L^p_μ-SPECTRA OF PSEUDO-DIFFERENTIAL OPERATORS, LOCALIZATION OPERATORS, AND WAVELET MULTIPLIERS INVOLVING CERTAIN INTEGRAL TRANSFORMS

Thesis submitted in partial fulfilment

for the Award of Degree

Doctor of Philosophy

by

Pragya Shukla

DEPARTMENT OF MATHEMATICAL SCIENCES
INDIAN INSTITUTE OF TECHNOLOGY
(BANARAS HINDU UNIVERSITY)
VARANASI -221005
INDIA

Roll No: 17121010 July 2022

CERTIFICATE

It is certified that the work contained in the thesis titled "***L^p_μ-spectra of pseudo-differential operators, localization operators, and wavelet multipliers involving certain integral transforms***" by ***Pragya Shukla*** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy and SOTA for the award of Ph.D. degree.

S K Upadhyay
25.07.2022

Dr. Santosh Kumar Upadhyay
(Supervisor)
Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

पर्यवेक्षक / Supervisor
गणितीय विज्ञान विभाग
Department of Mathematical Sciences
भारतीय प्रौद्योगिकी संस्थान
Indian Institute of Technology
(काशी हिन्दू विश्वविद्यालय)
(Banaras Hindu University)
वाराणसी / Varanasi-221005

DECLARATION BY THE CANDIDATE

I, ***Pragya Shukla***, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of ***Prof. Santosh Kumar Upadhyay*** from ***July, 2017*** to ***July, 2022*** at the ***Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi***. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

Date: 27/July/2022

Place: Varanasi

Pragya Shukla

(Pragya Shukla)

CERTIFICATE BY THE SUPERVISOR

It is certified that the above statement made by the student is correct to the best of my/our knowledge.

S K Upadhyay 27.07.2022

(Dr. Santosh Kumar Upadhyay)
Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

पर्यवेक्षक/Supervisor
गणितीय विज्ञान विभाग
Department of Mathematical Sciences
भारतीय प्रौद्योगिकी संस्थान
Indian Institute of Technology
(काशी हिन्दू विश्वविद्यालय)
(Banaras Hindu University)
वाराणसी/Varanasi-221005

27.07.2022

(Dr. S.K. Pandey)
Professor and Head
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

विभागाध्यक्ष/HEAD
गणितीय विज्ञान विभाग
Department of Mathematical Sciences
भारतीय प्रौद्योगिकी संस्थान
Indian Institute of Technology
(काशी हिन्दू विश्वविद्यालय)
(Banaras Hindu University)
वाराणसी/Varanasi-221005

COPYRIGHT TRANSFER CERTIFICATE

Title of the Thesis: *L^p_μ-spectra of pseudo-differential operators, localization operators, and wavelet multipliers involving certain integral transforms*

Name of the Student: *Pragya Shukla*

Copyright Transfer

Date: 27/July/2022

Place: Varanasi

Pragya Shukla

(Pragya Shukla)

ACKNOWLEDGEMENTS

I bow my head at the feet of Lord Shiva for the numerous blessings that have led to the successful integration of my work. The thesis is not a result of the sole efforts of a single individual. My Ph.D. journey has been marked by blessings and acknowledgement is one of the most pleasant task. In preparing this work, I would like to thanks all those who have helped in so many ways.

First of all, I would like to express my sincere thanks to my respected supervisor, ***Dr. Santosh Kumar Upadhyay, Professor, Department of Mathematical Sciences, IIT(BHU), Varanasi***, for his excellent guidance, continuous encouragement, patience and advice during the whole span of my Ph.D. He adequately guided and encouraged me to be professional and do the best even when the path was difficult. His inex-haustible working capacity, positive outlook and confidence towards my work inspired me and gave me confidence. I have been very lucky to have such a supervisor who cared so much about my work, has shaped my understanding on the subject and has given me confidence to work independently.

I express my cordial thanks to ***Prof. S. K. Pandey, Head of the Department of Mathematical Sciences, Dr. Vineet Kumar Singh, Convener, DPGC, of the Department of Mathematical Sciences*** for their supports throughout my research work. I also express my deep sense of gratitude to all faculty members of the department, especially ***Prof. K. N. Rai*** and ***Prof. Rekha Srivastav*** along with ***Prof. Subir Das, Prof. L. P. Singh, Prof. S. Mukhopadhyay, Dr. Ashok Ji Gupta, Dr. R. K. Pandey, Dr. Anuradha Banerjee, Dr. Debdas Ghosh, Dr. Sunil Kumar, Dr. Lavanya Sivakumar, Dr. Murali Krishna Vemuri, Dr. Aabhash K. Jha, Dr. Amit Kumar, Dr.***

Sheela Verma, Dr. Divya Goyal, Dr. Anoop and RPEC member ***Prof. A. K. Tripathi, Department of Computer Science and Engineering*** for their constant moral supports, suggestions, and encouragement.

I have very special thanks to my fellow lab mates Dr. Komal Khattarwani, Jay Singh Maurya, Kush Kumar Mishra, Mohd Sartaz, Sitaram Yadav, Amit, and Manjay Pal.

Further, I would like to extend my special thanks to my colleagues Diksha Gupta, Dipty Tripathi, Rahul Chaturvedi, Prashant Pandey, Shobhit, Jauny, Gourav, Robin, Dr. Manushi Gupta, Dr. Swati Yadav, Dr. Anuvedita, Dr. Pankaj, Dr. Avinash and, all the research scholars of the department for their moral supports.

