Chapter 1

Introduction

The theory of pseudo-differential operators is one of the most important tools in modern mathematics. It has found important applications in many mathematical developments. Utilizing the theory of the Fourier transform, pseudo-differential operators played an important role in studying problems in quantum mechanics, numerical analysis, functional analysis, and other areas of mathematics. This operator is the generalization of partial differential operators. Many authors studied the various properties of pseudo-differential operators by exploiting certain integral transform techniques and found many important observations. The calculus of pseudo-differential operators was originated by Kohn and Nirenberg [32] in 1965 and Hormander [29] did a significant contribution in the enhancement of this aforesaid theory and made well-structured calculus. Later on, Fefferman [19], Shubin [61], Taylor [63], Treves [65], Wong [75] and others established proper structures for the development of pseudo-differential operators and studied many properties by using the theory of the Fourier transform.

The spectral theory of a class of pseudo-differential operators was introduced by

Schechter [58], in 1970. Catchpole [10], Wong [72–74] and Weder [70], made important contributions about the spectral properties of a class of pseudo-differential operators by exploiting the theory of the Fourier transform. Wong [74] studied the spectrum and essential spectrum in $L^p(\mathbb{R}^n)$ of strongly Carleman pseudo-differential operators by taking a symbol of class $S^m_{\rho,0}$, $0 \leq \rho \leq 1$ and examined that both the aforesaid spectrums coincided with the symbol under the given range for $p \neq 2$. Absolute continuity of spectra of pseudo-differential operators was also found by Wong in [74]. However, the problem of finding eigen value is notoriously difficult due to the difficulties in proving convergence and L^2-boundedness.

The Hankel transform played an important role for finding the solution of cylindrical boundary value problems. Exploiting the theory of the Hankel transform, several results of pseudo-differential operators associated with certain class of symbols involving the Bessel operator were done by many authors [39–43, 48]. The pseudo-differential operators associated with the symbol class H^m and H^m_0 were defined by Pathak and Pandey [39], in 1995. It was shown that pseudo-differential operators associated with a symbol belonging to these classes are the continuous linear mapping of the Zemanian space H_μ into itself. Later on, Pathak and Pandey [40], in 1997 found the characterization of pseudo-differential operators in Sobolev type space $G^{s,p}_\mu$ associated with the help of distributional Hankel transform. In the same year, the properties of pseudo-differential operators associated with a homogeneous symbol by taking the Hankel transform technique were found by Pathak and Upadhyay [41].

Many authors have defined wavelet transforms associated with different integral transforms. From [44], Pathak et al. studied continuous and discrete Bessel wavelet transform by using the theory of Hankel convolution. In [68], Upadhyay et al. found the connection between the Bessel wavelet convolution product and Hankel convolution product involving Hankel transform technique.

Energized by the theory of the Fourier transform and the Hankel transform, Pseudo-differential operators can also be built from the other integral transforms. Using the Watson transformation theory, Pathak and Tiwari discussed the characterizations of pseudo-differential operators in [47].

Time-frequency analysis has rapidly evolved in the last two decades, as a result of the success of the Fourier transforms and wavelet transforms. Localization operators get their name in 1988 when Daubechies first used them as a mathematical tool to localize a signal on the time-frequency plane. Elena et al. [12] evaluated as a class of pseudo-differential operators known as time-frequency localization operators, Antiwick operators, Gabor-Toeplitz operators, or wave packets. In [2], among others, Baccar et al. proved that a class of pseudo-differential operators has been named as time-frequency analysis of localization operators, which depends on the symbol σ and two window functions g_1 and g_2.

Using the theory of localization operators, different works have been done by many authors by exploiting the various integral transform tools. Daubechies [14, 15], Elena [12, 13] discussed the theory of localization operators to study the class of bounded linear operators in signal analysis. Wong et al. [27, 34, 78], Upadhyay [66] examined the boundedness of the localization operator on various functional spaces in terms of the wavelet multiplier. From [27, p. 440], we see that pseudo-differential operators and wavelet multipliers are unitarily equivalent and since they are self-adjoint so they have the same spectrum.

