Chapter 2

Second Reformulated Zagreb Index for Graphs With Cyclomatic Number At Most Three

2.1 Introduction

In this chapter, we focus on finding the bounds of the second reformulated Zagreb index. In particular, we give the bounds for special graph classes having cyclomatic number at most three, namely the trees, unicyclic graphs, bicyclic graphs, and tricyclic graphs. To study the bounds for these graphs, we present a simple approach by using six graph operations/transformations. Further, we also find the extremal graphs which attain these bounds.

For a simple graph G, first and second Reformulated Zagreb indices [67] are defined as

$$EM_1(G) = \sum_{e \in E(G)} d_G^2(e), \quad EM_2(G) = \sum_{e \sim f} d_G(e).d_G(f),$$

where $e \sim f$ means the edge e and f are adjacent in the graph G, that is, the edge e and f have a common vertex. Further, if $e = uv$, then note that $d_G(e) = d_G(u) + d_G(v) - 2$.

The organization of the chapter is as follows: In Section 2.2, we give 6 transformations which increases and decreases the value of the second reformulated Zagreb index. In section 2.3.1, we compute the upper and lower bounds of $EM_2(G)$ for tree. In section 2.3.2, we compute the bounds of $EM_2(G)$ for unicyclic graphs. In section 2.3.4, we compute the bounds of $EM_2(G)$ for tricyclic graphs. In this process, we also find the graphs which attain these bounds.

2.2 Graph Transformations

In this section, we discuss graph transformation as defined in [102, 121] and show how they impact on the second reformulated Zagreb index.

2.2.1 Graph Transformations to Increase the Index Value

In this subsection, we discuss three graph transformations that are useful for finding the upper bounds of $EM_2(G)$.

Transformation A: Let G_0 be a non-trivial graph and u be any vertex of G_0. Let G_1 be a graph obtained from G_0 by joining u to the center of a star such that $d_{G_1}(v) \geq 2$, and $\{w_1, w_2, \ldots, w_t\}$ is the set of pendant neighbours of v in G_1, see Figure 2.1. Here, we say that G_2 is obtained from G_1 by **transformation A** and write as $G_2 = G_1 - \{vw_1, vw_2, \ldots, vw_t\} + \{uw_1, uw_2, \ldots, uw_t\}$.

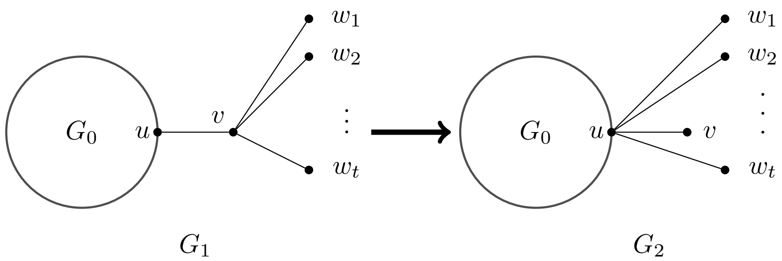


FIGURE 2.1: Representation of transformation A.

Lemma 2.1. *If G_2 is obtained by applying transformation A on G_1, then*

$$EM_2(G_1) < EM_2(G_2).$$

Proof: Let $N_{G_0}(u) = \{x \in V(G_0) : ux \in E(G_0)\}$. From the definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux) d_{G_1}(xy) \\ & + \sum_{x, y \in N_{G_0}(u)} d_{G_1}(xu) d_{G_1}(uy) + \sum_{x \in N_{G_0}(u)} d_{G_1}(xu) d_{G_1}(uv) \\ & + \sum_{i=1}^{t} d_{G_1}(uv) d_{G_1}(vw_i) + \frac{t(t-1)}{2} (d_{G_1}(v) - 1)^2, \end{aligned}$$

and

$$\begin{aligned}EM_2(G_2) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} (d_{G_1}(ux) + t) d_{G_1}(xy) \\ & + \sum_{x, y \in N_{G_0}(u)} (d_{G_1}(xu) + t)(d_{G_1}(uy) + t) + \sum_{x \in N_{G_0}(u)} (d_{G_1}(xu) + t) d_{G_1}(uv) \\ & + t \sum_{x \in N_{G_0}(u)} (d_{G_1}(xu) + t)(d_{G_1}(u) + t - 1) + \frac{t(t-1)}{2}(d_{G_1}(u) + t - 1)^2 \\ & + t(d_{G_1}(u) + t - 1)^2.\end{aligned}$$

Then,

$$\begin{aligned}EM_2(G_2) - EM_2(G_1) = & \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} t d_{G_1}(xy) + \sum_{x, y \in N_{G_0}(u)} t(d_{G_1}(xu) + d_{G_1}(uy) + t) \\ & + \sum_{x \in N_{G_0}(u)} t(d_{G_1}(u) + t - 1)(d_{G_1}(xu) + t + 1) - t^2(d_{G_1}(u) + t - 1) \\ & + \frac{t(t-1)}{2}((d_{G_1}(u) + 2t - 1)(d_{G_1}(u) - 1)) + t(d_{G_1}(u) + t - 1)^2 \\ \geq & \ t(d_{G_1}(u) + t - 1)^2 \\ > & \ 0.\end{aligned}$$

Transformation B: Let u and v be any two vertices of a simple connected graph G_1 such that there is a path of length $t \geq 1$, $P_t = u_0(= u), u_1, \ldots, u_{t-1}, u_t(= v)$ which connect the vertices u and v, see Figure 2.2. Here, we say that G_2 is obtained from G_1 by **transformation B** and write as $G_2 = G_1 - \{u_0u_1, u_1u_2, \ldots u_{t-1}u_t\} + \{wu_1, wu_2, \ldots, wu_t\}$, where $w = uov$.

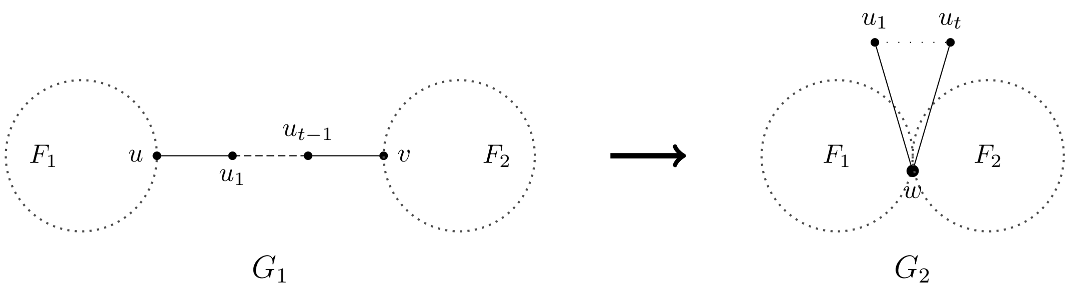


FIGURE 2.2: Representation of transformation B.

Lemma 2.2. *If G_2 is obtained from G_1 using transformation B, then*

$$EM_2(G_1) < EM_2(G_2).$$

Proof: Let $d_{G_1}(u) = a+1$, and $d_{G_1}(v) = b+1$, where $a \geq 1,\ b \geq 1$. Then the degree of edges $d_{G_1}(uu_1) = a+1$, $d_{G_1}(vu_{t-1}) = b+1$, and $d_{G_2}(wu_i) = (a+b+t-1)$, for $i = 1$ to t. We prove this lemma by making conditions on the path lengths t.