I am also grateful to my institute, the Indian Institute of Technology (BHU), for providing necessary resources throughout my research. I express my thanks to all non teaching staff members of the department for their supports.

I gratefully acknowledge University Grants Commission, India for providing the fellowship in form of Junior Research fellowship and Senior Research fellowship.

I am deeply indebted to my father ***Shri Harish Chandra Shukla***, my mother ***Smt. Gayatri Shukla***, my grandparents ***Shri T N Pandey***, my younger brother ***Ambuj Shukla***, my elder sister ***Prabha Shukla*** and all my cousins who always stood by my decisions and provided all kinds of supports, moral as well as financial. It was their love, care and patience which encouraged me to move on. The person with the greatest direct and indirect contribution to this work is my Sweet friends ***Diksha Gupta, Dipti Tripathi, Saraswati shah, Karishma Singh, Priyanka Divedi, Shruti Bajpai, Naina, Manisha, Megha Pandey***, for his deepest love, endless patience and continued support shown in course of my research work. This acknowledgement would be incomplete if the name of great visionary ***Pt. Madan Mohan Malaviya*** is not mentioned, who made this divine centre of knowledge. Deepest regards to him.

Above all, praises and thanks to the Lord Shiva, for his showers of blessings throughout my research work, who has made everything possible.

Date: 27/July/2022

Place: Varanasi

Pragya Shukla

Pragya Shukla

Contents

Abbreviations

$\mathbb{N}$	Set of natural numbers
$\mathbb{N}_0$	Set of non-negative integers
$\mathbb{R}^+$ or I	Open interval $(0, \infty)$
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	Usual Euclidean space of dimension n
$\mathbb{C}$	Set of complex numbers
$E(x)$ or $[x]$	Integer part of x
$\|x\|$	Norm of x
$D_x \equiv \frac{\partial}{\partial x}$	Partial derivative with respect to variable x
$a.e.$	Almost everywhere
R.H.S.	Right hand side
L.H.S.	Left hand side

PREFACE

The pseudo-differential operator is a generalization of the partial differential operator. Pseudo-differential operators are used extensively in the theory of partial differential equations and quantum field theory by exploiting the theory of the Fourier transform. This thesis treats different aspects and properties of the L^p_μ-spectra of pseudo-differential operators associated with the Bessel operator, Hankel wavelet multipliers, Watson wavelet multipliers, Watson wavelet convolution product and two-wavelet multipliers. This thesis consists of six chapters.

Chapter 1 is introductory, which provides the historical background of the pseudo-differential operators and their spectral properties. We state the definitions and properties of the Fourier transform, the Hankel transform, the Watson transform, the Zemanian space and other spaces. Definitions of localization operators, wavelet multipliers, unitary representation and their basic properties are given.

In chapter 2, the characterizations of the L^p_μ-spectra of pseudo-differential operators associated with the Bessel operator is investigated by exploiting the theory of the Hankel transform for $1 \leq p < \infty$. Some applications related to the essential spectrum of pseudo-differential operators involving the Hankel transform in the Sobolev-type space, and in the heat equation are given.

Chapter 3 describes the Hankel wavelet multiplier associated with the unitary representation and discussed its boundedness on L^p-space for $1 \leq p \leq \infty$, compactness and other properties. It is also shown that the Hankel wavelet multiplier is Hilbert-Schmidt operator and a unitarily equivalent to the Landau-Pollak Slepian operator by taking the Hankel transform technique.

In chapter 4, an L^p-boundedness, compactness and Hilbert-Schmidt class of wavelet multiplier associated with the Watson transform are investigated and its various properties studied. The Landau-Pollak Slepian operator associated with the Watson transform is discussed as an application of wavelet multiplier. The relation between the Watson wavelet multiplier and Sobolev-type space is given and the trace class of the Watson wavelet multiplier is also examined.

In chapter 5, the characterizations of localization operators associated with the integral representation of a locally compact group are discussed and with the help of the Watson transform, its relation with wavelet multipliers is found. We also obtained the trace class and Schatten-von Neumann property of localization operators.

In Chapter 6, utilizing the theory of Watson transform and Watson convolution, we explore the Watson wavelet convolution product and its related properties. The relation between the Watson Wavelet convolution product and Watson convolution is also computed. Watson wavelet transform and its inversion formula are analyzed heuristically. The Watson two-wavelet multipliers and their trace class are derived from the Watson wavelet convolution product.

Chapter 1

Introduction

The theory of pseudo-differential operators is one of the most important tools in modern mathematics. It has found important applications in many mathematical developments. Utilizing the theory of the Fourier transform, pseudo-differential operators played an important role in studying problems in quantum mechanics, numerical analysis, functional analysis, and other areas of mathematics. This operator is the generalization of partial differential operators. Many authors studied the various properties of pseudo-differential operators by exploiting certain integral transform techniques and found many important observations. The calculus of pseudo-differential operators was originated by Kohn and Nirenberg [32] in 1965 and Hormander [29] did a significant contribution in the enhancement of this aforesaid theory and made well-structured calculus. Later on, Fefferman [19], Shubin [61], Taylor [63], Treves [65], Wong [75] and others established proper structures for the development of pseudo-differential operators and studied many properties by using the theory of the Fourier transform.

The spectral theory of a class of pseudo-differential operators was introduced by