With the help of the above results and concepts, Our main interest in this thesis is to study the characterizations of the L^p_μ−spectra of pseudo-differential operators associated with the Bessel operator in Chapter 2 and also some applications related to the essential spectrum of pseudo-differential operators involving the Hankel transform in the Sobolev space, and in the heat equation are given. Chapter 3 examines the boundedness and compactness of the Hankel wavelet multiplier, as well as many other properties associated with the unitary representation. In chapter 4, an L^p-boundedness, compactness and Hilbert-Schmidt class of the wavelet multiplier associated with the Watson transform are investigated and its various properties studied. The Landau-Pollak Slepian operator associated with the Watson transform is discussed as an application of the wavelet multiplier. The relation between the Watson wavelet multiplier and Sobolev-type space is given and the trace class of the Watson wavelet multiplier is examined. Chapter 5 will be provided the various properties of localization operators related to wavelet multipliers based on the theory of the Watson transform. In Chapter 6, utilizing the theory of the Watson transform and Watson convolution, we explore the Watson wavelet convolution product and its related properties. The relation between the Watson Wavelet convolution product and Watson convolution is also computed. Watson wavelet transform and its inversion formula are analyzed heuristically. The Watson two-wavelet multipliers and their trace class are derived from the Watson wavelet convolution product.

Now, from Betancor [4], Braksma [8], Wing [71], Pathak [45, 49], Schechter [58, 59], Titchmarsh [64], Wong [27, 34, 72–74, 76, 80], and Zemanian [83], we are giving some important definitions, formulae and properties in form of sections that will be used in the subsequent chapters.

1.1 Fourier transform

In this section, we will discuss various definitions, formulae, and properties of the Fourier transform, which are the basis for other subsequent chapters in this thesis, as follows:

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(\omega) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i(t,\omega)} f(t)dt. \tag{1.1.1}$$

If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, then the inverse Fourier transform of $\hat{f}$ is given by

$$f(t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i(t,\omega)} \hat{f}(\omega)d\omega, \quad a.e. \tag{1.1.2}$$

where $(t,\omega) = t_1\omega_1 + t_2\omega_2 + ... + t_n\omega_n$.

Fourier transform in $L^2(\mathbb{R}^n)$-space

The Fourier transform of $f \in L^2(\mathbb{R}^n)$ is defined by

$$\hat{f}(\omega) = \underset{N\to\infty}{l.i.m.} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-N}^{N} e^{-i(t,\omega)} f(t)dt, \tag{1.1.3}$$

and the corresponding inversion formula of the Fourier transform is defined by

$$f(t) = \underset{N\to\infty}{l.i.m.} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-N}^{N} e^{i(t,\omega)} \hat{f}(\omega)d\omega, \tag{1.1.4}$$

(1.1.4) defines convergence in $L^2(\mathbb{R}^n)$ and is called the limit in the mean($l.i.m.$).

Properties of Fourier transform

1. Let $f, g \in L^1(\mathbb{R}^n)$, then the convolution of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy. \tag{1.1.5}$$

2. Let $f, g \in L^2(\mathbb{R}^n)$, then the Parseval formula of the Fourier transform is

$$\langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{R}^n)} = \langle f, g \rangle_{L^2(\mathbb{R}^n)}. \tag{1.1.6}$$

3. If $f = g$, then (1.1.6)

$$\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}, \tag{1.1.7}$$

where the inner product in $L^2(\mathbb{R}^n)$ space is defined by

$$\langle f, g \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(t)\overline{g(t)}dt. \tag{1.1.8}$$

4. Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$. Then for almost every $x \in \mathbb{R}^n$,

$$||f * g||_p \leq ||f||_1 ||g||_p. \tag{1.1.9}$$

5. Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ for $1 \leq p, q \leq r \leq \infty$, then

$$||f * g||_r \leq ||f||_p ||g||_q, \tag{1.1.10}$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$.

6. Let $f, g \in L^1(\mathbb{R}^n)$, then

$$(f * g)\hat{}(\omega) \leq (2\pi)^{\frac{n}{2}} \hat{f}(\omega)\hat{g}(\omega). \tag{1.1.11}$$

7. Let $f \in L^1(\mathbb{R}^n)$, then

   (a) $\hat{f}$ is continuous on $\mathbb{R}^n$.

   (b) $\lim_{|\omega|\to\infty} \hat{f}(\omega) = 0$.

   (c) $f_j \to f$ in $L^1(\mathbb{R}^n)$ implies $\hat{f}_j \to \hat{f}$ uniformly on $\mathbb{R}^n$.

