Chapter 4

SDD Index for Bicyclic graphs

4.1 Introduction

In CGT, cycles exist in aromatic compounds which contain the Kekule structure. The corresponding graph representation involves the study of perfect matching as it plays an essential role in analyzing the resonance energy and stability of hydrocarbons. Such an application also propels our interest in studying the SDD index's behavior for the bicyclic graphs having a perfect matching. In this chapter, we present the first five lower bounds of the SDD index for all bicyclic graphs that have a perfect matching and the graphs that attain the bounds. Further, we also compute an upper bound of the SDD index for bicyclic graphs with a maximum degree of four, which admits a perfect matching.

The organization of the chapter is as follows. In Section 4.2 we present the first five lower bounds of the SDD index for all bicyclic graphs with perfect matching. In Section 4.3, we compute the upper bound of bicyclic graphs that admit a perfect matching and having maximum degree at most four.

Before proving the desired results, first we prove a lemma which helps in finding the bounds of SDD index.

Lemma 4.1. $S(1,x) = \dfrac{x^2+1}{x}$, $x \geq 2$, *is a monotone increasing function.*

Proof. Let $f(x) = \dfrac{x^2+1}{x}$ then $f'(x) = 1 - \dfrac{1}{x^2} > 0$, since $x \geq 2$. Hence $S(1,x) = \dfrac{x^2+1}{x}$ is an increasing function. □

Remark 4.2. Note that the minimum value of $S(x,y) = \dfrac{x^2+y^2}{xy} \geq 2$ and equality holds iff $x = y$.

4.1.1 Notations and Definitions

We state some of the notations and definitions required in this chapter.

Let β_{2n} denote the set of all bicyclic graphs which have a *perfect matching* on $2n$ vertices. Next, we define three of its subsets which also form a partition of β_{2n}.

1. Let $\beta_{2n}^1 \subset \beta_{2n}$ denote the set of bicyclic graphs on $2n$ vertices such that if $G \in \beta_{2n}^1$ then the two cycles in G are joined by a path as shown in Figure 4.1(a).

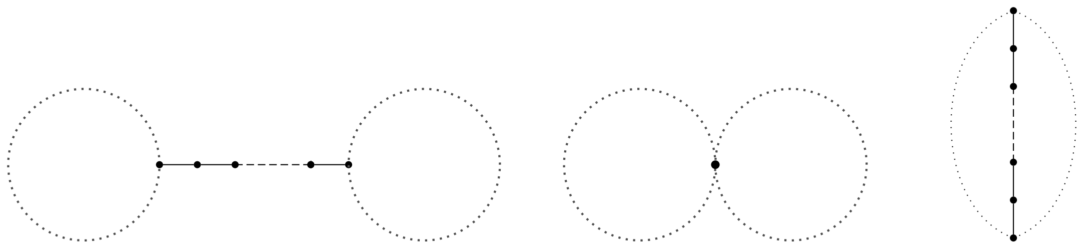
2. Let $\beta_{2n}^2 \subset \beta_{2n}$ denote the set of bicyclic graphs on $2n$ vertices such that if $G \in \beta_{2n}^2$ then the two cycles in G are joined by a common vertex, see Figure 4.1(b).

3. Let $\beta_{2n}^3 \subset \beta_{2n}$ denote the set of bicyclic graphs such that for $G \in \beta_{2n}^3$, the two cycles of G have a common path as shown in Figure 4.1(c).

Note that any graph $G \in \beta_{2n}$ belongs to exactly one of the three subsets β_{2n}^1, β_{2n}^2 or β_{2n}^3, and hence

$$\beta_{2n} = \beta_{2n}^1 \cup \beta_{2n}^2 \cup \beta_{2n}^3.$$

A representative for each of the graph classes defined above is shown in Figure 4.1.



(a) $G \in \beta_{2n}^1$ (b) $G \in \beta_{2n}^2$ (c) $G \in \beta_{2n}^3$

FIGURE 4.1: Bicyclic graphs.

4.2 Lower Bounds

In this section, we compute the first five minimum values of SDD index for all the bicyclic graphs that admit a perfect matching. To this end, we identify those graphs which possess the smallest value for the SDD index in each of the subclasses β_{2n}^1, β_{2n}^2 and β_{2n}^3.

4.2.1 Subclass β_{2n}^1

Before proving the required bounds for graphs in β_{2n}^1, we define some special classes of bicyclic graphs in β_{2n}^1 which play a primary role in our proof.

Let $F_1^1(2n) \subset \beta_{2n}^1$, $n \geq 3$ be a collection such that for any $G \in F_1^1(2n)$, the edge-degree partition of G is given by $E_1^1(G) = \{e_{22} = 2n-4, e_{23} = 4, e_{33} = 1\}$;

For $n \geq 4$ and $i = 2, 3$, let $F_i^1(2n) \subset \beta_{2n}^1$ represent those graphs G with edge-degree partitions $E_2^1(G) = \{e_{22} = 2n-5, e_{23} = 6\}$ and $E_3^1(G) = \{e_{12} = 1, e_{23} = 5, e_{22} = 2n-7, e_{33} = 2\}$, respectively.

For $n \geq 5$ and $i = 4, 5, 6$, let $F_i^1(2n) \subset \beta_{2n}^1$ be defined by the edge-degree partitions $E_4^1(G) = \{e_{12} = 2, e_{23} = 4, e_{22} = 2n - 9, e_{33} = 4\}$, $E_5^1(G) = \{e_{12} = 1, e_{23} = 7, e_{22} = 2n - 8, e_{33} = 1\}$, and $E_6^1(G) = \{e_{12} = 1, e_{23} = 9, e_{22} = 2n - 9\}$, respectively.

Let $F_7^1(2n) \subset \beta_{2n}^1$, $n \geq 4$ be defined such that for any $G \in F_7^1(2n)$, the edge-degree partition is given by $E_7^1(G) = \{e_{13} = 1, e_{22} = 2n - 6, e_{23} = 4, e_{33} = 2\}$.

Finally, let us define $F_8^1(2n) \subset \beta_{2n}^1$, for $n \geq 6$ to be the collections of the bicyclic graphs G, whose edge-degree partition is given by $E_8^1(G) = \{e_{12} = 2, e_{23} = 6, e_{22} = 2n - 10, e_{33} = 3\}$.

A representative for each of the graph classes defined above is shown in Figure 4.2(a) to 4.2(h).

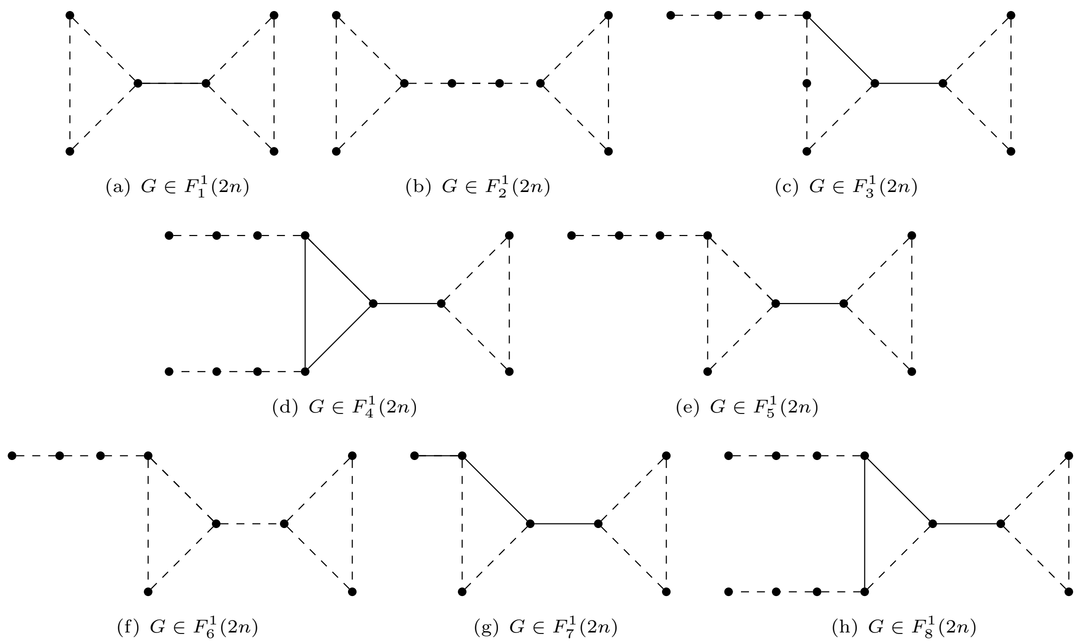


FIGURE 4.2: Representation of graphs corresponding to edge-degree partition $E_i^1(G)$ of bicyclic graphs $F_i^1(2n)$, $i = 1, 2, \ldots, 8$ respectively.