Case 1: When $t \geq 3$. From the definition of $EM_2(G)$, we have

$$\begin{aligned}
EM_2(G_1) = & \sum_{e_i \sim e_j, e_i, e_j \in E(F_1 \backslash \{u\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{e_i \sim e_j, e_i, e_j \in E(F_2 \backslash \{v\})} d_{G_1}(e_i) d_{G_1}(e_j) \\
& + \sum_{x \in N_{F_1}(u)} \sum_{y \in N_{F_1}(x) \backslash \{u\}} d_{G_1}(ux) d_{G_1}(xy) + \sum_{x \in N_{F_2}(v)} \sum_{y \in N_{F_2}(x) \backslash \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\
& + \sum_{x,y \in N_{F_1}(u)} d_{G_1}(xu) d_{G_1}(uy) + \sum_{x,y \in N_{F_2}(v)} d_{G_1}(xv) d_{G_1}(vy) + \sum_{x \in N_{F_1}(u)} d_{G_1}(xu) d_{G_1}(uu_1) \\
& + \sum_{x \in N_{F_2}(v)} d_{G_1}(xv) d_{G_1}(vu_{t-1}) + 2d_{G_1}(uu_1) + 2d_{G_1}(u_{t-1}v) + 4(t-3).
\end{aligned}$$

$$\begin{aligned}
EM_2(G_2) = & \sum_{e_i \sim e_j, e_i, e_j \in E(F_1 \backslash \{w\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{e_i \sim e_j, e_i, e_j \in E(F_2 \backslash \{w\})} d_{G_1}(e_i) d_{G_1}(e_j) \\
& + \sum_{x \in N_{F_1}(w)} \sum_{y \in N_{F_1}(x) \backslash \{w\}} d_{G_2}(wx) d_{G_1}(xy) + \sum_{x \in N_{F_2}(w)} \sum_{y \in N_{F_2}(x) \backslash \{w\}} d_{G_2}(wx) d_{G_1}(xy) \\
& + \sum_{x,y \in N_{F_1}(w)} d_{G_2}(xw) d_{G_2}(wy) + \sum_{x,y \in N_{F_2}(w)} d_{G_2}(xw) d_{G_2}(wy) \\
& + t \sum_{x \in N_{F_1}(w)} d_{G_2}(xw)(d_{G_2}(w) - 1) + t \sum_{x \in N_{F_2}(w)} d_{G_2}(xw)(d_{G_2}(w) - 1) \\
& + \sum_{x \in N_{F_1}(w)} \sum_{y \in N_{F_2}(w)} d_{G_2}(xw) d_{G_2}(wy) + \frac{1}{2} t(t-1)(d_{G_2}(w) - 1)^2,
\end{aligned}$$

since $d_{G_2}(w) = d_{G_1}(u) + d_{G_1}(v) + t - 2$, $N_{F_1}(w) = N_{F_1}(u)$ and $N_{F_2}(w) = N_{F_2}(v)$. Then

$$\begin{aligned}
EM_2(G_2) - EM_2(G_1) > & \sum_{x \in N_{F_1}(w)} \sum_{y \in N_{F_2}(w)} d_{G_2}(xw) d_{G_2}(wy) + \frac{1}{2} t(t-1)(d_{G_2}(w) - 1)^2 \\
& - 2d_{G_1}(uu_1) - 2d_{G_1}(u_{t-1}v) - 4(t-3) \\
& > (a+b+t-1)^2 - 2(a+1) - 2(b+1) - 4(t-3) \\
& \geq 2(a+1)(b+1) \\
& > 0.
\end{aligned}$$

Case 2: When $t = 2$. From the definition of $EM_2(G)$, we have

$$\begin{aligned}EM_2(G_1) = &\sum_{e_i \sim e_j, e_i, e_j \in E(F_1\setminus\{u\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{e_i \sim e_j, e_i, e_j \in E(F_2\setminus\{v\})} d_{G_1}(e_i)d_{G_1}(e_j)\\ &+ \sum_{x\in N_{F_1}(u)} \sum_{y \in N_{F_1}(x)\setminus\{u\}} d_{G_1}(ux)d_{G_1}(xy) + \sum_{x\in N_{F_2}(v)} \sum_{y \in N_{F_2}(x)\setminus\{v\}} d_{G_1}(vx)d_{G_1}(xy)\\ &+ \sum_{x,y\in N_{F_1}(u)} d_{G_1}(xu)d_{G_1}(uy) + \sum_{x,y\in N_{F_2}(v)} d_{G_1}(xv)d_{G_1}(vy) + \sum_{x\in N_{F_1}(u)} d_{G_1}(xu)d_{G_1}(uu_1)\\ &+ \sum_{x\in N_{F_2}(v)} d_{G_1}(xv)d_{G_1}(vu_1) + d_{G_1}(uu_1)d_{G_1}(u_1v).\end{aligned}$$

$$\begin{aligned}EM_2(G_2) = &\sum_{e_i \sim e_j, e_i, e_j \in E(F_1\setminus\{w\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{e_i \sim e_j, e_i, e_j \in E(F_2\setminus\{w\})} d_{G_1}(e_i)d_{G_1}(e_j)\\ &+ \sum_{x\in N_{F_1}(w)} \sum_{y \in N_{F_1}(x)\setminus\{w\}} d_{G_2}(wx)d_{G_1}(xy) + \sum_{x\in N_{F_2}(w)} \sum_{y \in N_{F_2}(x)\setminus\{w\}} d_{G_2}(wx)d_{G_1}(xy)\\ &+ \sum_{x,y\in N_{F_1}(w)} d_{G_2}(xw)d_{G_2}(wy) + \sum_{x,y\in N_{F_2}(w)} d_{G_2}(xw)d_{G_2}(wy)\\ &+ 2\sum_{x\in N_{F_1}(w)} d_{G_2}(xw)(d_{G_2}(w)-1) + 2\sum_{x\in N_{F_2}(w)} d_{G_2}(xw)(d_{G_2}(w)-1)\\ &+ \sum_{x\in N_{F_1}(w)} \sum_{y\in N_{F_2}(w)} d_{G_2}(xw)d_{G_2}(wy) + (d_{G_2}(w)-1)^2,\end{aligned}$$

since $d_{G_2}(w) = d_{G_1}(u) + d_{G_1}(v)$, $N_{F_1}(w) = N_{F_1}(u)$ and $N_{F_2}(w) = N_{F_2}(v)$. Then

$$\begin{aligned}EM_2(G_2) - EM_2(G_1) &> \sum_{x\in N_{F_1}(w)} \sum_{y\in N_{F_2}(w)} d_{G_2}(xw)d_{G_2}(wy) + (d_{G_2}(w)-1)^2\\ &\quad - d_{G_1}(uu_1)d_{G_1}(u_1v)\\ &> (a+b+1)^2 - (a+1)(b+1)\\ &> 0.\end{aligned}$$

Case 3: When $t = 1$. From the definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) = &\sum_{e_i \sim e_j, e_i, e_j \in E(F_1 \backslash \{u\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{e_i \sim e_j, e_i, e_j \in E(F_2 \backslash \{v\})} d_{G_1}(e_i) d_{G_1}(e_j) \\ &+ \sum_{x \in N_{F_1}(u)} \sum_{y \in N_{F_1}(x) \backslash \{u\}} d_{G_1}(ux) d_{G_1}(xy) + \sum_{x \in N_{F_2}(v)} \sum_{y \in N_{F_2}(x) \backslash \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\ &+ \sum_{x, y \in N_{F_1}(u)} d_{G_1}(xu) d_{G_1}(uy) + \sum_{x, y \in N_{F_2}(v)} d_{G_1}(xv) d_{G_1}(vy) + \sum_{x \in N_{F_1}(u)} d_{G_1}(xu) d_{G_1}(uv) \\ &+ \sum_{x \in N_{F_2}(v)} d_{G_1}(xv) d_{G_1}(vu). \end{aligned}$$