1.2 Spectal theory of pseudo-differential operators

In this section, various definitions and properties of the pseudo-differential operator associated with the Fourier transform which are useful for our further investigations are discussed:

Definition 1.2.1. $\mathscr{S}(\mathbb{R}^n)$, is the set of all infinitely differentiable functions ϕ defined on $\mathbb{R}^n$, such that for all multi-indices α and β

$$\gamma_{\alpha,\beta}(\phi) = \sup_{x\in\mathbb{R}^n} \left| x^\alpha (D^\beta \phi)(x) \right| < \infty, \tag{1.2.1}$$

then the space $\mathscr{S}(\mathbb{R}^n)$ is called Schwartz test function space.

Definition 1.2.2. A complex-valued continuous function σ defined on $\mathbb{R}^n$ is called a symbol S^m if there exist a constant $C > 0$ and $m \in (-\infty, \infty)$ such that $|\sigma(\xi)| \leq C(1 + |\xi|)^m$ for all $\xi \in \mathbb{R}^n$.

If σ is a symbol, then the pseudo-differential operator T_σ is defined by

$$(T_\sigma \phi)(x) = \int_{\mathbb{R}^n} e^{ix\xi} \sigma(\xi) \hat{\phi}(\xi) d\xi, \qquad \phi \in \mathscr{S}(\mathbb{R}^n) \tag{1.2.2}$$

where $\hat{\phi}(\xi)$ denotes the Fourier transform of ϕ.

Definition 1.2.3. A symbol σ is said to be strongly Carleman, if there is a positive number b such that

$$\frac{1}{\sigma(\xi)} = O(|\xi|^{-b}) \quad \text{as} \quad |\xi| \to \infty. \tag{1.2.3}$$

Definition 1.2.4. Let $X \neq \{0\}$ be a complex normed space and $T \in B(X)$(bounded linear operator on X). A number $\lambda \in \mathbb{C}$ is said to be a regular value of T if

1. $T - \lambda I$ is one-to-one.
2. $(T - \lambda I)^{-1}$ is defined on X, and
3. $(T - \lambda I)^{-1}$ is bounded.

where $\mathbb{C}$ is the set of all complex number and I is the identity operator from X into X. The set of all regular value of T is denoted by $\rho(T)$, is called the resolvent set of T.

The spectrum $\sum(T)$ of T is defined to be the set of complement of $\rho(T)$ in $\mathbb{C}$.

Definition 1.2.5. The essential spectrum $\sum_e(T_\sigma)$ of the operator $T_\sigma : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is defined to be the set of all complex number λ, if there is a sequence $\langle x_n \rangle$ of elements of $D(T_\sigma)$, where D is the dense domain of the operator T_σ such that $||x_n|| = 1$, $||(T_\sigma - \lambda I)x_n|| \to 0$ and $\langle x_n \rangle$ has no convergent subsequence, then $\lambda \in \sum_e(T_\sigma)$.

The minimal and maximal pseudo-differential operators

Let X and Y be complex Banach space with norms denoted by $\| \ \|_X$ and $\| \ \|_Y$ respectively. We are concerned with linear operators A mapping a dense subspace of X, usually denoted by $D(A)$ into Y. We call $D(A)$ the domain of the operator A.

Definition 1.2.6. Let X and Y be Banach space, then an operator A is said to be closed if for any sequence $< x_k >$ of vectors in $D(A)$ such that $x_k \to x$ in X and $Ax_k \to y$ in Y as $k \to \infty$, we have $x \in D(A)$ and $Ax = y$.

Definition 1.2.7. The operator A is said to be closable if for any sequence $< x_k >$ of vectors in $D(A)$ such that $x_k \to 0$ in X and $Ax_k \to y$ as $k \to \infty$, we have $y = 0$.

Definition 1.2.8. Let $\sigma \in S$. Then for $1 \leq p \leq \infty$, the pseudo-differential operator $T_\sigma : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is closable in $L^p(\mathbb{R}^n)$. The closure in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, of $T_\sigma : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is denoted by $T_{\sigma p}$ and called the minimal pseudo-differential operator.

Definition 1.2.9. Let u and f be functions in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. We say that u lies in $D(T_{\sigma,1})$ and $T_{\sigma,1}u = f$ iff $\langle u, T_\sigma^* \phi \rangle = \langle f, \phi \rangle$, ϕ in $\mathscr{S}$, where T_σ^* is the formal adjoint of T_σ.