Theorem 4.3. *Let $G \in \beta_{2n}$.*

1. *If $G \in \beta_{2n}^1$, then $SDD(G) \geq 4n + \dfrac{8}{3}$. Equality holds iff $G \in F_1^1(2n)$, $n \geq 3$.*
2. *If $G \in \beta_{2n}^1 \setminus \{F_1^1(2n)\}$, then $SDD(G) \geq 4n + 3$. Equality holds iff $G \in F_2^1(2n)$, $n \geq 4$.*
3. *If $G \in \beta_{2n}^1 \setminus \{F_1^1(2n), F_2^1(2n)\}$, then $SDD(G) \geq 4n + \dfrac{10}{3}$. Equality holds iff $G \in F_3^1(2n)$, $n \geq 4$.*

4. *If $G \in \beta^1_{2n} \setminus \{F^1_1(2n), F^1_2(2n), F^1_3(2n)\}$, then $SDD(G) \geq 4n + \frac{11}{3}$. Equality holds iff $G \in F^1_4(2n)$ or $G \in F^1_5(2n)$, $n \geq 5$.*

5. *If $G \in \beta^1_{2n} \setminus \{F^1_1(2n), F^1_2(2n), F^1_3(2n), F^1_4(2n), F^1_5(2n)\}$, then $SDD(G) \geq 4(n+1)$. Equality holds iff $G \in F^1_6(2n)$, $n \geq 5$ or $G \in F^1_7(2n)$, $n \geq 4$ or $G \in F^1_8(2n)$, $n \geq 6$.*

Proof. We prove this theorem by taking conditions on the number of pendant paths K in the bicyclic graph $G \in \beta^1_{2n}$.

Case 1: If $K = 0$, then $\triangle = 3$ and $G \in F^1_1(2n)$ with $n \geq 3$ or $G \in F^1_2(2n)$ with $n \geq 4$. Note that by direct computation, $SDD(G) = 4n + \frac{8}{3}$, if $G \in F^1_1(2n)$, and $SDD(G) = 4n + 3$, if $G \in F^1_2(2n)$.

Case 2: If $K = 1$, then $3 \leq \triangle \leq 4$ and we need to consider following two subcases:

(2.1) - when the length of the pendant path is one and

(2.2) when the length of the pendant path is at least two.

Subcase 2.1: If the length of the pendant path is one, then by taking condition on maximum degree $\triangle$, we have:

(i) If $\triangle = 3$, then G has exactly three vertices w_1, w_2, w_3 of degree three. Now, again studying the vertices w_1, w_2, w_3 we see that among these vertices G has at most two pairs of them are adjacent, since $G \in \beta^1_{2n}$.

  (a) Suppose G have two pair of adjacent vertices among w_1, w_2, w_3, then $G \in F^1_7(2n)$ with $n \geq 4$ and $SDD(G) = 4(n+1)$.

  (b) Suppose that at most one pair of vertices are adjacent among w_1, w_2, w_3, then G has at least 6 edges which connect the vertices of degree two and three. Since, contribution of an edge uv is at least 2, we get

$$SDD(G) \geq 6S(2,3) + S(1,3) + 4(n-3) = 4n + \frac{13}{3} > 4(n+1).$$

(ii) If $\triangle = 4$, then G has atleast two edges which connect the vertices of degree two and $\triangle$. Then,

$$SDD(G) \geq 2S(2,4) + S(1,4) + 4(n-1) = 4n + \frac{21}{4} > 4(n+1).$$

Subcase 2.2: If the length of the pendant path is at least two, then we again make conditions on the maximum degree.

(i) Let $\triangle = 3$, then G has exactly three vertices w_1, w_2, w_3 of degree three. Again we observe that G can have at most two pairs of adjacent vertices among w_1, w_2, w_3, since $G \in \beta^1_{2n}$.

(a) Suppose two pairs of vertices among w_1, w_2, w_3 are adjacent, then $G \in F^1_3(2n)$ with $n \geq 4$ and $SDD(G) = 4n + \dfrac{10}{3}$.

(b) Suppose one pair of the vertices among w_1, w_2, w_3 are adjacent, then $G \in F^1_5(2n)$ with $n \geq 5$ and $SDD(G) = 4n + \dfrac{11}{3}$.

(c) Suppose that no pair of vertices w_1, w_2, w_3 are adjacent, then $G \in F^1_6(2n)$ with $n \geq 5$ and $SDD(G) = 4(n + 1)$.

(ii) Let $\triangle = 4$, then G has at least three edges which connect the vertices of degree two and four. In this subcase, we have

$$SDD(G) \geq 3S(2, 4) + S(1, 2) + 2S(2, 3) + 2(2n - 5) = 4n + \frac{13}{3} > 4(n + 1).$$

Case 3: If $K = 2$, then $3 \leq \triangle \leq 5$. Now, we need to consider two subcases.

Subcase 3.1: If G has at least one pendant path of length one, then from Lemma 4.1, we have

$$SDD(G) \geq S(1, 3) + S(1, 2) + 2S(2, 3) + 2(2n - 3) = 4n + \frac{25}{6} > 4(n + 1).$$

Subcase 3.2: Both the pendant paths have length atleast two.

(i) If $\triangle = 3$, then G has four vertices w_1, w_2, w_3, w_4 of degree three. Now, we analyze the position of these vertices w_1, w_2, w_3, w_4 in G. Since $G \in \beta^1_{2n}$, cycles are joined by a path, so among the vertices w_1, w_2, w_3, w_4, at most four pair of vertices are adjacent.

(a) Suppose G has four pairs of adjacent vertices among w_1, w_2, w_3, w_4, then G will have exactly four edges which connect the vertices of degree two and three. Hence $G \in F^1_4(2n)$ with $n \geq 5$ and $SDD(G) = 4n + \dfrac{11}{3}$.

(b) Suppose G has three pairs of adjacent vertices from w_1, w_2, w_3, w_4, then G will have exactly 6 edges which connect the vertices of degree two and three. In that case, $G \in F^1_8(2n)$ with $n \geq 6$ and $SDD(G) = 4(n + 1)$.

(c) Suppose that G has at most two pairs of adjacent vertices among w_1, w_2, w_3, w_4, then G has at least 8 edges which connect the vertices of degree two and three. Then, $SDD(G) \geq 8S(2, 3) + 2S(1, 2) + 2(2n - 9) = 4n + \dfrac{13}{3} > 4(n + 1)$.

(ii) If $\triangle \geq 4$, then G has at least two edges which connect the vertices of degree two and $\triangle$. Then, $SDD(G) \geq 2S(1,2)+2S(2,\triangle)+2S(2,3)+2(2n-5) \geq 2S(1,2)+2S(2,4)+2S(2,3)+2(2n-5) = 4n + \frac{13}{3} > 4(n+1)$.

Case 4: If $K = 3$, then $3 \leq \triangle \leq 6$ and we consider the following two subcases:

(4.1) - when atleast one pendant path has length one, and

(4.2) - all three pendant path have length atleast two.

Subcase 4.1: If G has at least one pendant path of length one then from Remark 4.2 and Lemma 4.1, we have

$$SDD(G) \geq S(1,3) + 2S(1,2) + 4(n-1) = 4n + \frac{13}{3} > 4(n+1).$$

Subcase 4.2: If all three pendant path in G have length atleast two then we have following cases based on the maximum degree $\triangle$.

(i) If $\triangle = 3$, then G has five vertices w_1, w_2, w_3, w_4, w_5 of degree three. Since $G \in \beta^1_{2n}$ is bicyclic graph in which cycles are joined by a path, then G has atmost 5 pair of adjacent vertices among w_1, w_2, w_3, w_4, w_5. Then G has at least 5 edges which connect the vertices of degree two and three. Since G has three pendant paths, then

$$SDD(G) \geq 3S(1,2) + 5S(2,3) + 2(2n-7) = 4n + \frac{13}{3} > 4(n+1).$$

(ii) If $\triangle \geq 4$, then G has at least one edge which connect the vertices of degree two and $\triangle$, then $SDD(G) \geq 3S(1,2)+S(2,\triangle)+2S(2,3)+2(2n-5) \geq 3S(1,2)+S(2,4)+2S(2,3)+2(2n-5) = 4n + \frac{13}{3} > 4(n+1)$.

Case 5: If $K \geq 4$, then from Lemma 3.1.1, $SDD(G) \geq \frac{2}{3}K + 2|E(G)| \geq \frac{2}{3} \times 4 + 2(2n+1) = 4n + \frac{14}{3} > 4(n+1)$. □

4.2.2 Subclass β^2_{2n}

Before proving the required bounds for graphs in β^2_{2n}, first we identify a special class of graphs in this collection which is required for our proof.