$$\begin{aligned} EM_2(G_2) = &\sum_{e_i \sim e_j, e_i, e_j \in E(F_1 \backslash \{w\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{e_i \sim e_j, e_i, e_j \in E(F_2 \backslash \{w\})} d_{G_1}(e_i) d_{G_1}(e_j) \\ &+ \sum_{x \in N_{F_1}(w)} \sum_{y \in N_{F_1}(x) \backslash \{w\}} d_{G_2}(wx) d_{G_1}(xy) + \sum_{x \in N_{F_2}(w)} \sum_{y \in N_{F_2}(x) \backslash \{w\}} d_{G_2}(wx) d_{G_1}(xy) \\ &+ \sum_{x, y \in N_{F_1}(w)} d_{G_2}(xw) d_{G_2}(wy) + \sum_{x, y \in N_{F_2}(w)} d_{G_2}(xw) d_{G_2}(wy) \\ &+ \sum_{x \in N_{F_1}(w)} d_{G_2}(xw)(d_{G_2}(w) - 1) + \sum_{x \in N_{F_2}(w)} d_{G_2}(xw)(d_{G_2}(w) - 1) \\ &+ \sum_{x \in N_{F_1}(w)} \sum_{y \in N_{F_2}(w)} d_{G_2}(xw) d_{G_2}(wy), \end{aligned}$$

since $d_{G_2}(w) = d_{G_1}(u) + d_{G_1}(v) - 1$, $N_{F_1}(w) = N_{F_1}(u)$ and $N_{F_2}(w) = N_{F_2}(v)$. Then

$$\begin{aligned} EM_2(G_2) - EM_2(G_1) > &\sum_{x \in N_{F_1}(w)} \sum_{y \in N_{F_2}(w)} d_{G_2}(xw) d_{G_2}(wy) \\ > &\ 0. \end{aligned}$$

Transformation C: Suppose u and v are any two vertices of a simple connected graph G_0. Let G_1 be the graph obtained from G_0 by adding pendant vertices $\{v_1, v_2, \ldots, v_t\}$, $\{u_1, u_2, \ldots, u_r\}$ at v and u respectively as shown in Figure 2.3. Now we have two possibilities for a transformation namely, $G_2 = G_1 - \{uu_1, uu_2, \ldots, uu_r\} + \{vu_1, vu_2, \ldots, vu_r\}$, and $G_3 = G_1 - \{vv_1, vv_2, \ldots, vv_t\} + \{uv_1, uv_2, \ldots, uv_t\}$. Here, we say that G_2 and G_3 are obtained from G_1 by **transformation C.**

Lemma 2.3. *If graphs G_2 and G_3 are obtained from graph G_1 by transformation C, then either $EM_2(G_1) < EM_2(G_2)$ or $EM_2(G_1) < EM_2(G_3)$.*

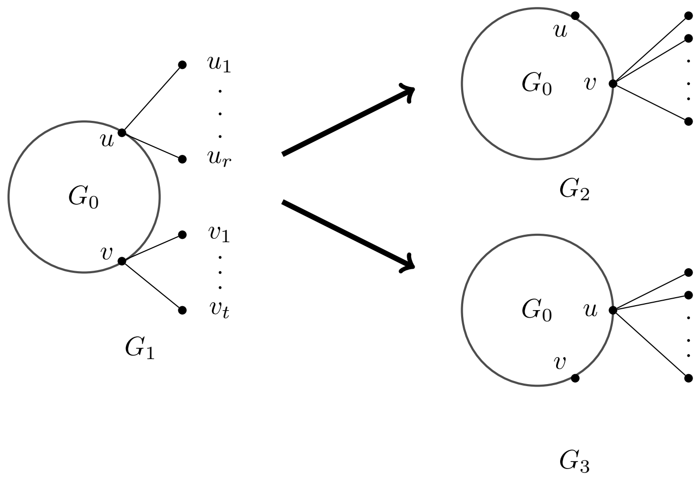


Figure 2.3: Representation of transformation C.

Proof: Let $d_{G_0}(u) = p$, $d_{G_0}(v) = q$. Without loss of generality, we assume that $d_{G_0}(v) \leq d_{G_0}(u)$, let $p = q + a$, where $a \geq 0$. Now, we prove this lemma by making conditions on the vertices u and v.

Case 1: If vertex u and v are not adjacent. Then, we have two subcases.

Subcase 1.1: If $N_{G_0}(u) \cap N_{G_0}(v) = \emptyset$, then from the definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u)} d_{G_1}(xu) d_{G_1}(uy) + \sum_{x \in N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vy) + r(q+a+r-1) \sum_{x \in N_{G_0}(u)} d_{G_1}(xu) \\ & + t(q+t-1) \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) + \frac{r(r-1)}{2}(q+a+r-1)^2 + \frac{t(t-1)}{2}(q+t-1)^2, \end{aligned}$$

$$\begin{aligned} EM_2(G_2) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} (d_{G_1}(ux) - r) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) - r)(d_{G_1}(uy) - r) + \sum_{x \in N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} (d_{G_1}(vx) + r) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v)} (d_{G_1}(xv) + r)(d_{G_1}(vy) + r) + (r+t)(q+r+t-1) \sum_{x \in N_{G_0}(v)} (d_{G_1}(xv) + r) \\ & + \frac{(r+t)(r+t-1)}{2}(q+r+t-1)^2, \end{aligned}$$

and $$EM_2(G_3) = \sum_{e_i\sim e_j, e_i,e_j\in E(G_0\backslash\{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x\in N_{G_0}(u)}\sum_{y\in N_{G_0}(x)\backslash\{u\}} (d_{G_1}(ux)+t)d_{G_1}(xy)$$
$$+ \sum_{x,y\in N_{G_0}(u)} (d_{G_1}(xu)+t)(d_{G_1}(uy)+t) + \sum_{x\in N_{G_0}(v)}\sum_{y\in N_{G_0}(x)\backslash\{v\}} (d_{G_1}(vx)-t)d_{G_1}(xy)$$
$$+ \sum_{x,y\in N_{G_0}(v)} (d_{G_1}(xv)-t)(d_{G_1}(vy)-t) + \frac{(r+t)(r+t-1)}{2}(q+a+r+t-1)^2$$
$$+ (r+t)(q+a+r+t-1)\sum_{x\in N_{G_0}(u)} (d_{G_1}(xu)+t).$$

Then, $$EM_2(G_2) - EM_2(G_1) = -r\sum_{x\in N_{G_0}(u)}\sum_{y\in N_{G_0}(x)\backslash\{u\}} d_{G_1}(xy)$$
$$- r\sum_{x,y\in N_{G_0}(u)} (d_{G_1}(xu)+d_{G_1}(uy)-r) - r(q+r+a-1)\sum_{x\in N_{G_0}(u)} d_{G_1}(xu)$$
$$+ r\sum_{x\in N_{G_0}(v)}\sum_{y\in N_{G_0}(x)\backslash\{v\}} d_{G_1}(xy) + r\sum_{x,y\in N_{G_0}(v)} (d_{G_1}(xv)+d_{G_1}(vy)+r)$$
$$+ r(q+r+2t-1)\sum_{x\in N_{G_0}(v)} d_{G_1}(xv) + qr(r+t)(q+r+t-1) + rt(q+r+t-1)^2$$
$$+ \frac{1}{2}rt(t-1)(2q+r+2t-2) + \frac{1}{2}r(r-1)(t-a)(2q+2r+t+a-2), \tag{2.1}$$

and $$EM_2(G_3) - EM_2(G_1) = t\sum_{x\in N_{G_0}(u)}\sum_{y\in N_{G_0}(x)\backslash\{u\}} d_{G_1}(xy)$$
$$+ t\sum_{x,y\in N_{G_0}(u)} (d_{G_1}(xu)+d_{G_1}(uy)+t) + t(q+2r+t+a-1)\sum_{x\in N_{G_0}(u)} d_{G_1}(xu)$$
$$- t\sum_{x\in N_{G_0}(v)}\sum_{y\in N_{G_0}(x)\backslash\{v\}} d_{G_1}(xy) - t\sum_{x,y\in N_{G_0}(v)} (d_{G_1}(xv)+d_{G_1}(vy)-t)$$
$$- t(q+t-1)\sum_{x\in N_{G_0}(v)} d_{G_1}(xv) + t(q+a)(r+t)(q+r+t+a-1)$$
$$+ \frac{1}{2}rt(r-1)(2q+2r+t+2a-2) + rt(q+r+t+a-1)^2$$
$$+ \frac{1}{2}t(t-1)(r+a)(2q+r+2t+a-2). \tag{2.2}$$