Definition 1.2.10. $T_{\sigma,1}$ is the largest closed extension of T_σ having $\mathscr{S}$ contained in the domain of its adjoint. In other words, if B is any closed extension of T_σ such that $\mathscr{S} \subseteq D(B^t)$, then $T_{\sigma,1}$ is an extension of B called the maximal operator of T_σ.

Properties of Pseudo-differential operator

1. Let σ be a symbol. Then T_σ maps Schwartz space $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$ itself.

2. Let $\sigma \in S^0$. Then T_σ, initially defined on $\mathscr{S}(\mathbb{R}^n)$, can be uniquely extended to a bounded linear operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

3. Let σ be a symbol in S^m and T_σ be its associated pseudo-differential operator. Suppose there exists a linear operator T_σ^* a formal adjoint of the operator $T_\sigma^* : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ such that

$$\langle T_\sigma \phi, \psi \rangle = \langle \phi, T_\sigma^* \psi \rangle \qquad \phi, \psi \in \mathscr{S}(\mathbb{R}^n) \tag{1.2.4}$$

   Then we call T_σ^* is a formal adjoint of T_σ.

4. Let σ be a symbol. Then for $1 \leq p \leq \infty$, the pseudo-differential operator T_σ is closable.

5. $T_{\sigma 1}$ is a closed extension of $T_{\sigma p}$ for $1 \leq p \leq \infty$.

6. Let $\sigma \in S$. Then $T_{\sigma p} = T_{\sigma 1}$ for $1 \leq p < \infty$ i.e for $\sigma \in \mathscr{S}(\mathbb{R}^n)$, the minimal and maximal extensions in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, of $T_\sigma : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ coincide.

1.3 Hankel transform

In this section, various definitions, properties and formulae of the Hankel transform are discussed:

Let $f \in L^1(0,\infty)$, then the Hankel transform is defined by

$$(h_\mu f)(y) = \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy) f(x) dx, \quad \mu \geq -\frac{1}{2} \tag{1.3.1}$$

where J_μ denotes the Bessel function of first kind and of order μ.

If $f \in L^1(0,\infty)$ and $h_\mu f \in L^1(0,\infty)$, then the inverse Hankel transform is given by

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy)(h_\mu f)(y) dy, \text{ for } \mu \geq -\frac{1}{2}. \tag{1.3.2}$$

The H_μ space consists of all complex-valued infinitely differentiable functions ϕ defined on $I = (0,\infty)$ satisfying

$$\gamma^\mu_{n,l}(\phi) = \sup_{x \in I} \left| x^n (x^{-1} D_x)^l x^{-\mu-1/2} \phi(x) \right| < \infty, \qquad \forall\, n, l \in \mathbb{N} \cup \{0\}. \tag{1.3.3}$$

From [6], let $x^{\mu+\frac{1}{2}}f(x), x^{\mu+\frac{1}{2}}g(x) \in L^1(0,\infty)$, then the Hankel convolution is defined by

$$(f\#g)(x) = \int_0^\infty f(y)(\tau_x g)(y)dy, \tag{1.3.4}$$

where

$$(\tau_x g)(y) = \tilde{g}(x,y) = \int_0^\infty g(z)D_\mu(x,y,z)dz, \tag{1.3.5}$$

is the Hankel translation and $D_\mu(x,y,z)$ is the basic function

$$D_\mu(x,y,z) = \int_0^\infty t^{-\mu-\frac{1}{2}}(xt)^{\frac{1}{2}}J_\mu(xt)(yt)^{\frac{1}{2}}J_\mu(yt)(zt)^{\frac{1}{2}}J_\mu(zt)dt, \tag{1.3.6}$$

where $x, y, z \in I = (0,\infty)$.