Let $J^2_{2n} \subset \beta^2_{2n}$, be a collection of bicyclic graphs on $2n$ vertices, such that if $G \in J^2_{2n}$, then G has edge-degree partition $E(G) = \{e_{24} = 4, e_{22} = 2n-3\}$, $n \geq 3$, see Figure 4.1(b).

Theorem 4.4. *If $G \in \beta_{2n}^2$, then $SDD(G) \geq 4(n+1)$. Equality holds iff $G \in J_{2n}^2$, $n \geq 3$.*

Proof. We prove this theorem by taking condition on the number of pendant paths K in the bicyclic graph $G \in \beta_{2n}^2$.

Case 1: If $K = 0$, then $\triangle = 4$ and $G \in J_{2n}^1$ with $n \geq 3$. Note that $SDD(G) = 4(n+1)$ if $G \in J_{2n}^1$ for $n \geq 3$.

Case 2: If $K = 1$, then $4 \leq \triangle \leq 5$ and G has exactly one pendant paths. Now, we need to consider following two subcases:

(2.1) When the length of pendant path is one and

(2.2) When the length of pendant path is at least two.

Subcase 2.1: If the length of the pendant path is one, then there are at least three edges of G connect the vertices of degree two and $\triangle$. From the definition of SDD index, we get

$$SDD(G) \geq 3S(2,\triangle) + S(1,3) + 2(n-3) \geq 3S(2,4) + S(1,3) + 2(2n-3) = 2n + \frac{29}{6} > 4(n+1).$$

Subcase 2.2: If the length of the pendant path is at least two, then we make conditions on the maximum degree $\triangle$.

(i) Let $\triangle = 5$, then G has five edges which connect the vertices of degree two and $\triangle$. Since G has a pendant path,

$$SDD(G) = S(1,2) + 5S(2,5) + 2(2n-5) = 4n + 7 > 4(n+1).$$

(i) Let $\triangle = 4$, then G has atleast three edges which connect the vertices of degree two and $\triangle$ and one edge which connect the vertices of degree two and three. Then, we have

$$SDD(G) \geq S(1,2) + S(2,3) + 3S(2,4) + 4(n-2) = 4n + \frac{25}{6} > 4(n+1).$$

Case 3: If $K = 2$, then $4 \leq \triangle \leq 6$. Now, we need to consider two subcases.

Subcase 3.1: If G has at least one pendant path of length one, then from Lemma 4.1, we have

$$\begin{aligned} SDD(G) &\geq 2S(2,\triangle) + S(1,3) + S_{(1,2)} + 2(2n-3) \geq 2S(2,4) + S(1,3) + S(1,2) + 2(2n-3) \\ &= 4n + \frac{29}{6} > 4(n+1). \end{aligned}$$

Subcase 3.2: Both the pendant paths have length atleast two.

$$SDD(G) \geq 2S(2, \triangle) + 2S(1,2) + 2S(2,3) + 2(2n-5) \geq 2S(2,4) + 2S(1,2) + 2S(2,3) + 2(2n-5)$$
$$= 4n + \frac{13}{3} > 4(n+1).$$

Case 4: If $K = 3$, then $4 \leq \triangle \leq 7$ and we consider the following two subcases:

(4.1) - when atleast one pendant path has length one, and

(4.2) - all three pendant path have length atleast two.

Subcase 4.1: If G has at least one pendant path of length one then from Remark 4.2 and Lemma 4.1, we have

$$SDD(G) \geq S(2, \triangle) + 2S(1,2) + S(1,3) + 2(2n-3) \geq S(2,4) + 2S(1,2) + S(1,3) + 2(2n-3)$$
$$= 4n + \frac{29}{6} > 4(n+1).$$

Subcase 4.2: If all three pendant path in G have length atleast two. Then, we get

$$SDD(G) \geq S(2, \triangle) + 3S(1,2) + 3S_{(2,3)} + 2(2n-6) \geq S(2,4) + 3S(1,2) + 3S(2,3) + 2(2n-6)$$
$$= 4n + \frac{27}{6} > 4(n+1).$$

Case 5: If $K \geq 4$, then from Lemma 3.1.1, $SDD(G) \geq \frac{2}{3}K + 2|E(G)| \geq 4n + \frac{14}{3} > 4(n+1)$. □

4.2.3 Subclass β^3_{2n}

Before proving the required bounds for graphs in β^3_{2n}, first we identify and define some special classes of bicyclic graphs in β^3_{2n} which are required for our proof.

Let $H^3_1(2n) \subset \beta^3_{2n}$, $n \geq 2$; $H^3_2(2n)$, $H^3_3(2n) \subset \beta^3_{2n}$, $n \geq 3$; $H^3_4(2n) \subset \beta^3_{2n}$, $n \geq 2$; $H^3_5(2n)$, $H^3_6(2n) \subset \beta^3_{2n}$, $n \geq 4$; $H^3_7(2n) \subset \beta^3_{2n}$, $n \geq 3$; $H^3_8(2n) \subset \beta^3_{2n}$, $n \geq 4$; $H^3_9(2n) \subset \beta^3_{2n}$, $n \geq 5$; $H^3_{10}(2n) \subset \beta^3_{2n}$, $n \geq 3$; $H^3_{11}(2n) \subset \beta^3_{2n}$, $n \geq 5$, and $H^3_{12}(2n) \subset \beta^3_{2n}$, $n \geq 6$ be the collections of bicyclic graphs which has a *perfect matching* such that if $G \in H^3_i(2n)$, $i = 1, 2, \ldots, 12$, then it has following edge-degree partition

$$E_1^3(G) = \{e_{22} = 2n - 4, e_{23} = 4, e_{33} = 1\};$$
$$E_2^3(G) = \{e_{22} = 2n - 5, e_{23} = 6\};$$
$$E_3^3(G) = \{e_{12} = 1, e_{23} = 3, e_{22} = 2n - 6, e_{33} = 3\};$$
$$E_4^3(G) = \{e_{12} = 2, e_{23} = 4, e_{22} = 2n - 9, e_{33} = 4\};$$
$$E_5^3(G) = \{e_{12} = 2, e_{23} = 2, e_{22} = 2n - 8, e_{33} = 5\};$$
$$E_6^3(G) = \{e_{12} = 1, e_{23} = 5, e_{22} = 2n - 7, e_{33} = 2\};$$
$$E_7^3(G) = \{e_{13} = 1, e_{22} = 2n - 5, e_{23} = 2, e_{33} = 3\};$$
$$E_8^3(G) = \{e_{12} = 1, e_{23} = 7, e_{22} = 2n - 8, e_{33} = 1\};$$
$$E_9^3(G) = \{e_{12} = 1, e_{23} = 9, e_{22} = 2n - 9\};$$
$$E_{10}^3(G) = \{e_{13} = 1, e_{23} = 4, e_{22} = 2n - 6, e_{33} = 2\};$$
$$E_{11}^3(G) = \{e_{12} = 2, e_{23} = 6, e_{22} = 2n - 10, e_{33} = 3\};$$
$$E_{12}^3(G) = \{e_{12} = 3, e_{23} = 3, e_{22} = 2n - 11, e_{33} = 6\}$$

respectively, see Figure 4.3 for a graph representing each of these classes.

Theorem 4.5. *1. If $G \in \beta_{2n}^3$, then $SDD(G) \geq 4n + \frac{8}{3}$. Equality holds iff $G \in H_1^3(2n)$, $n \geq 4$.*

2. If $G \in \beta_{2n}^3 \setminus \{H_1^3(2n)\}$, then $SDD(G) \geq 4n+3$. Equality holds iff $G \in H_2^3(2n)$ or $G \in H_3^3(2n)$, $n \geq 3$ or $G \in H_4^3(2n)$, $n \geq 5$.

