-s-1)....(-s-l+1)x^{-s-l}ds\right) \\
&\qquad\quad \times\sigma_t(\zeta)W(\phi\psi)(\zeta)d\zeta| \\
&\quad = |(-s)(-s-1)....(-s-l+1)x^{-s-l}x^{c+l}\int_0^\infty\left(\int_{c-i\infty}^{c+i\infty}K(s)(\zeta)^{-s}ds\right) \\
&\qquad\quad \times\sigma_t(\zeta)W(\phi\psi)(\zeta)d\zeta|.
\end{aligned}
$$

Put $s = c + iu$, we get

$$
\begin{aligned}
|x^{c+l}D_x^l(W_t(\phi\psi))(x)| &\leq A_{s,l}|x^{-iu}|\int_0^\infty\left(\int_{c-i\infty}^{c+i\infty}|K(c+iu)|du\right) \\
&\qquad \times|(\zeta)^{-c-iu}\sigma_t(\zeta)W(\phi\psi)(\zeta)|d\zeta| \\
&\leq A_{s,l}D\int_0^\infty\left(\int_{c-i\infty}^{c+i\infty}|K(c+iu)|du\right) \\
&\qquad \times|\zeta^{-c}\sigma_t(\zeta)W(\phi\psi)(\zeta)|d\zeta| \\
&\leq A_{s,l}D'sup|\zeta^{-c}W(\phi\psi)(\zeta)|\int_0^\infty|\sigma_t(\zeta)|d\zeta. \\
&\leq A_{s,l}D'sup|\zeta^{-c}W(\phi\psi)(\zeta)|\int_0^\infty|(1+\zeta^2)^{-t/2}|d\zeta.
\end{aligned}
$$

Thus for $t > 0$ and $\phi\psi \in T(\lambda, \mu)$ we get the required result

$$|x^{c+l}D_x^l(W_t(\phi\psi)(x)| < \infty.$$

□

With the help of all above Theorems given in this section and from [40], we find a Watson wavelet multiplier on the Sobolev space $G^{s,2}_{\mu}$.

Theorem 4.5.6. *Let $P_{\sigma_t,f}$ is a Watson wavelet multiplier on $L^p(I)$ and $f \in O_M$, which satisfies the norm inequality*

$$\|(1+x^2)^{-t/2}f(x)\|_2 \leq C_t \quad for \quad t > 0. \tag{4.5.5}$$

Then for $u \in L^1(0,\infty)$, $P_{\sigma_t,f}$ can be estimated by the

$$\|P_{\sigma_t,f}u\|_1 \leq C_s\|\sigma_t(\zeta)W(uf)(\zeta)\|_{t,2}. \tag{4.5.6}$$

Proof. From Theorem 4.5.1, we have

$$\begin{aligned}
\|P_{\sigma_t,f}u\|_1 &= \left(\int_0^\infty |f(x)W^{-1}[\sigma_t(\zeta)W(uf)(\zeta)](x)|dx\right)\\
&= \left(\int_0^\infty |(1+x^2)^{-t}f(x)(1+x^2)^{t}W^{-1}[\sigma_t(\zeta)W(uf)(\zeta)](x)|dx\right)\\
&\leq \|(1+x^2)^{-t}f(x)\|_2\left(\int_0^\infty |(1+x^2)^{t}W^{-1}[\sigma_t(\zeta)W(uf)(\zeta)](x)|^2dx\right)^{1/2}\\
&\leq C_t\left(\int_0^\infty |(1+x^2)^{t}W^{-1}[\sigma_t(\zeta)W(uf)(\zeta)](x)|^2dx\right)^{1/2}\\
&\leq C_t\|\sigma_t(\zeta)W(uf)(\zeta)\|_{t,2}.
\end{aligned}$$

□

4.6 Trace class of the Watson wavelet multiplier

This Section introduces the trace class associated with Watson transform and with the help of Wong [76, p. 14], it is shown that the Watson wavelet multiplier is in trace class S_1.

Proposition 4.6.1. *Let $A : X \to X$ be a bounded linear operator on a Hilbert space X and let $\{\phi_k : k = 1, 2, 3 \ldots\}$ be any orthonormal basis for X. Then the series $\sum_{k=1}^{\infty}\langle A\phi_k, \phi_k\rangle$ is absolutely convergent and the sum is independent of the choice of the orthonormal basis $\{\phi_k : k = 1, 2, 3 \ldots\}$.*

In view of the Proposition 4.6.1, we can define the trace class S_1 of any linear operator $A : X \to X$ on a Hilbert space X by

$$tr(A) = \sum_{k=1}^{\infty}\langle A\phi_k, \phi_k\rangle. \tag{4.6.1}$$

Theorem 4.6.2. *Let $\sigma \in L^1(0,\infty)$. Then the trace class of Watson wavelet multiplier $P_{\sigma,f} : L^2(0,\infty) \to L^2(0,\infty)$ is satisfied the following inequality*

$$|tr(P_{\sigma,f})| \le K\frac{1}{C_f}\int_0^{\infty}|\sigma(\xi)|d\xi. \tag{4.6.2}$$

Proof. Let $\{\phi_k\}$ be a sequence of orthonormal basis for $L^2(0,\infty)$. Then, using (4.6.1) we have

$$|tr(P_{\sigma,f}\phi_k)| = |\sum_{k=1}^{\infty}\langle P_{\sigma,f}\phi_k, \phi_k\rangle|.$$

From (4.1.6), we get

$$\begin{aligned}|tr(P_{\sigma,f}\phi_k)| &= |\sum_{k=1}^{\infty}\frac{1}{C_f}\int_0^{\infty}\sigma(\xi)\langle\phi_k, \pi_\xi f\rangle\langle\pi_\xi f, \phi_k\rangle d\xi| \\ &\le \sum_{k=1}^{\infty}|\frac{1}{C_f}\int_0^{\infty}\sigma(\xi)\langle\phi_k, \pi_\xi f\rangle\overline{\langle\phi_k, \pi_\xi f\rangle}d\xi|.\end{aligned}$$

With the help of [54, p. 409], we get

$$|tr(P_{\sigma,f}\phi_k)| \le \frac{1}{C_f}\int_0^{\infty}|\sigma(\xi)|\cdot\|\pi_\xi f\|^2 d\xi.$$

Using (4.1.5), we get

$$\begin{aligned} |tr(P_{\sigma,f}\phi_k)| &\leq \frac{1}{C_f}\int_0^\infty |\sigma(\xi)| \cdot \|k(x\xi)f(x)\|^2 d\xi \\ &\leq K\frac{1}{C_f}\|f\|_2^2\int_0^\infty |\sigma(\xi)|d\xi \qquad (4.6.3) \\ &< \infty. \end{aligned}$$

□

4.7 Conclusions

Keeping a view of past, recent, and future researches on the wavelet multipliers exploiting by Fourier, and other integral transforms, the contribution of the Authors is the characterization of the Watson wavelet multiplier associated with the unitary representation. This aforesaid theory is heavily correlated with the L^p-boundedness, Hilbert-Schmidt classes, compactness, trace classes, and Sobolev spaces. Like other integral transforms, Watson transform has a nice mathematical background, and the associated wavelet multiplier is expressed in the form of Pseudo-differential operators. This theory is significant in the problems of signal processing and many areas of mathematics.

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