Properties

From [6, 39, 43], we shall use the following formulae:

1. The Hankel transform h_μ is an automorphism on the Zemanian space H_μ.

2. Let $f \in L^1(0,\infty)$ and $g \in L^1(0,\infty)$, then $||(f\#g)||_1 \leq ||f||_1||g||_1$.

3. If $x^{\mu+\frac{1}{2}}f(x)$ and $x^{\mu+\frac{1}{2}}g(x) \in L^1(0,\infty)$ then

$$h_\mu(f\#g)(x) = x^{-\mu-\frac{1}{2}}(h_\mu f)(x)(h_\mu g)(x). \tag{1.3.7}$$

4. $S_\mu = S_{\mu,x} = \dfrac{d^2}{dx^2} + \dfrac{1-4\mu^2}{4x^2}$. (1.3.8)

5. $h_\mu(S_\mu\phi) = (-y^2)h_\mu\phi$. (1.3.9)

6. $S^r_{\mu,x}\phi(x) = \sum_{j=0}^{r} b_j x^{2j+\mu+1/2}(x^{-1}D_x)^{r+j}(x^{-\mu-1/2}\phi(x))$. (1.3.10)

From Wing [71] and Kerr [31], for $\mu \geq -\frac{1}{2}$, the Hankel transform of function $f \in L^2(0,\infty)$ is defined by

$$(h_\mu f)(y) = \underset{N\to\infty}{l.i.m.} \int_0^N (xy)^{\frac{1}{2}} J_\mu(xy) f(x)dx, \tag{1.3.11}$$

and the corresponding inverse Hankel transform is

$$f(x) = \underset{N\to\infty}{l.i.m.} \int_0^N (xy)^{\frac{1}{2}} J_\mu(xy)(h_\mu f)(y)dy, \tag{1.3.12}$$

where $l.i.m.$ denote convergence in $L^2(0,\infty)$.

h_μ is isometric on $L^2(0,\infty)$, $h_\mu^{-1} h_\mu f = f$, then the Parseval's formula of the Hankel transformation for $f, g \in L^2(0,\infty)$ is given by

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty (h_\mu f)(y)(h_\mu g)(y)dy. \tag{1.3.13}$$

Pseudo-Differential Operator associated with the Hankel transform

A complex-valued continuous function $\sigma \in C^\infty(0,\infty)$, is a symbol belong to the class H^m, if it satisfies the following inequality

$$|(\xi^{-1}D_\xi)^\alpha \sigma(\xi)| \leq C_\alpha (1+\xi)^{m-2\alpha}, \quad \forall\ \xi \in I, \tag{1.3.14}$$

for $C_\alpha > 0$, and m is any fixed real number.

The pseudo-differential operator associated with the symbol $\sigma \in H^m$, is defined by

$$(h_{\mu,\sigma}\phi)(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\sigma(\xi)(h_\mu\phi)(\xi)d\xi, \quad \phi \in H_\mu. \tag{1.3.15}$$

1.4 Watson transform

From Schuitman [8] and Titchmarsh [64], Watson transform is the generalization of the Fourier transform and the Hankel transform. Various definitions and properties of the Watson transform, which are used in other subsequent chapters, are given below:

The Watson transform of function $f \in L^1(0,\infty)$ is defined by

$$(Wf)(x) = \int_0^\infty k(xt)f(t)dt. \tag{1.4.1}$$

If $f \in L^1(0,\infty)$ and $Wf \in L^1(0,\infty)$, then the inversion of the Watson transform is

$$f(t) = \int_0^\infty k(xt)(Wf)(x)dx, \tag{1.4.2}$$

where $k(x)$ is called the kernel of the Watson transform which would be in the following form:

$$k(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K(s)x^{-s}ds \quad and \quad s = c+it, \tag{1.4.3}$$

where $K(s)$ be an analytic function on $\lambda < \text{Re } s < \mu$ such that $K(c+it) \in L^1(-\infty,\infty)$ for some c with $\lambda < c < \mu$. Assume further that for every pair (a,b) such that $\lambda < a \leq b < \mu$, there exists a real number γ such that $K(s) = O(s^\gamma)$ as $|s| \to \infty$, uniformly on $a \leq \text{Re } s \leq b$, for $\lambda, \mu \in \mathbb{R}^*, \lambda < \mu$.

We denote $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$, $\mathbb{C}$ is the set of complex numbers and $\mathbb{N}_0 = \{0, 1, 2...\}$. Let $\lambda, \mu \in \mathbb{R}^*, \lambda < \mu$. Let $\{\lambda_n\}_{n=0}^\infty$ and $\{\mu_n\}_{n=0}^\infty$ be sequence of real numbers with $\lambda_n \downarrow \lambda, \mu_n \uparrow \mu$ and $\lambda_n < \mu_n$ for all $n \in \mathbb{N}_0$.