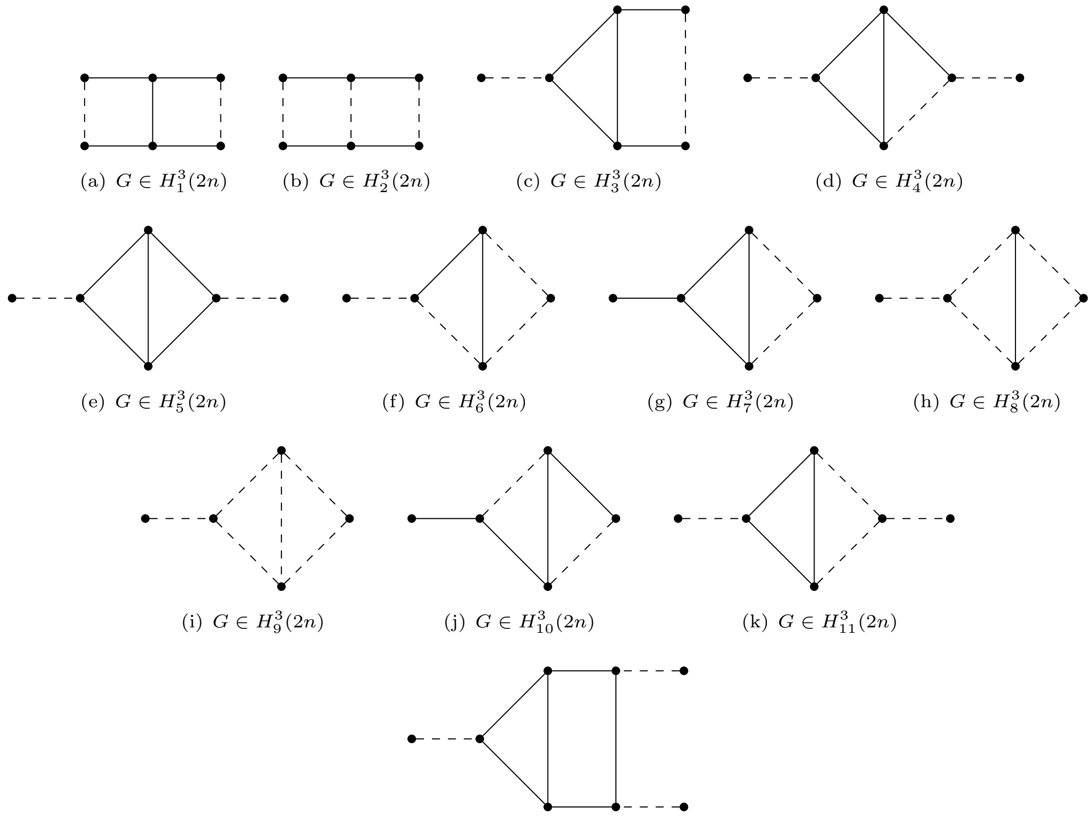
3. If $G \in \beta_{2n}^3 \setminus \{H_i^3(2n)\}$, $i = 1, 2, 3, 4$, then $SDD(G) \geq 4n + \frac{10}{3}$. Equality holds iff $G \in H_5^3(2n)$ or $G \in H_6^3(2n)$, $n \geq 4$.

4. If $G \in \beta_{2n}^3 \setminus \{H_i^3(2n)\}$, $i = 1, ..., 6$, then $SDD(G) \geq 4n + \frac{11}{3}$. Equality holds iff $G \in H_7^3(2n)$, $n \geq 3$ or $G \in H_8^3(2n)$, $n \geq 4$.

5. If $G \in \beta_{2n}^3 \setminus \{H_i^3(2n)\}$, $i = 1, ..., 8$, then $SDD(G) \geq 4(n+1)$. Equality holds iff $G \in H_{10}^3(2n)$, $n \geq 4$ or $G \in H_9^3(2n)$ or $G \in H_{11}^3(2n)$, $n \geq 5$ or $G \in H_{12}^3(2n)$, $n \geq 6$.

Proof. We prove this theorem by taking condition on pendant paths K in bicyclic graph $G \in \beta_{2n}^3(p, q, l)$.

Case 1: If $K = 0$, then $\triangle = 3$ and $G \in H_1^3(2n)$ with $n \geq 2$ or $G \in H_2^3(2n)$ with $n \geq 3$. Note that by direct computation, $SDD(G) = 4n + \frac{8}{3}$ if $G \in H_1^3(2n)$ and $SDD(G) = 4n + 3$ if $G \in H_2^3(2n)$.



(a) $G \in H_1^3(2n)$ (b) $G \in H_2^3(2n)$ (c) $G \in H_3^3(2n)$ (d) $G \in H_4^3(2n)$

(e) $G \in H_5^3(2n)$ (f) $G \in H_6^3(2n)$ (g) $G \in H_7^3(2n)$ (h) $G \in H_8^3(2n)$

(i) $G \in H_9^3(2n)$ (j) $G \in H_{10}^3(2n)$ (k) $G \in H_{11}^3(2n)$

(l) $G \in H_{12}^3(2n)$

FIGURE 4.3: Representation of graphs corresponding to edge-degree partition $E_i^3(G)$ of bicyclic graphs in $H_i^3(2n)$, $i = 1, 2, \ldots, 12$.

Case 2: If $K = 1$, then $3 \leq \triangle \leq 4$ and we need to consider following two subcases:

(2.1) - when the length of the pendant path is one and

(2.2) when the length of the pendant path is at least two.

Subcase 2.1: If the length of the pendant path is one, then by taking condition on maximum degree $\triangle$, we have:

(i) If $\triangle = 3$, then G has exactly three vertices w_1, w_2, w_3 of degree three. Now, again studying the vertices w_1, w_2, w_3 we see that among these vertices G has at most three pairs of them are adjacent, since $G \in \beta_{2n}^3(p, q, l)$.

   (a) Suppose three pair of vertices w_1, w_2, w_3 are adjacent, then $G \in H_7^3(2n)$ with $n \geq 7$ and $SDD(H_7^3(2n)) = 4n + \frac{11}{3}$.

   (b) Suppose G have two pair of adjacent vertices among w_1, w_2, w_3, then G has four edges which connect the vertices of degree two and three. In that case $G \in H_{10}^3(2n)$ with $n \geq 4$ and $SDD(G) = 4(n + 1)$

(c) Suppose that at most one pair of vertices are adjacent among w_1, w_2, w_3, then G has atleast six edges which connect the vertices of degree two and three. Since G has a pendant path, then we have

$$SDD(G) \geq S(1,3) + 6S(2,3) + 4(n-3) = 4n + \frac{13}{3} > 4(n+1).$$

(ii) If $\triangle = 4$, then G has atleast two edges which connect the vertices of degree two and $\triangle$. Then,

$$SDD(G) \geq 2S(2,4) + S(1,4) + 4(n-1) = 4n + \frac{21}{4} > 4(n+1).$$

Subcase 2.2: If the length of the pendant path is at least two, then we again make conditions on the maximum degree.

(i) Let $\triangle = 3$, then G has exactly three vertices w_1, w_2, w_3 of degree three. Again we observe that G can have at most three pairs of adjacent vertices among w_1, w_2, w_3, since $G \in \beta^3_{2n}(p,q,l)$.

(a) Suppose three pair of vertices w_1, w_2, w_3 are adjacent, then $G \in H^3_3(2n)$ with $n \geq 3$ and $SDD(G) = 4n + 3$.

(b) Suppose two pair of vertices w_1, w_2, w_3 are adjacent then $G \in H^3_6(2n)$ with $n \geq 4$ and $SDD(G) = 4n + \frac{10}{3}$.

(c) Suppose one pair of vertices w_1, w_2, w_3 are adjacent, then $G \in H^3_8(2n)$ with $n \geq 4$ and $SDD(G) = 4n + \frac{11}{3}$.

(d) Assume that no pair of vertices w_1, w_2, w_3 are adjacent, then $G \in H^3_9(2n)$ with $n \geq 5$ and $SDD(H^2_9(2n)) = 4(n+1)$.

(ii) Let $\triangle = 4$, then G has at least three edges which connect the vertices of degree two and four. In this subcase, we have

$$SDD(G) \geq 3S(2,4) + S(1,2) + 2S(2,3) + 2(2n-5) = 4n + \frac{13}{3} > 4(n+1).$$

Case 3: If $K = 2$, then $3 \leq \triangle \leq 5$. Now, we need to consider two subcases.

Subcase 3.1: If G has at least one pendant path of length one, then from Lemma 4.1, we have

$$SDD(G) \geq S(1,3) + S(1,2) + 2S(2,3) + 2(2n-3) = 4n + \frac{25}{6} > 4(n+1).$$

Subcase 3.2: Both the pendant paths have length atleast two.

(i) If $\triangle = 3$, then G has four vertices w_1, w_2, w_3, w_4 of degree three. Now, we analyze the position of these vertices w_1, w_2, w_3, w_4 in G. Since $G \in \beta^3_{2n}(p, q, l)$ is a bicyclic graph in which cycles have a common path, so among the vertices w_1, w_2, w_3, w_4, at most five pair of vertices are adjacent.

(a) Suppose G has five pair of adjacent vertices w_1, w_2, w_3, w_4, then G has exactly two edges which connect the vertices of degree two and three. In that case $G \in H^3_5(2n)$ with $n \geq 4$ and $SDD(G) = 4n + \frac{10}{3}$.