Note that if either of the expressions (2.1) or (2.2) is greater than 0, then the lemma holds. Suppose $EM_2(G_2) - EM_2(G_1) \le 0$, then we will show that $EM_2(G_3) - EM_2(G_1)$ has to be greater than 0. We will now find a relation from Equation (2.1) by letting $EM_2(G_2) - EM_2(G_1) \le 0$ and use the same to prove our claim. That is, we get

$$\begin{aligned}
& q(r+t)(q+r+t-1)+t(q+r+t-1)^2+\frac{1}{2}(r-1)(t-a)(2q+2r+t+a-2) \\
& \quad +\frac{1}{2}t(t-1)(2q+r+2t-2) \\
\leq & \sum_{x\in N_{G_0}(u)} \sum_{y\in N_{G_0}(x)\setminus\{u\}} d_{G_1}(xy)+\sum_{x,y\in N_{G_0}(u)} (d_{G_1}(xu)+d_{G_1}(uy)-r) \\
& +(q+r+a-1)\sum_{x\in N_{G_0}(u)} d_{G_1}(xu)-\sum_{x\in N_{G_0}(v)} \sum_{y\in N_{G_0}(x)\setminus\{v\}} d_{G_1}(xy) \\
& -\sum_{x,y\in N_{G_0}(v)} (d_{G_1}(xv)+d_{G_1}(vy)+r)-(q+r+2t-1)\sum_{x\in N_{G_0}(v)} d_{G_1}(xv) \\
\leq & \sum_{x\in N_{G_0}(u)} \sum_{y\in N_{G_0}(x)\setminus\{u\}} d_{G_1}(xy)+\sum_{x,y\in N_{G_0}(u)} (d_{G_1}(xu)+d_{G_1}(uy)+t) \\
& +(q+2r+t+a-1)\sum_{x\in N_{G_0}(u)} d_{G_1}(xu)-\sum_{x\in N_{G_0}(v)} \sum_{y\in N_{G_0}(x)\setminus\{v\}} d_{G_1}(xy) \\
& -\sum_{x,y\in N_{G_0}(v)} (d_{G_1}(xv)+d_{G_1}(vy)-t)-(q+t-1)\sum_{x\in N_{G_0}(v)} d_{G_1}(xv). \qquad (2.3)
\end{aligned}$$

Substituting for the first six terms of the right hand side of Equation (2.2) with the inequality (2.3), we have

$$\begin{aligned}
EM_2(G_3)-EM_2(G_1) \geq\ & t\{q(r+t)(q+r+t-1)+(q+a)(r+t)(q+r+t+a-1) \\
& +\frac{1}{2}(r-1)(t-a)(2q+2r+t+a-2)+\frac{1}{2}t(t-1)(2q+r+2t-2) \\
& +\frac{1}{2}r(r-1)(2q+2r+t+2a-2)+\frac{1}{2}(t-1)(r+a)(2q+r+2t+a-2) \\
& +t(q+r+t-1)^2+r(q+r+t+a-1)^2\} \\
>\ & 0, \quad \text{since } q\geq 2, \text{ and } r,t\geq 1.
\end{aligned}$$

Subcase 1.2: If $N_{G_0}(u)\cap N_{G_0}(v)\neq\emptyset$, then from the definition of $EM_2(G)$, we have

$$\begin{aligned}
EM_2(G_1) = & \sum_{e_i\sim e_j, e_i,e_j\in E(G_0\setminus\{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j)+\sum_{x\in N_{G_0}(u)\setminus N_{G_0}(v)} \sum_{y\in N_{G_0}(x)\setminus\{u\}} d_{G_1}(ux)d_{G_1}(xy) \\
& +\sum_{x,y\in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uy)+\sum_{x,y\in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vy)+t(q+t-1)\sum_{x\in N_{G_0}(v)} d_{G_1}(xv) \\
& +r(q+a+r-1)\sum_{x\in N_{G_0}(u)} d_{G_1}(xu)+\sum_{x\in N_{G_0}(v)\setminus N_{G_0}(u)} \sum_{y\in N_{G_0}(x)\setminus\{v\}} d_{G_1}(vx)d_{G_1}(xy) \\
& +\sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)} d_{G_1}(ux)d_{G_1}(xv)+\frac{r(r-1)}{2}(q+a+r-1)^2+\frac{t(t-1)}{2}(q+t-1)^2, \qquad (2.4)
\end{aligned}$$

$$
\begin{aligned}
EM_2(G_2) = & \sum_{e_i \sim e_j . e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x,y \in N_{G_0}(v)} (d_{G_1}(xv) + r)(d_{G_1}(vy) + r) \\
& + \sum_{x \in (N_{G_0}(u) \setminus N_{G_0}(v))} \sum_{y \in (N_{G_0}(x) \setminus \{u\})} (d_{G_1}(ux) - r) d_{G_1}(xy) + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) - r)(d_{G_1}(uy) - r) \\
& + \sum_{x \in N_{G_0}(v) \setminus N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} (d_{G_1}(vx) + r) d_{G_1}(xy) + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} (d_{G_1}(ux) - r)(d_{G_1}(xv) + r) \\
& + (r+t)(q+r+t-1) \sum_{x \in N_{G_0}(v)} (d_{G_1}(xv) + r) + \frac{(r+t)(r+t-1)}{2} (q+r+t-1)^2, \qquad (2.5)
\end{aligned}
$$

and

$$
\begin{aligned}
EM_2(G_3) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) + t)(d_{G_1}(uy) + t) \\
& + \sum_{x \in (N_{G_0}(u) \setminus N_{G_0}(v))} \sum_{y \in (N_{G_0}(x) \setminus \{u\})} (d_{G_1}(ux) + t) d_{G_1}(xy) \\
& + (r+t)(q+a+r+t-1) \sum_{x \in N_{G_0}(u)} (d_{G_1}(xu) + t) \\
& + \sum_{x \in N_{G_0}(v) \setminus N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} (d_{G_1}(vx) - t) d_{G_1}(xy) + \sum_{x,y \in N_{G_0}(v)} (d_{G_1}(xv) - t)(d_{G_1}(vy) - t) \\
& + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} (d_{G_1}(ux) + t))(d_{G_1}(xv) - t) + \frac{(r+t)(r+t-1)}{2} (q+a+r+t-1)^2. \qquad (2.6)
\end{aligned}
$$