Then $T(\lambda,\mu)$ is the space of all functions $\phi \in C^{\infty}(0,\infty)$ with the property that

$$\beta_n(\phi) = \sup_{\substack{t>0\\ \rho=0,1,2....n\\ \lambda_n\leq c\leq\mu_n}} |t^{c+p}\phi^{p}(t)| < \infty \quad \forall\; n \in \mathbb{N}_0. \tag{1.4.4}$$

A complex-valued continuous function σ defined on $I = (0,\infty)$ is a symbol belongs to the class T_m if and only if there exists a constant $C > 0$ such that

$$|\sigma(y)| \leq C(1+y)^{m}, \tag{1.4.5}$$

where m is a fixed real number.

Then the pseudo-differential operator $A(x,D)$ associated with the symbol $\sigma(y)$ is defined by

$$A(x,D) = \int_0^{\infty} k(xy)\sigma(y)(W\phi)(y)dy, \quad \phi \in T(\lambda,\mu), \tag{1.4.6}$$

where k is defined in (1.4.3) and W denotes the Watson transform.

The basic function is defined by

$$w(h,g,t) = \int_0^{\infty} k(h\xi)k(g\xi)k(t\xi)d\xi, \tag{1.4.7}$$

provided integral (1.4.3) being convergent under the assumption $k \in L^1(0,\infty) \cap L^{\infty}(0,\infty)$ and assume that $k(0) = 1$ and $w(h,g,t) > 0$ for every $h,g,t \in (0,\infty)$.

From (1.4.3), we have

$$k(h\xi)k(g\xi) = \int_0^{\infty} w(h,g,t)k(t\xi)dt. \tag{1.4.8}$$

Setting $\xi = 0$ in (1.4.8),

$$\int_0^{\infty} w(h,g,t)dt = 1. \tag{1.4.9}$$

Using (1.4.7), (1.4.8) and (1.4.9), the Watson translation is given by

$$f(h,g) = (\tau_g f) = \int_0^\infty f(t)w(h,g,t)dt. \tag{1.4.10}$$

Let $f \in L^1(0,\infty)$ and $\psi \in L^1(0,\infty)$. Then Watson convolution is defined by

$$(f\#\psi)(h) = \int_0^\infty f(h,g)\psi(g)dg. \tag{1.4.11}$$

If $f, g \in L^1(0,\infty) \cap L^2(0,\infty)$, then the Parseval relation is given as

$$\int_0^\infty (Wf)(t)(Wg)(t)dt = \int_0^\infty f(x)g(x)dx. \tag{1.4.12}$$

Properties of Watson transform

1. The map $W : T(1-\mu, 1-\lambda) \to T(\lambda,\mu)$, defined by

$$(Wf)(x) = \int_0^\infty k(xt)f(t)dt \tag{1.4.13}$$

   is linear and continuous.

2. Let $f \in L^1(0,\infty)$ and $\psi \in L^1(0,\infty)$ then

$$W(f\#\psi) = W(f)W(\psi). \tag{1.4.14}$$

3. Let $f \in L^1(0,\infty)$ and $\psi \in L^1(0,\infty)$ then

$$\|f\#\psi\|_{L^1(0,\infty)} \leq \|f\|_{L^1(0,\infty)}\|\psi\|_{L^1(0,\infty)}. \tag{1.4.15}$$

4. Let $f \in L^1(0,\infty)$ and $\psi \in L^p(0,\infty)$ and then

$$\|f\#\psi\|_{L^p(0,\infty)} \leq \|f\|_{L^1(0,\infty)}\|\psi\|_{L^p(0,\infty)}. \tag{1.4.16}$$

5. Let $f \in L^p(0,\infty)$ and $\psi \in L^q(0,\infty)$ then

$$\|f\#\psi\|_{L^r(0,\infty)} \leq \|f\|_{L^p(0,\infty)}\|\psi\|_{L^q(0,\infty)}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1. \tag{1.4.17}$$

1.5 Localization operator and Wavelet multiplier

From [34], and with the help of the Fourier transform, we introduced definitions and formulae of the unitary representation, localization operators, and wavelet multipliers.