(b) Suppose G has four pair of adjacent vertices w_1, w_2, w_3, w_4, then G has exactly four edges which connect the vertices of degree two and three. In that case $G \in H^3_4(2n)$ with $n \geq 5$ and $SDD(G) = 4n + 3$.

(c) Suppose G has three pair of adjacent vertices w_1, w_2, w_3, w_4, then G has exactly six edges which connect the vertices of degree two and three. In that case $G \in H^3_{11}(2n)$ with $n \geq 5$ and $SDD(G) = 4(n + 1)$.

(d) Assume that G has atmost two pair of adjacent vertices w_1, w_2, w_3, w_4, then G has atleast eight edges which connect the vertices of degree two and three.

$$SDD(G) \geq 8S(2, 3) + 2S(1, 2) + 2(2n - 9) = 4n + \frac{13}{3} > 4(n + 1).$$

(ii) If $\triangle \geq 4$, then G has at least two edges which connect the vertices of degree two and $\triangle$. Then, $SDD(G) \geq 2S(1, 2) + 2S(2, \triangle) + 2S(2, 3) + 2(2n-5) \geq 2S(1, 2) + 2S(2, 4) + 2S(2, 3) + 2(2n - 5) = 4n + \dfrac{13}{3} > 4(n + 1)$.

Case 4: If $K = 3$, then $3 \leq \triangle \leq 6$ and we consider the following two subcases:

(4.1) - when atleast one pendant path has length one, and

(4.2) - all three pendant path have length atleast two.

Subcase 4.1: If G has at least one pendant path of length one then from Remark 4.2 and Lemma 4.1, we have

$$SDD(G) \geq S(1, 3) + 2S(1, 2) + 4(n - 1) = 4n + \frac{13}{3} > 4(n + 1).$$

Subcase 4.2: If all three pendant path in G have length atleast two then we have following cases based on the maximum degree $\triangle$.

(i) If $\triangle = 3$, then G has five vertices w_1, w_2, w_3, w_4, w_5 of degree three. Since $G \in \beta^3_{2n}(p, q, l)$ is a bicyclic graph in which cycles have a common path, then G has atmost 6 pair of adjacent vertices among w_1, w_2, w_3, w_4, w_5.

(a) Suppose G has six pair of adjacent vertices w_1, w_2, w_3, w_4, w_5, then $G \in H^3_{12}(2n)$ with $n \geq 6$ and $SDD(G) = 4(n + 1)$.

(b) Assume that G has atmost five pair of adjacent vertices w_1, w_2, w_3, w_4, w_5, then G has atleast five edges which connect the vertices of degree two and three. Since G has three pendant paths,

$$SDD(G) \geq 3S(1, 2) + 5S(2, 3) + 2(2n - 7) = 4n + \frac{13}{3} > 4(n + 1).$$

(ii) If $\triangle \geq 4$, then G has at least one edge which connect the vertices of degree two and $\triangle$, then $SDD(G) \geq 3S(1, 2)+S(2, \triangle)+2S(2, 3)+2(2n-5) \geq 3S(1, 2)+S(2, 4)+2S(2, 3)+2(2n-5) = 4n + \frac{13}{3} > 4(n + 1)$.

Case 5: If $K \geq 4$, then from Lemma 3.1.1, $SDD(G) \geq \frac{2}{3}K + 2|E(G)| \geq \frac{2}{3} \times 4 + 2(2n + 1) = 4n + \frac{14}{3} > 4(n + 1)$. □

Now, combining the above three theorems, we are ready with first five minimum values for the SDD index of all bicyclic graphs which have a *perfect matching*.

Theorem 4.6. *Let $G \in \beta_{2n}$ be a bicyclic graph which has a perfect matching.*

1. *The minimum value of $SDD(G)$ is $4n + \frac{8}{3}$ and equality holds iff $G \in F^1_1(2n)$, $n \geq 3$ or $G \in H^3_1(2n)$, $n \geq 4$.*

2. *The second-minimum value for $SDD(G)$ is $4n + 3$ and equality holds iff $G \in H^3_2(2n)$ or $G \in H^3_3(2n)$, $n \geq 3$ or $G \in F^1_2(2n)$ or $G \in H^3_4(2n)$, $n \geq 4$.*

3. *The third-minimum value of $SDD(G)$ is $4n + \frac{10}{3}$ and equality holds iff $G \in F^1_3(2n)$ or $G \in H^3_5(2n)$ or $G \in H^3_6(2n)$, $n \geq 4$.*

4. *The fourth-minimum value of $SDD(G)$ is $4n + \frac{11}{3}$ and equality holds iff $G \in H^3_7(2n)$, for $n \geq 3$ or $H^3_8(2n)$, for $n \geq 4$ or $G \in F^1_4(2n)$ or $G \in F^1_5(2n)$, $n \geq 5$.*

5. *The fifth-minimum value of $SDD(G)$ is $4(n + 1)$ and equality holds iff $G \in J^2_{2n}$, $n \geq 3$ or $F^1_7(2n)$ or $G \in H^3_{10}(2n)$, $n \geq 4$ or $G \in F^1_6(2n)$ or $G \in H^3_9(2n)$ or $G \in H^3_{11}(2n)$, $n \geq 5$ or $G \in F^1_8(2n)$ or $G \in H^3_{12}(2n)$, $n \geq 6$.*

Proof. The theorem follows directly from Theorems 4.3, 4.4 and 4.5. □

4.3 Upper Bounds

In this section, we compute the upper bounds of SDD index for bicyclic graphs, which has maximum degree four and that admits a *perfect matching.* Before proving the results we identify and define some interesting class of graphs which play a crucial role in computation of upper bounds. Let $D_i^1(2n) \subset \beta_{2n}^1$, $D_i^3(2n) \subset \beta_{2n}^3$ $i = 1, 2, 3$ be the set of bicyclic graphs such that, if $G \in D_1^3(2n)$, $n \geq 6$ or $G \in D_1^1(2n)$, $n \geq 10$, then depending on n being even or odd, we have two sets of edge-degree partition of G. When n is even, then the edge-degree partition is given by

$$E(G) = \{e_{12} = \frac{n-2}{2}, e_{14} = \frac{n+2}{2}, e_{24} = \frac{n-2}{2}, e_{44} = \frac{n+4}{2}\},$$

see Figure 4.4(a) and Figure 4.4(c). When n is odd, the edge-degree partition is

$$E(G) = \{e_{12} = \frac{n-3}{2}, e_{13} = 1, e_{14} = \frac{n+1}{2}, e_{24} = \frac{n-3}{2}, e_{34} = 2, e_{44} = \frac{n+1}{2}\},$$

see Figure 4.4(b) and Figure 4.4(d).

If $G \in D_2^3(2n)$, $n \geq 4$, or $G \in D_2^1(2n)$, $n \geq 6$, then edge-degree partition of G is

$$E(G) = \{e_{13} = n-2, e_{14} = 2, e_{33} = n-4, e_{34} = 4, e_{44} = 1\},$$

see Figure 4.5(a) and Figure 4.5(c).

For $G \in D_3^3(2n)$, $n \geq 5$, or $G \in D_3^1(2n)$, $n \geq 7$, the edge-degree partition of bicyclic graph G is

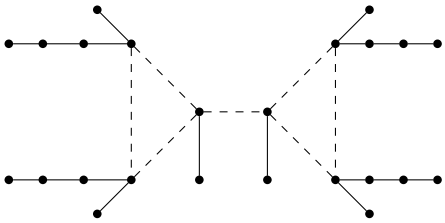
$$E(G) = \{e_{13} = n-2, e_{14} = 2, e_{33} = n-5, e_{34} = 6\},$$

see Figure 4.5(b) and Figure 4.5(d).

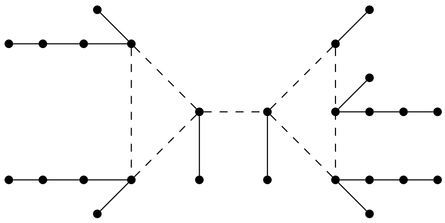
Theorem 4.7. *Let $G \in \beta_{2n}$, for $n \geq 6$ and G have maximum degree at most four. Then*

$$SDD(G) \leq \begin{cases} \frac{1}{8}(45n+25): & n \textit{ is odd,} \\ \frac{1}{8}(45n+26): & n \textit{ is enen.} \end{cases}$$

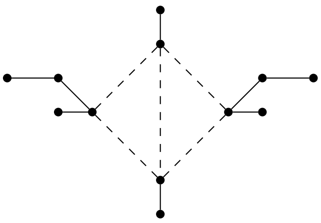
Equality holds if and only if $G \in D_1^3(2n)$, $n \geq 6$ or $G \in D_1^1(2n)$, $n \geq 9$.