Then,

$$
\begin{aligned}
EM_2(G_2) - EM_2(G_1) = & -r \sum_{x \in N_{G_0}(u) \setminus N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(xy) \\
& - r \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) + d_{G_1}(uy) - r) + r \sum_{x,y \in N_{G_0}(v)} (d_{G_1}(xv) + d_{G_1}(vy) + r) \\
& - r(q+r+a-1) \sum_{x \in N_{G_0}(u)} d_{G_1}(xu) + r(q+r+2t-1) \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) \\
& + r \sum_{x \in N_{G_0}(v) \setminus N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(xy) - r \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} [d_{G_1}(vx) - d_{G_1}(xu)] \\
& + rt(q+r+t-1)^2 - r^2 |N_{G_0}(u) \cap N_{G_0}(v)| + qr(r+t)(q+r+t-1) \\
& + \frac{1}{2} r(r-1)(t-a)(2q+2r+t+a-2) \frac{1}{2} rt(t-1)(2q+r+2t-2). \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
EM_2(G_3) - EM_2(G_1) = t & \sum_{x \in N_{G_0}(u) \setminus N_{G_0}(v)} \; \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(xy) \\
& + t \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) + d_{G_1}(uy) + t) - t \sum_{x,y \in N_{G_0}(v)} (d_{G_1}(xv) + d_{G_1}(vy) - t) \\
& + t(q + 2r + t + a - 1) \sum_{x \in N_{G_0}(u)} d_{G_1}(xu) - t(q + t - 1) \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) \\
& - t \sum_{x \in N_{G_0}(v) \setminus N_{G_0}(u)} \; \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(xy) + t \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} [d_{G_1}(vx) - d_{G_1}(xu)] \\
& - t^2 |N_{G_0}(u) \cap N_{G_0}(v)| + t(q + a)(r + t)(q + r + t + a - 1) + rt(q + r + t + a - 1)^2 \\
& + \frac{1}{2} rt(r - 1)(2q + 2r + t + 2a - 2) + \frac{1}{2} t(t - 1)(r + a)(2q + r + 2t + a - 2). \qquad (2.8)
\end{aligned}
$$

Again, if one of the expressions (2.7) and (2.8) is greater than 0, then the lemma holds. Suppose $EM_2(G_2) - EM_2(G_1) \leq 0$, then we will show that $EM_2(G_3) - EM_2(G_1)$ has to be greater than 0. Similar to Subcase 1.1, we will first obtain a relation from Equation (2.7) by letting $EM_2(G_2) - EM_2(G_1) \leq 0$. That is, we get

$$
\begin{aligned}
& q(r + t)(q + r + t - 1) + t(q + r + t - 1)^2 + \frac{1}{2}(r - 1)(t - a)(2q + 2r + t + a - 2) \\
& \quad + \frac{1}{2} t(t - 1)(2q + r + 2t - 2) - r|N_{G_0}(u) \cap N_{G_0}(v)| \\
& \leq \Big\{ \sum_{x \in N_{G_0}(u) \setminus N_{G_0}(v)} \; \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(xy) + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) + d_{G_1}(uy) - r) \\
& \quad + (q + r + a - 1) \sum_{x \in N_{G_0}(u)} d_{G_1}(xu) - \sum_{x \in N_{G_0}(v) \setminus N_{G_0}(u)} \; \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(xy) \\
& \quad - \sum_{x,y \in N_{G_0}(v)} (d_{G_1}(xv) + d_{G_1}(vy) + r) - (q + r + 2t - 1) \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) \\
& \quad + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} (d_{G_1}(vx) - d_{G_1}(xu)) \Big\} \\
& \leq \Big\{ \sum_{x \in N_{G_0}(u) \setminus N_{G_0}(v)} \; \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(xy) + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) + d_{G_1}(uy) + t) \\
& \quad + (q + 2r + t + a - 1) \sum_{x \in N_{G_0}(u)} d_{G_1}(xu) - \sum_{x \in N_{G_0}(v) \setminus N_{G_0}(u)} \; \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(xy) \\
& \quad - \sum_{x,y \in N_{G_0}(v)} [d_{G_1}(xv) + d_{G_1}(vy) - t] - (q + t - 1) \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) \\
& \quad + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} [d_{G_1}(vx) - d_{G_1}(xu)] \Big\}. \qquad (2.9)
\end{aligned}
$$

Substituting for the first seven terms in the right hand side of Equation (2.8) by the inequality (2.9), we have

$$
\begin{aligned}
EM_2(G_3) - EM_2(G_1) \geq\ & t\{q(r+t)(q+r+t-1) + t(q+r+t-1)^2 \\
& + r(q+r+t+a-1)^2 + \frac{1}{2}(r-1)(t-a)(2q+2r+t+a-2) \\
& + (q+a)(r+t)(q+r+t+a-1) + \frac{1}{2}r(r-1)(2q+2r+t+2a-2) \\
& + \frac{1}{2}(t-1)(r+a)(2q+r+2t+a-2) + \frac{1}{2}t(t-1)(2q+r+2t-2) \\
& - (r+t)|N_{G_0}(u) \cap N_{G_0}(v)|\} \\
>\ & 0, \text{ since } |N_{G_0}(u) \cap N_{G_0}(v)| \leq q, \quad q \geq 2, \text{ and } r, t \geq 1.
\end{aligned}
$$

Case 2: If u and v are adjacent, then from the definition of $EM_2(G)$, we have

$$
\begin{aligned}
EM_2(G_1) =\ & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x,y \in N_{G_0}(u) \setminus \{v\}} d_{G_1}(xu) d_{G_1}(uy) \\
& + \sum_{x \in N_{G_0}(u) \setminus \{v, N_{G_0}(v)\}} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux) d_{G_1}(xy) + \sum_{x,y \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv) d_{G_1}(vy) \\
& + \sum_{x \in N_{G_0}(v) \setminus \{u, N_{G_0}(u)\}} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx) d_{G_1}(xy) + r(q+a+r-1) \sum_{x \in N_{G_0}(u) \setminus \{v\}} d_{G_1}(xu) \\
& + t(q+t-1) \sum_{x \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv) + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} d_{G_1}(ux) d_{G_1}(xv) \\
& + \sum_{x \in N_{G_0}(u) \setminus \{v\}} d_{G_1}(xu) d_{G_1}(uv) + \sum_{x \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv) d_{G_1}(vu) + t(q+t-1) d_{G_1}(uv) \\
& + r(r+q+a-1) d_{G_1}(uv) + \frac{r(r-1)}{2}(q+a+r-1)^2 + \frac{t(t-1)}{2}(q+t-1)^2, \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
EM_2(G_2) =\ & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x,y \in N_{G_0}(u) \setminus \{v\}} (d_{G_1}(xu) - r)(d_{G_1}(uy) - r) \\
& + \sum_{x \in N_{G_0}(u) \setminus \{v, N_{G_0}(v)\}} \sum_{y \in N_{G_0}(x) \setminus \{u\}} (d_{G_1}(ux) - r) d_{G_1}(xy) + \sum_{x \in N_{G_0}(u) \setminus \{v\}} (d_{G_1}(xu) - r) d_{G_1}(uv) \\
& + \sum_{x \in N_{G_0}(v) \setminus \{u, N_{G_0}(u)\}} \sum_{y \in N_{G_0}(x) \setminus \{v\}} (d_{G_1}(vx) + r) d_{G_1}(xy) + \sum_{x \in N_{G_0}(v) \setminus \{u\}} (d_{G_1}(xv) + r) d_{G_1}(uv) \\
& + \sum_{x,y \in N_{G_0}(v) \setminus \{u\}} (d_{G_1}(xv) + r)(d_{G_1}(vy) + r) + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} (d_{G_1}(ux) - r)(d_{G_1}(xv) + r) \\
& + (r+t)(q+r+t-1) \sum_{x \in N_{G_0}(v) \setminus \{u\}} (d_{G_1}(xv) + r)
\end{aligned}
$$

$$+(r+t)(q+r+t-1)d_{G_1}(uv)+\frac{(r+t)((r+t-1))}{2}(q+r+t-1)^2, \qquad (2.11)$$