Definition 1.5.1. Let $\pi : \mathbb{R}^n \to U(L^2(\mathbb{R}^n))$ be the unitary representation of the multiplicative group $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ is defined by

$$(\pi(\xi)u)(x) = e^{ix\xi}u(x), \qquad x, \xi \in \mathbb{R}^n \tag{1.5.1}$$

for all functions u in $L^2(\mathbb{R}^n)$, and $U(L^2(\mathbb{R}^n))$ is the group of all unitary operators on $L^2(\mathbb{R}^n)$.

Let ϕ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $||\phi||_2 = 1$, where $||\cdot||_p$ denotes the norm in $L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$. Then it is proved that

$$\langle \phi u, \phi v\rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} \langle u, \pi(\xi)\phi\rangle\langle\pi(\xi)\phi, v\rangle d\xi \tag{1.5.2}$$

for all functions u and v in the Schwartz space $\mathscr{S}$ and $\langle\ \rangle$ denotes the inner product.

Definition 1.5.2. Let $\sigma \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then for $u, v \in \mathscr{S}(\mathbb{R}^n)$, we give the definition of localization operator which is defined by

$$\langle P_{\sigma,\phi} u, v\rangle = \int_{\mathbb{R}} \sigma(\zeta)\langle u, \pi(\zeta)\phi\rangle\langle \pi(\zeta)\phi, v\rangle d\zeta, \quad \zeta \in \mathbb{R}^n, \tag{1.5.3}$$

where ϕ plays the role of admissible wavelet in the localization operator $P_{\sigma,\phi}$.

Remark: The function ϕ in the bounded linear operator $P_{\sigma,\phi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ plays the role of the admissible wavelet in a localization operator, then the bounded linear operator $P_{\sigma,\phi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a wavelet multiplier.

Definition 1.5.3. Hilbert-schmidt operator:

From Wong [80, p. 17], let $T : H \to H$ be a bounded linear operator in a Hilbert space H such that,

$$\sum_k \|T\phi_k\|^2 < \infty, \tag{1.5.4}$$

for all orthonormal bases $\{\phi_k : k \in \mathbb{N}\}$ in H. Then $T : H \to H$ is in the Hilbert-Schmidt class S_2 and satisfies the following norm

$$\|T\|_{S_2}^2 = \sum_k \|T\phi_k\|^2. \tag{1.5.5}$$

With the help of Wong [76, p. 14], we state the definition of trace class S_1.

Proposition 1.5.4. *Let $A : X \to X$ be a bounded linear operator on a Hilbert space X and let $\{\phi_k : k = 1, 2, 3 \ldots\}$ be any orthonormal basis for X. Then the series $\sum_{k=1}^{\infty} \langle A\phi_k, \phi_k\rangle$ is absolutely convergent and the sum is independent of the choice of the orthonormal basis $\{\phi_k : k = 1, 2, 3 \ldots\}$.*

Definition 1.5.5. Trace class:

In view of the Proposition 1.5.4, we can define the trace class S_1 of any linear operator $A : X \to X$ on a Hilbert space X by

$$tr(A) = \sum_{k=1}^{\infty} \langle A\phi_k, \phi_k \rangle. \tag{1.5.6}$$

Properties

1. Let $\sigma \in L^\infty(\mathbb{R}^n)$. Then from [34, p. 1010], we find that the bounded linear operators $P_{\sigma,\phi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\phi T_\sigma \bar{\phi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are unitarily equivalent.

2. The function ϕ in $L^2(\mathbb{R}^n)$ satisfying $||\phi||_2 = 1$ and

$$\int_{\mathbb{R}^n} |\langle \phi, \pi(\xi)\phi \rangle|^2 d\xi < \infty \tag{1.5.7}$$

   is said to be an admissible wavelet of $\pi : \mathbb{R}^n \to U(L^2(\mathbb{R}^n))$.

3. For every admissible wavelet ϕ, the wavelet constant c_ϕ is given by

$$c_\phi = \int_{\mathbb{R}^n} |\langle \phi, \pi(\xi)\phi \rangle|^2 d\xi. \tag{1.5.8}$$

4. The set of admissible wavelets for $\pi : \mathbb{R}^n \to U(L^2(\mathbb{R}^n))$ consists of all functions ϕ in $L^2(\mathbb{R}^n) \cap L^4(\mathbb{R}^n)$ for which $||\phi||_2 = 1$, and for every admissible wavelet ϕ,

$$c_\phi = (2\pi)^n ||\phi||_4^4. \tag{1.5.9}$$

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