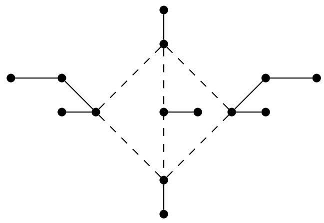
(a) $G \in D_1^1(2n)$, n is an even number



(b) $G \in D_1^1(2n)$, n is an odd number

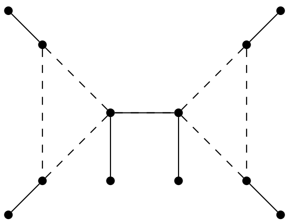


(c) $G \in D_1^3(2n)$, n is an even number

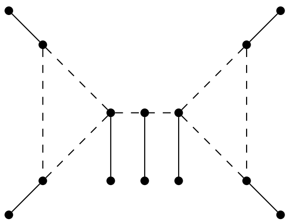


(d) $G \in D_1^3(2n)$, n is an odd number

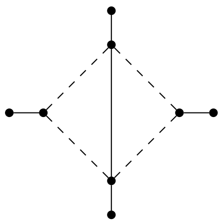
FIGURE 4.4: Representation of bicyclic graphs which attains maximum SDD index.



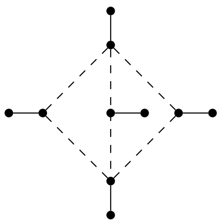
(a) $Q_1^1 \in D_2^1(2n)$



(b) $Q_2^1 \in D_3^1(2n)$



(c) $Q_1^3 \in D_2^3(2n)$



(d) $Q_2^3 \in D_3^3(2n)$

FIGURE 4.5: Representation of graphs corresponding to edge-degree partition $E(G)$ of bicyclic graphs $D_2^1(2n)$, $D_3^1(2n)$, $D_2^3(2n)$, $D_3^3(2n)$ respectively.

Proof. Let

$$\Psi(n) = \begin{cases} \frac{1}{8}(45n+25) : & n \text{ is odd,} \\ \frac{1}{8}(45n+26) : & n \text{ is enen.} \end{cases} \tag{4.1}$$

We prove this theorem by considering two cases depending on the number of pendant vertices in $G \in \beta_{2n}$: (1) G has exactly n pendant vertices and (2) G has at most $n-1$ pendant vertices.

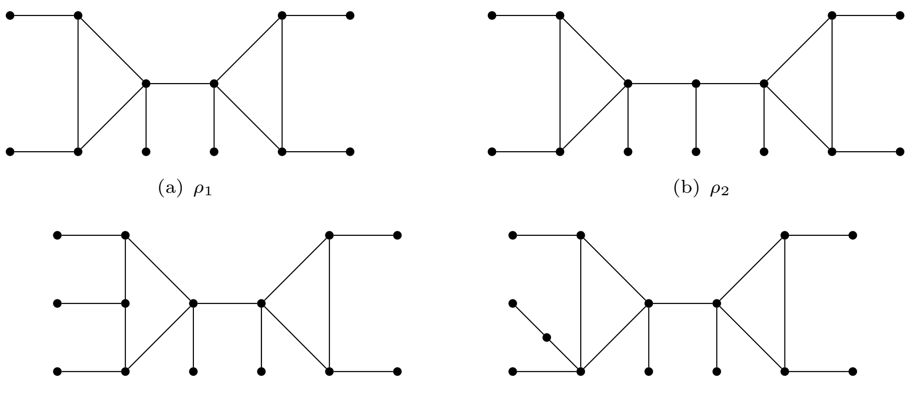
Case. 1: When G has n pendant verticecs, then each non-pendant vertex of G is adjacent to a vertex of degree one, and that in this case, either $G \in \beta^1_{2n}$ or $G \in \beta^3_{2n}$, and $G \notin \beta^2_{2n}$ as the graphs under study have maximum degree at most four.

Now we consider two subcases: (1.1) If $G \in \beta^1_{2n}$ or, (1.2) If $G \in \beta^3_{2n}$.

Subcase. 1.1: Suppose $G \in \beta^1_{2n}$. We prove this case by the method of induction.

(i) When $n = 6$, then $G \cong \rho_1$ (as shown in Figure 4.6(a)) and $SDD(\rho_1) = 36.16 < \Psi(6)$.

(ii) When $n = 7$, then $G \cong \rho_2$ (as in Figure 4.6(b),) or $G \cong \rho_3$ (Figure 4.6(c)) or $G \cong \rho_4$ (Figure 4.6(d)) and $SDD(\rho_2) = 41.66 < \Psi(7) = 42.5$, $SDD(\rho_3) = 41.5 < \Psi(7)$ and $SDD(\rho_4) = 42.08 < \Psi(7)$.



(a) ρ_1 (b) ρ_2 (c) ρ_3 (d) ρ_4

FIGURE 4.6: Bicyclic graphs discussed in Subcase 1.1 and having either 12 or 14 vertices.

(iii) For $n = 8, 9, 10$, G is one of the graphs $\beta^1_{16}, \beta^1_{18}, \beta^1_{20}$ which have edge-degree partitions as given in Table 4.1, Table 4.2 and Table 4.3, respectively.

(a) If $G \in \beta^1_{16}$, then from Table 4.1, $SDD(G) < \Psi(8) = 48.25$.

Classes	e_{12}	e_{13}	e_{14}	e_{23}	e_{24}	e_{33}	e_{34}	e_{44}	SDD
I	0	6	2	0	0	3	6	0	47
II	0	6	2	0	0	4	4	1	46.83
III	1	4	3	0	1	1	6	1	47.58
IV	1	4	3	0	1	2	4	2	47.41

TABLE 4.1: Edge-degree partition for graphs in β^1_{16}.

(b) If $G \in \beta^1_{18}$, then from Table 4.2, $SDD(G) \leq \Psi(9) = 53.75$ and equality is attained by graphs in the class (XIII) from Table 4.2 whose edge-degree partition represents $D^1_1(18)$ that is, equality holds if $G \in D^1_1(18)$.

Classes	e_{12}	e_{13}	e_{14}	e_{23}	e_{24}	e_{33}	e_{34}	e_{44}	SDD
I	0	7	2	0	0	4	6	0	52.33
II	0	7	2	0	0	5	4	1	52.166
III	1	5	3	0	1	2	6	1	52.91
IV	1	5	3	0	1	3	4	2	52.75
V	1	5	3	0	1	1	8	0	53.08
VI	1	5	3	1	0	2	5	2	52.5
VII	1	5	3	1	0	1	7	1	52.66
VIII	2	3	4	0	2	1	4	3	53.33
IX	2	3	4	0	2	1	5	2	53.41
X	2	3	4	0	2	0	6	2	53.5
XI	2	3	4	0	2	2	2	4	53.166
XII	2	3	4	1	1	0	5	3	53.08
XIII	**3**	**1**	**5**	**0**	**3**	**0**	**2**	**5**	**53.75**

TABLE 4.2: Edge-degree partition for graphs in β^1_{18}.

(c) If $G \in \beta^1_{20}$, then from Table 4.3, $SDD(G) \leq \Psi(10) = 59.5$ and equality is attained by graphs in the class (X) from Table 4.3 whose edge-degree partition represents $D^1_1(20)$, that is, equality holds if $G \in D^1_1(20)$.

Classes	e_{12}	e_{13}	e_{14}	e_{23}	e_{24}	e_{33}	e_{34}	e_{44}	SDD
I	0	8	2	0	0	5	6	0	57.66
II	0	8	2	0	0	6	4	1	57.5
III	1	6	3	0	1	2	8	0	58.41
IV	2	4	4	0	2	2	4	3	58.66
V	2	4	4	0	2	1	6	2	58.833
VI	1	6	3	0	1	3	6	1	58.25
VII	3	2	5	0	3	1	2	5	59.083
VIII	3	2	5	0	3	0	4	4	59.25
IX	1	6	3	0	1	4	4	2	58.08
X	**4**	**0**	**6**	**0**	**4**	**0**	**0**	**7**	**59.5**
XI	3	2	5	1	2	0	3	5	58.833
XII	2	4	4	2	0	0	6	3	58.166
XIII	1	6	3	1	0	3	5	2	57.833
XIV	2	4	4	1	1	1	5	3	58.41
XV	2	4	4	1	1	2	3	4	58.25
XVI	3	2	5	0	3	0	4	1	53.25
XVII	2	4	4	2	0	1	4	4	58
XVIII	2	4	4	1	1	0	7	2	58.58
XIX	1	6	3	1	0	2	7	1	58
XX	2	4	4	0	2	0	8	1	59
XXI	2	4	4	0	2	3	2	4	58.5
XXII	1	6	3	1	0	1	9	0	58.166

TABLE 4.3: Edge-degree partition for graphs in β^1_{20}.