$$\text{and } EM_2(G_3)=\sum_{e_i\sim e_j,e_i,e_j\in E(G_0\setminus\{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j)+\sum_{x,y\in N_{G_0}(u)\setminus\{v\}}(d_{G_1}(xu)+t)(d_{G_1}(uy)+t)$$
$$+\sum_{x\in N_{G_0}(u)\setminus\{v,N_{G_0}(v)\}}\sum_{y\in N_{G_0}(x)\setminus\{u\}}(d_{G_1}(ux)+t)d_{G_1}(xy)+\sum_{x\in N_{G_0}(u)\setminus\{v\}}(d_{G_1}(xu)+t)d_{G_1}(uv)$$
$$+\sum_{x\in N_{G_0}(v)\setminus\{u,N_{G_0}(u)\}}\sum_{y\in N_{G_0}(x)\setminus\{v\}}(d_{G_1}(vx)-t)d_{G_1}(xy)+\sum_{x\in N_{G_0}(v)\setminus\{u\}}(d_{G_1}(xv)-t)d_{G_1}(uv)$$
$$+(r+t)(q+a+r+t-1)\sum_{x\in N_{G_0}(u)\setminus\{v\}}(d_{G_1}(xu)+t)(r+t)(p+r+t-1)d_{G_1}(uv)$$
$$+\sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)}(d_{G_1}(ux)+t))(d_{G_1}(xv)-t)+\frac{(r+t)((r+t-1))}{2}(q+a+r+t-1)^2. \qquad (2.12)$$

Then,

$$EM_2(G_2)-EM_2(G_1)=-r\sum_{x\in N_{G_0}(u)\setminus\{v,N_{G_0}(v)\}}\sum_{y\in N_{G_0}(x)\setminus\{u\}}d_{G_1}(xy)$$
$$-r\sum_{x,y\in N_{G_0}(u)\setminus\{v\}}[d_{G_1}(xu)+d_{G_1}(uy)-r]-r(q+r+a-1)\sum_{x\in N_{G_0}(u)\setminus\{v\}}d_{G_1}(xu)$$
$$+r\sum_{x\in N_{G_0}(v)\setminus\{u,N_{G_0}(u)\}}\sum_{y\in N_{G_0}(x)\setminus\{v\}}d_{G_1}(xy)+r\sum_{x,y\in N_{G_0}(v)\setminus\{u\}}(d_{G_1}(xv)$$
$$+rt(q+r+t-1)^2+d_{G_1}(vy)+r)+r(q+r+2t-1)\sum_{x\in N_{G_0}(v)\setminus\{u\}}d_{G_1}(xv)$$
$$-r\sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)}(d_{G_1}(vx)-d_{G_1}(xu))-r^2|N_{G_0}(u)\cap N_{G_0}(v)|+2r(t-a)d_{G_1}(uv)$$
$$+r(q-1)(r+t)(q+r+t-1)+\frac{r(r-1)}{2}(t-a)(2q+2r+t+a-1)$$
$$+\frac{rt(t-1)}{2}(2q+r+2t-2) \qquad (2.13)$$

and

$$EM_2(G_3)-EM_2(G_1)=t\sum_{x\in N_{G_0}(u)\setminus\{v,N_{G_0}(v)\}}\sum_{y\in N_{G_0}(x)\setminus\{u\}}d_{G_1}(xy)$$
$$+t\sum_{x,y\in N_{G_0}(u)\setminus\{v\}}(d_{G_1}(xu)+d_{G_1}(uy)+t)+t(q+2r+t+a-1)\sum_{x\in N_{G_0}(u)\setminus\{v\}}d_{G_1}(xu)$$
$$-t\sum_{x\in N_{G_0}(v)\setminus\{u,N_{G_0}(u)\}}\sum_{y\in N_{G_0}(x)\setminus\{v\}}d_{G_1}(xy)-t\sum_{x,y\in N_{G_0}(v)\setminus\{u\}}(d_{G_1}(xv)+d_{G_1}(vy)-t)$$

$$
\begin{aligned}
&- t(q+t-1)\sum_{x\in N_{G_0}(v)\backslash\{u\}} d_{G_1}(xv) + t\sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)} (d_{G_1}(vx)-d_{G_1}(xu))\\
&- t^2|N_{G_0}(u)\cap N_{G_0}(v)| + 2t(r+a)d_{G_1}(uv) + t(q+a-1)(r+t)(q+r+t+a-1)\\
&+ rt(q+r+t+a-1)^2 + \frac{rt(r-1)}{2}(2q+2r+t+2a-2)\\
&+ \frac{t(t-1)}{2}(r+a)(2q+r+2t+a-2). \qquad (2.14)
\end{aligned}
$$

If either of the expressions in (2.13) or (2.14) is greater than 0, then the lemma holds. Suppose $EM_2(G_2) - EM_2(G_1) \leq 0$. We will show that $EM_2(G_3) - EM_2(G_1)$ has to be greater than 0.

If $EM_2(G_2) - EM_2(G_1) \leq 0$ in Equation (2.13), we have

$$
\begin{aligned}
&(q-1)(r+t)(q+r+t-1) + \frac{t(t-1)}{2}(2q+r+2t-2) + 2(t-a)d_{G_1}(uv)\\
&\quad + t(q+r+t-1)^2 + \frac{1}{2}(r-1)(t-a)(2q+2r+t+a-1) - r|N_{G_0}(u)\cap N_{G_0}(v)|\\
&\leq \sum_{x\in N_{G_0}(u)\backslash\{v,N_{G_0}(v)\}}\ \sum_{y\in N_{G_0}(x)\backslash\{u\}} d_{G_1}(xy) + \sum_{x,y\in N_{G_0}(u)\backslash\{v\}} (d_{G_1}(xu)+d_{G_1}(uy)-r)\\
&\quad + (q+r+a-1)\sum_{x\in N_{G_0}(u)\backslash\{v\}} d_{G_1}(xu) - \sum_{x\in N_{G_0}(v)\backslash\{u,N_{G_0}(u)\}}\ \sum_{y\in N_{G_0}(x)\backslash\{v\}} d_{G_1}(xy)\\
&\quad - \sum_{x,y\in N_{G_0}(v)\backslash\{u\}} (d_{G_1}(xv)+d_{G_1}(vy)+r) - (q+r+2t-1)\sum_{x\in N_{G_0}(v)\backslash\{u\}} d_{G_1}(xv)\\
&\quad + \sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)} (d_{G_1}(vx)-d_{G_1}(xu))\\
&\leq \sum_{x\in N_{G_0}(u)\backslash\{v,N_{G_0}(v)\}}\ \sum_{y\in N_{G_0}(x)\backslash\{u\}} d_{G_1}(xy) + \sum_{x,y\in N_{G_0}(u)\backslash\{v\}} (d_{G_1}(xu)+d_{G_1}(uy)+t)\\
&\quad + (q+2r+t+a-1)\sum_{x\in N_{G_0}(u)\backslash\{v\}} d_{G_1}(xu) - \sum_{x\in N_{G_0}(v)\backslash\{u,N_{G_0}(u)\}}\ \sum_{y\in N_{G_0}(x)\backslash\{v\}} d_{G_1}(xy)\\
&\quad - \sum_{x,y\in N_{G_0}(v)\backslash\{u\}} (d_{G_1}(xv)+d_{G_1}(vy)-t) - (q+t-1)\sum_{x\in N_{G_0}(v)\backslash\{u\}} d_{G_1}(xv)\\
&\quad + \sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)} (d_{G_1}(vx)-d_{G_1}(xu)). \qquad (2.15)
\end{aligned}
$$

Now, substituting for the first seven terms in the right hand side of Equation (2.14) by the inequality (2.15), we have

$$\begin{aligned} EM_2(G_3) - EM_2(G_1) \geq \ & t\{2(t-a)d_{G_1}(uv) + (q-1)(r+t)(q+r+t-1) \\ & + \frac{1}{2}(r-1)(t-a)(2q+2r+t+a-1) + \frac{t(t-1)}{2}(2q+r+2t-2) \\ & + (q+a-1)(r+t)(q+r+t+a-1) + 2(r+a)d_{G_1}(uv) + r(q+r+t+a-1)^2 \\ & + \frac{r(r-1)}{2}(2q+2r+t+2a-2) + \frac{(t-1)}{2}(r+a)(2q+r+2t+a-2) \\ & + t(q+r+t-1)^2 - (r+t)|N_{G_0}(u) \cap N_{G_0}(v)|\}. \end{aligned}$$

Since $|N_{G_0}(u) \cap N_{G_0}(v)| \leq q$ and $d_{G_1}(uv) = (2q+r+t+a-2)$, the above inequality can be written as

$$\begin{aligned} EM_2(G_3) - EM_2(G_1) \geq \ & t\{2(t+a)(2q+r+t+a-2) + (q-1)(r+t)(q+r+t-1) \\ & + \frac{1}{2}(r-1)(t-a)(2q+2r+t+a-1) + \frac{t(t-1)}{2}(2q+r+2t-2) + t(q+r+t-1)^2 \\ & + (q+a-1)(r+t)(q+r+t+a-1) + \frac{r(r-1)}{2}(2q+2r+t+2a-2) \\ & + \frac{(t-1)}{2}(r+a)(2q+r+2t+a-2) + r(q+r+t+a-1)^2 - (r+t)q\} \\ > \ & 0, \quad \text{since } q \geq 2, \text{ and } r, t \geq 1. \end{aligned}$$

Remark 2.4. If $G_0 = C_k$, where C_k is the cycle graph on k vertices, then in Lemma 2.3, vertices u and v have the same degree in graph G_0.

Remark 2.5. If G_0 is a bicyclic graph which has no pendant vertex, see Figure 2.7. Then, in Lemma 2.3, possible degree of vertex u and v in graph G_0 are $p = q$ or $p = q + 1$ or $p = q + 2$.

To estimate the upper bounds of $EM_2(G)$ for trees, unicyclic, and bicyclic graphs we will use the transformations **A**, **B** and **C**.

2.2.2 Graph Transformations to Decrease the Index Value

In this subsection, we discuss three graph transformations that are useful for finding the lower bounds of $EM_2(G)$.

Transformation D: Let G_0 be a non-trivial graph and u be any vertex of G_0. Let G_1 be a graph obtained from G_0 by attaching at u two paths $P_1 = uu_1u_2 \ldots u_r$ of length r and $P_2 = uv_1v_2 \ldots v_t$

of length t. Let $G_2 = G_1 - \{uu_1\} + \{v_t u_1\}$; see Figure 2.4. Here G_2 is said to be obtained from G_1 by **transformation D**.

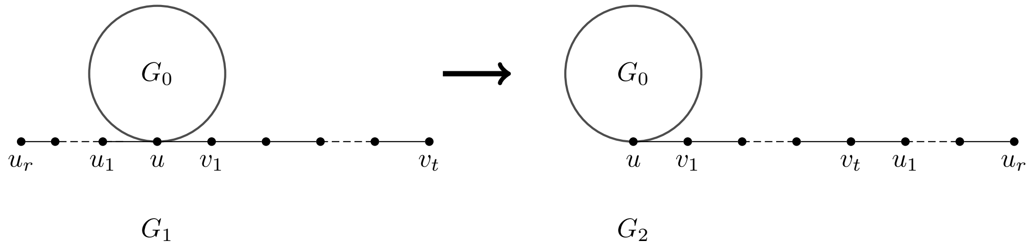


FIGURE 2.4: Representation of transformation D

Lemma 2.6. *Let G_2 be a graph obtained by Transformation D from G_1 as shown in Figure 2.4. Then, $EM_2(G_1) > EM_2(G_2)$.*

Proof: Let $N_{G_0}(u)$ denote the neighborhood of the vertex of u in G_0. Now, we prove this lemma by taking condition on path length r, t.

Case 1: If $r, t \geq 3$ then by definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) = & \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(xy) + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) + d_{G_1}(uy) - 1) \\ & + \sum_{x \in N_{G_0}(u)} ((d_{G_1}(u)+1) d_{G_1}(ux) + (d_{G_1}(u) - 1)) + d^2_{G_1}(u) + 2d_{G_1}(u) - 8 \\ > & \ \ 0, \quad \text{since } d_{G_1}(u) > 2. \end{aligned}$$

Case 2: If $r = 2, t \geq 3$, then by definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) = & \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(xy) + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) + d_{G_1}(uy) - 1) \\ & + \sum_{x \in N_{G_0}(u)} ((d_{G_1}(u)+1) d_{G_1}(ux) + (d_{G_1}(u) - 1)) + d^2_{G_1}(u) + d_{G_1}(u) - 6 \\ & > 0. \end{aligned}$$

Case 3: If $r = 1, t \geq 3$, then by definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) = &\sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(xy) + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) + d_{G_1}(uy) - 1) \\ &+ \sum_{x \in N_{G_0}(u)} (d_{G_1}(u) d_{G_1}(ux) + d_{G_1}(u) - 1) + d^2_{G_1}(u) - d_{G_1}(u) - 2 \\ &> 0. \end{aligned}$$

Case 4: If $r = 2, t = 2$, then by definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) = &\sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(xy) + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) + d_{G_1}(uy) - 1) \\ &+ \sum_{x \in N_{G_0}(u)} ((d_{G_1}(u) + 1) d_{G_1}(ux) + (d_{G_1}(u) - 1)) + d^2_{G_1}(u) - 4 \\ &> 0. \end{aligned}$$

Case 5: If $r = 1, t = 2$, then by definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) = &\sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(xy) + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) + d_{G_1}(uy) - 1) \\ &+ \sum_{x \in N_{G_0}(u)} (d_{G_1}(u) d_{G_1}(ux) + d_{G_1}(u) - 1) + d^2_{G_1}(u) - 2d_{G_1}(u) \\ &> 0. \end{aligned}$$

Case 6: If $r = 1, t = 1$, then by definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) = &\sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(xy) + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) + d_{G_1}(uy) - 1) \\ &+ \sum_{x \in N_{G_0}(u)} ((d_{G_1}(u) - 1) d_{G_1}(ux) + d_{G_1}(u) - 1) + d^2_{G_1}(u) - 3d_{G_1}(u) + 2 \\ &> 0. \end{aligned}$$

Remark 2.7. Note that, we can convert any tree into a path graph by repeating transformation D.

Transformation E: Let G_0 be a non-trivial graph having vertices u and v. Let G_1 be a graph obtain from G_0 by adding two paths $< u, u_1, u_2, \ldots, u_r >$ and $< v, v_1, v_2, \ldots, v_t >$ at vertices u

and v of length r and t, respectively. Let $G_2 := G_1 - \{uu_1\} + \{v_t u_1\}$. Here, G_2 is said to be obtained from G_1 by **transformation E**; see Figure 2.5.

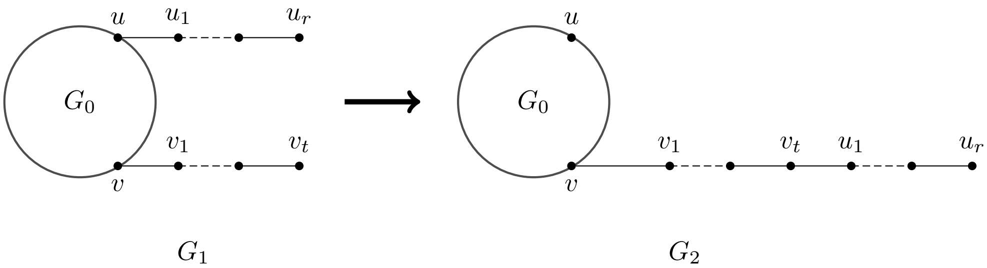


FIGURE 2.5: Representation of transformation E.