Thus the results holds for $6 \leq n \leq 10$.

(iv) For $n > 10$, we prove the theorem by induction by assuming that the result holds for $G \in \beta^1_{2m}$, for $10 < m < n$, where each non-pendant vertex of G has a pendant-neighbor.

Let M be a *perfect matching* of $G \in \beta^1_{2n}$ and let each non-pendant vertex of G have a pendant-neighbor. Suppose $x_1, \dots, x_n$ are the pendant vertices which are adjacent to the vertices $y_1, \dots, y_n$, respectively, where $d(y_i) \geq 2$, $1 \leq i \leq n$. Then $\{x_i y_i : 1 \leq i \leq n\} \in M$. We complete the proof of this case by considering two subcases.

Subcase. 1.1(iv).1: If G has at least one vertex $y \in \{y_1, \dots y_n\}$ such that $d(y) = 2$.

Without loss of generality, let $y := y_1$. Let x_1 be its neighboring pendant vertex, where $\{x_1 y_1\} \in M$. In this subcase, suppose $y_2 (\neq x_1)$ is the other neighbor of $y_1 \in G$, then $d(y_2) \geq 3$.

(A) Suppose $d(y_2) = 3$ with $N_G(y_2) = \{y_1, x_2, y_3\}$, where $d(y_3) \geq 3$.

If G has no vertex of degree four, then G will not be a bicyclic graph as each non-pendant vertex of G has a pendant neighbor, and so we get a contradiction. Hence, there exist a vertex y_{2+k}, $k \geq 1$ of degree four in G, such that $y_2, y_3, \dots, y_{k+1}$ are vertices of degree three in G. Let $\Upsilon_1 = G + \{y_1 y_{2+k}\} \backslash \{x_2, y_2, x_3, y_3, \dots, x_{k+1}, y_{k+1}\}$ and $M_1 = M \backslash \{x_2 y_2, x_3 y_3, \dots, x_{k+1} y_{k+1}\}$. Note that $\Upsilon_1 \in \beta^1_{2(n-k)}$ and M_1 is a *perfect matching* of Υ_1; see Figure 4.7. By induction hypothesis, we have

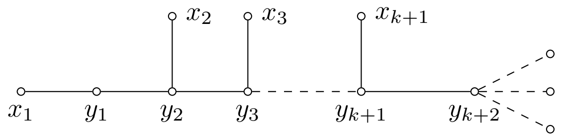
$$\begin{aligned} SDD(G) &= SDD(\Upsilon_1) + kS(1,3) + (k-1)S(3,3) + S(2,3) + S(3,4) - S(2,4) \\ &\leq \Psi(n-k) + \frac{1}{12}(64k - 3). \end{aligned}$$

(a) If $n - k$ is even, then from Equation 4.1, we have

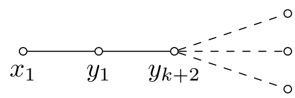
$$SDD(G) \leq \frac{1}{8}\{45(n-k) + 26\} + \frac{1}{12}(64k - 3) = \Psi(n) - \frac{1}{24}(7k + 6) < \Psi(n).$$

(b) If $n - k$ is odd, then from Equation 4.1, we have

$$SDD(G) \leq \frac{1}{8}\{45(n-k) + 25\} + \frac{1}{12}(64k - 3) = \Psi(n) - \frac{1}{24}(7k + 6) < \Psi(n).$$



(a) G



(b) Υ_1

FIGURE 4.7: Illustration of induction in Case (A).

(B) When $d(y_2) = 4$: and let us denote the neighbors as $N_G(y_2) = \{y_1, x_2, y_3, y_n\}$, where $d(x_2) = 1$, $d(y_3), d(y_n) \geq 2$. Since $n > 10$, either y_3, or y_n has degree greater than or equal to three. Without loss of generality, let $d(y_3) \geq 3$. Now, we need to take condition on $d(y_3)$, and $d(y_n)$.

(a) Suppose $d(y_n) = 2$ and $d(y_3) \geq 3$. Let $N_G(y_n) = \{x_n, y_2\}$ and let $\{y_2, x_3, y_4\}$ be the three neighbors of y_3, such that $d(y_4) \geq 3$. Note that, if G has no vertex of degree four other than $\{y_2\}$, then G cannot be a bicyclic graph. Hence there exist a vertex of degree four in G, say y_{k+2} where $k \geq 1$ is the least. that is, either y_3 is degree 4 or the vertices $y_3, \ldots, y_{k+1}$ are having degree three.

Let $\Upsilon_2 := G + \{y_n y_{k+2}\} \setminus \{x_1, y_1, x_2, y_2, \ldots, x_{k+1}, y_{k+1}\}$ and let $M_2 := M \setminus \{x_1y_1, x_2y_2, \ldots, x_{k+1}y_{k+1}\}$. Now, $\Upsilon_2 \in \beta^1_{2(n-k-1)}$ and M_2 is a *perfect matching* of Υ_2; see Figure 4.8. Hence, by induction, we have

i. When $k = 1$, we have

$$
\begin{aligned}
SDD(G) =& SDD(\Upsilon_2) + S(1,2) + S(2,4) + S(4,1) + S(2,4) + S(4,4) \\
& - S(2,4) \leq \Psi(n-2) + \frac{45}{4} \leq \Psi(n).
\end{aligned}
$$

ii. For $k > 1$, we have

$$
\begin{aligned}
SDD(G) =& SDD(\Upsilon_2) + S(1,2) + S(2,4) + S(1,4) + (k-1)S(1,3) \\
& + S(3,4) + (k-2)S(3,3) + S(2,4) + S(3,4) - S(2,4) \\
& \leq \Psi(n-k-1) + \frac{1}{12}(64k + 73) \\
& \leq \Psi(n) - \frac{1}{24}(7k - 11), \text{(From Equation 4.1)} \\
& < \Psi(n).
\end{aligned}
$$

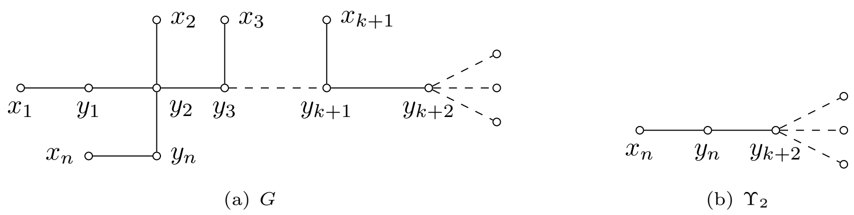


FIGURE 4.8: Illustration for the Case (B)(a).

(b) When $d(y_n) = d(y_3) = 3$: Denote the neighbors of y_n and y_3 by $N_G(y_n) = \{y_{n-1}, x_n, y_2\}$ and $N_G(y_3) = \{y_2, x_3, y_4\}$, respectively, where x_n, x_3 are pendant vertices, $d(y_{n-1}) \geq 2$ and $d(y_4) \geq 2$.

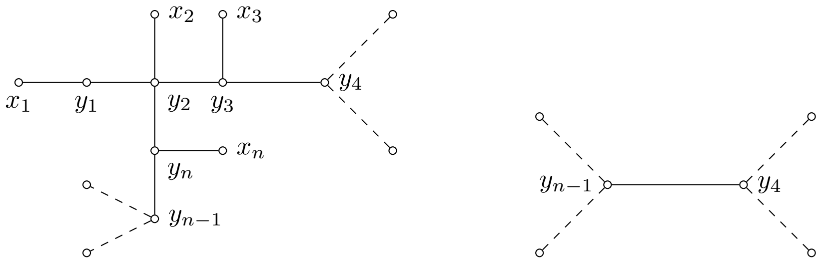
Let $\Upsilon_3 = G + \{y_{n-1}y_4\} \setminus \{x_1, y_1, y_2, x_2, x_n, y_n, x_3, y_3\}$ and let $M_3 := M \setminus \{x_n y_n, x_1 y_1, x_2 y_2, x_3 y_3\}$. We have $\Upsilon_3 \in \beta^1_{2(n-4)}$ and M_3 is a *perfect matching* of Υ_3; see Figure 4.9. Now by induction, we have

$$SDD(G) = SDD(\Upsilon_3) + S(1,2) + S(2,4) + S(1,4) + 2S(3,4) + 2S(1,3) \\ + S(3, d(y_{n-1})) + S(3, d(y_4)) - S(d(y_{n-1}), d(y_4)).$$

Since $d(y_{n-1}), d(y_4) \geq 2$ and $S(3,2) > S(3,4) > S(z,z)$, where $z \geq 2$. Then, we have

$$SDD(G) \leq \Psi(n-4) + \frac{269}{12} \leq \Psi(n) - \frac{1}{12} < \Psi(n).$$

This follows from Equation 4.1.



(a) G (b) Υ_3

FIGURE 4.9: Illustration for the Case (B)(b).

(c) When $d(y_n) \geq 3$ and $d(y_3) = 4$: Let $\Upsilon_4 = G + \{y_n y_3\} \setminus \{x_1, y_1, x_2, y_2\}$ and $M_4 := M \setminus \{x_1 y_1, x_2 y_2\}$. Note that $\Upsilon_4 \in \beta^1_{2(n-2)}$ and M_4 is a *perfect matching* of Υ_4; see Figure 4.10. By induction hypothesis, we have

$$SDD(G) = SDD(\Upsilon_4) + S(1,2) + S(2,4) + S(1,4) + S(3,4) + S(4,4) \\ - S(3,4) \leq \Psi(n-2) + \frac{45}{4} \leq \Psi(n),$$

which follows from Equation 4.1.

Hence in that subcase result is true.

Subcase. 1.1(iv).2: If no pendant vertex has a degree two neighbor in $G \in \beta^1_{2n}$.

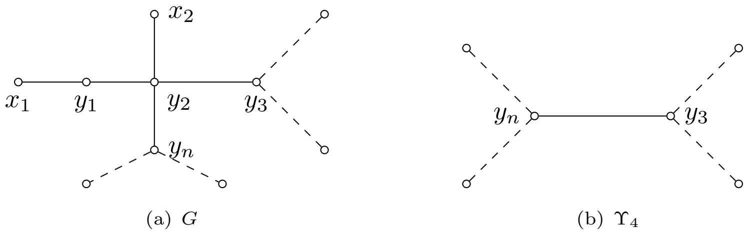


FIGURE 4.10: Illustration for the Case (B)(c).

Since G is a bicyclic graph where each of its non pendant vertex has a pendant neighbor, it follows immediately that G is isomorphic to one of the graphs in the subcollection Q_1^1, Q_2^1, as shown in Figure 4.5(a) and 4.5(b), that is, $G \cong Q_1^1$ or $G \cong Q_2^1$.
By direct computation, we find that $SDD(Q_1^1) = \frac{1}{6}(32n+25)$, and $SDD(Q_2^1) = \frac{1}{6}(32n+26)$.

(a) If n is even, then from Equation 4.1, we have

$$\Psi(n) - SDD(Q_1^1) = \frac{1}{8}(45n+26) - \frac{1}{6}(32n+25) = \frac{1}{24}(7n-22) > 0,$$
$$\text{since} \quad n \geq 6.$$

(b) If n is odd, then from Equation 4.1, we have

$$\Psi(n) - SDD(Q_2^1) = \frac{1}{8}(45n+25) - \frac{1}{6}(32n+25) = \frac{1}{24}(7n-25) > 0,$$
$$\text{since} \quad n \geq 6.$$

Thus implying the result is true in this subcase.

Subcase. 1.2: If $G \in \beta_{2n}^3$ and each non pendant vertex of G has a pendant neighbor.

(i) If $n = 6$, then graphs of β_{12}^3 have edge-degree partition as given by Table 4.4. From direct observation, we have $SDD(G) \leq \Psi(6) = 37$ and equality is attained by class (VI) in Table 4.4. Note the graph of class (VI) represents $D_1^3(12)$, that is, equality holds if $G \in D_1^3(12)$.

Classes	e_{12}	e_{13}	e_{14}	e_{23}	e_{24}	e_{33}	e_{34}	e_{44}	SDD
I	0	4	2	0	0	1	6	0	36.33
II	0	4	2	0	0	2	4	1	36.166
III	1	2	3	0	1	0	4	2	36.75
IV	1	2	3	0	1	1	2	3	36.58
V	1	2	3	1	0	0	3	3	36.33
VI	**2**	**0**	**4**	**0**	**2**	**0**	**0**	**5**	**37**

TABLE 4.4: Edge-degree partition for graphs in β_{12}^3.

(ii) If $n = 7$, then graphs of β^3_{14} have edge-degree partitions as given in Table 4.5.From Table 4.5, $SDD(G) \leq \Psi(7) = 42.5$ and equality is attained by the graphs in class (VIII) of Table 4.5 whose edge-degree partition represents $D^3_1(14)$, that is, equality holds if $G \in D^3_1(14)$. Thus

Classes	e_{12}	e_{13}	e_{14}	e_{23}	e_{24}	e_{33}	e_{34}	e_{44}	SDD
I	0	5	2	0	0	2	6	0	41.66
II	0	5	2	0	0	3	4	1	41.5
III	1	3	3	0	1	0	6	1	42.25
IV	1	3	3	0	1	1	4	2	42.08
V	1	3	3	1	0	0	5	2	41.83
VI	1	3	3	1	0	1	3	3	41.66
VII	2	1	4	1	1	0	1	5	40.08
VIII	**2**	**1**	**4**	**0**	**2**	**0**	**2**	**4**	**42.5**

TABLE 4.5: Edge-degree partition for graphs in β^3_{14}.

the results hold for $n = 6$ and $n = 7$.

(iii) For $n \geq 8$, we prove by induction by assuming that the result holds for β^3_{2m}, $8 \leq m < n$, where each non pendant vertex of $G \in \beta^3_{2m}$ has a pendant neighbor.

Let M be the *perfect matching* of $G \in \beta^3_{2n}$ where each non-pendant vertex of G has a pendant neighbor. Let $x_1, x_2, \ldots, x_n$ be the pendant vertices which are adjacent to the vertices $y_1, y_2, \ldots, y_n$, respectively, where $d(y_i) \geq 2$, $i = 1, 2, \ldots, n$. Note that $\{x_i y_i\} \subseteq M$, for $i = 1, 2, \ldots, n$. Similar to Case 1.1(iv), we consider the following two subcases to complete the proof.

(a) If $G \in \beta^3_{2n}$ has a vertex $y \in \{y_1, \ldots, y_n\}$ such that $d(y) = 2$.

Proof of this subcase is similar to *Subcase. 1.1(iv).1.*

(b) If no pendant vertex has a degree two neighbor in $G \in \beta^3_{2n}$.

In this subcase, we find that G is isomorphic to one of the graphs in the subcollection Q^3_1, Q^3_2, that is, either $G \cong Q^3_1$ (see Figure. 4.5(c)) or $G \cong Q^3_2$(see Figure. 4.5(d)).

By direct computation, we have that $SDD(Q^3_1) = \frac{1}{6}(32n+25)$, and $SDD(Q^3_2) = \frac{1}{6}(32n+26)$.

i. If n is even, then from Equation 4.1, we have

$$\Psi(n) - SDD(Q^3_1) = \frac{1}{24}(7n - 22) > 0, \qquad \text{since} \quad n \geq 4.$$

ii. If n is odd, then from Equation 4.1, we have

$$\Psi(n) - SDD(Q^3_2) = \frac{1}{24}(7n - 26) > 0, \qquad \text{since} \quad n \geq 4.$$

Hence, if each non pendant vertex of a bicyclic graph $G \in \beta_{2n}$ have a pendant neighbor, then $SDD(G) \leq \Psi(n)$.

Case. 2: Suppose G has at most $n-1$ pendant vertex, then G has at least one vertex which is not adjacent to a vertex of degree one.

From Lemma 3.2.1, it is immediate that the contribution of a vertex in SDD index is maximum, if that vertex has a pendant neighbor. Further, in *Case. 1*, we have just shown that $SDD(G) \leq \Psi(n)$, when each non pendant vertex of a bicyclic graph G have a pendent neighbor, which implies that $SDD(G) \leq \Psi(n)$, if G have at most $n-1$ vertex.

Hence, to summarize, if $G \in \beta_{2n}^1$ then $SDD(G) \leq \Psi(n)$ and equality holds if and only if $G \in D_1^1(2n)$, $n \geq 9$. If $G \in \beta_{2n}^2$, then $SDD(G) < \Psi(n)$ and in that case equality does not hold. Finally, if $G \in \beta_{2n}^3$, then $SDD(G) \leq \Psi(n)$ and equality holds if and only if $G \in D_1^3(2n)$, $n \geq 6$. □

4.4 Summary

In this chapter, we have studied the first five minimum values of the SDD index attained by the bicyclic graphs having a perfect matching. In this study, one of our main contributions is in identifying the graphs that attain the stated bounds. Further, we have also computed an upper bound of the SDD index for bicyclic graphs with a maximum degree of four, which admits a perfect matching, and have shown that the given bound is tight.

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