Lemma 2.8. *Let G_2 be the graph obtained from G_1 by using transformation E, then* $EM_2(G_1) > EM_2(G_2)$.

Proof: We prove this lemma by making conditions on the path lengths r and t.

Case 1: If $r \geq 3$ and $t \geq 3$. Now we have the following subcases.

Subcase 1.1: If vertex u and v are not adjacent, then we have two subcases.

Subcase 1.1.1: If $N_{G_0}(u) \cap N_{G_0}(v) = \emptyset$, then from definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u)} d_{G_1}(xu) d_{G_1}(uy) + \sum_{x \in N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vy) + \sum_{x \in N_{G_0}(u)} d_{G_1}(xu) d_{G_1}(uu_1) \\ & + \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vv_1) + 2d_{G_1}(uu_1) + 2d_{G_1}(vv_1) + 4(r-3) + 4(t-3) + 4, \end{aligned}$$

and

$$\begin{aligned} EM_2(G_2) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} (d_{G_1}(ux) - 1) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) - 1)(d_{G_1}(uy) - 1) + \sum_{x \in N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vy) + \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vv_1) + 2d_{G_1}(vv_1) \\ & + 4(r+t-6) + 14, \end{aligned}$$

then,

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) = & \sum_{x\in N_{G_0}(u)} \sum_{y\in N_{G_0}(x)\setminus\{u\}} d_{G_1}(xy) + \sum_{x\in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uu_1) \\ & + \sum_{x,y\in N_{G_0}(u)} (d_{G_1}(xu) + d_{G_1}(uy) - 1) + 2d_{G_1}(uu_1) - 10 \\ \geq & \ d_{G_1}(xu)d_{G_1}(uu_1) + 2d_{G_1}(uu_1) - 10 \\ > & \ 0, \quad \text{since} \ \ d_{G_1}(xu) \geq 2, \ d_{G_1}(uu_1) \geq 3. \end{aligned}$$

Subcase 1.1.2: If $N_{G_0}(u) \cap N_{G_0}(v) \neq \emptyset$, then from definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) = & \sum_{e_i\sim e_j, e_i, e_j\in E(G_0\setminus\{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x\in N_{G_0}(u)\setminus N_{G_0}(v)} \sum_{y\in N_{G_0}(x)\setminus\{u\}} d_{G_1}(ux)d_{G_1}(xy) \\ & + \sum_{x,y\in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uy) + \sum_{x\in N_{G_0}(v)\setminus N_{G_0}(u)} \sum_{y\in N_{G_0}(x)\setminus\{v\}} d_{G_1}(vx)d_{G_1}(xy) \\ & + \sum_{x,y\in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vy) + \sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)} d_{G_1}(ux)d_{G_1}(xv) \\ & + \sum_{x\in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uu_1) + \sum_{x\in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vv_1) + 2d_{G_1}(uu_1) \\ & + 2d_{G_1}(vv_1) + 4(r-3) + 4(t-3) + 4, \end{aligned}$$

and

$$\begin{aligned} EM_2(G_2) = & \sum_{e_i\sim e_j, e_i, e_j\in E(G_0\setminus\{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x\in N_{G_0}(u)\setminus N_{G_0}(v)} \sum_{y\in N_{G_0}(x)\setminus\{u\}} (d_{G_1}(ux) - 1)d_{G_1}(xy) \\ & + \sum_{x,y\in N_{G_0}(u)} (d_{G_1}(xu) - 1)(d_{G_1}(uy) - 1) + \sum_{x\in N_{G_0}(v)\setminus N_{G_0}(u)} \sum_{y\in N_{G_0}(x)\setminus\{v\}} d_{G_1}(vx)d_{G_1}(xy) \\ & + \sum_{x,y\in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vy) + \sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)} (d_{G_1}(ux) - 1)d_{G_1}(xv) \\ & + \sum_{x\in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vv_1) + 2d_{G_1}(vv_1) + 4(r+t-6) + 14, \end{aligned}$$

then,

$$
\begin{aligned}
EM_2(G_1) - EM_2(G_2) = & \sum_{x \in N_{G_0}(u) \setminus N_{G_0}(v)} \ \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(xy) + \sum_{x \in N_{G_0}(u)} d_{G_1}(xu) d_{G_1}(uu_1) \\
& + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) + d_{G_1}(uy) - 1) + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} d_{G_1}(xv) + 2d_{G_1}(uu_1) - 10 \\
\geq & \ d_{G_1}(xu) d_{G_1}(uu_1) + 2d_{G_1}(uu_1) - 10 \\
> & \ 0, \quad \text{since } \ d_{G_1}(xu) \geq 2, \ d_{G_1}(uu_1) \geq 3.
\end{aligned}
$$

Subcase 1.2 If vertex u and v are adjacent then from definition of $EM_2(G)$, we have

$$
\begin{aligned}
EM_2(G_1) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u) \setminus \{v, N_{G_0}(v)\}} \ \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux) d_{G_1}(xy) \\
& + \sum_{x,y \in N_{G_0}(u) \setminus \{v\}} d_{G_1}(xu) d_{G_1}(uy) + \sum_{x \in N_{G_0}(v) \setminus \{u, N_{G_0}(u)\}} \ \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\
& + \sum_{x,y \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv) d_{G_1}(vy) + \sum_{x \in N_{G_0}(u) \setminus \{v\}} d_{G_1}(xu) d_{G_1}(uu_1) \\
& + \sum_{x \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv) d_{G_1}(vv_1) + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} d_{G_1}(uy) d_{G_1}(yv) \\
& + \sum_{x \in N_{G_0}(u) \setminus \{v\}} d_{G_1}(xu) d_{G_1}(uv) + \sum_{x \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv) d_{G_1}(vu) \\
& + d_{G_1}(uu_1) d_{G_1}(uv) + d_{G_1}(vv_1) d_{G_1}(uv) + 2d_{G_1}(uu_1) + 2d_{G_1}(vv_1) \\
& + 4(r-3) + 4(t-3) + 4,
\end{aligned}
$$

and

$$
\begin{aligned}
EM_2(G_2) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x,y \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv) d_{G_1}(vy) \\
& + \sum_{x,y \in N_{G_0}(u) \setminus \{v\}} (d_{G_1}(xu) - 1)(d_{G_1}(uy) - 1) + \sum_{x \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv) d_{G_1}(vv_1) \\
& + \sum_{x \in N_{G_0}(u) \setminus \{v, N_{G_0}(v)\}} \ \sum_{y \in N_{G_0}(x) \setminus \{u\}} (d_{G_1}(ux) - 1) d_{G_1}(xy) \\
& + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} (d_{G_1}(ux) - 1) d_{G_1}(xv) + \sum_{x \in N_{G_0}(u) \setminus \{v\}} (d_{G_1}(xu) - 1)(d_{G_1}(uv) - 1)
\end{aligned}
$$

$$+\sum_{x\in N_{G_0}(v)\setminus\{u,N_{G_0}(u)\}}\ \sum_{y\in N_{G_0}(x)\setminus\{v\}} d_{G_1}(vx)d_{G_1}(xy)+\sum_{x\in N_{G_0}(v)\setminus\{u\}} d_{G_1}(xv)(d_{G_1}(vu)-1)$$
$$+\,d_{G_1}(vv_1)(d_{G_1}(uv)-1)+2d_{G_1}(vv_1)+4(r+t-6)+14,$$

then

$$\begin{aligned}EM_2(G_1)-EM_2(G_2) &= \sum_{x\in N_{G_0}(u)\setminus\{v,N_{G_0}(v)\}}\ \sum_{y\in N_{G_0}(x)\setminus\{u\}} d_{G_1}(xy)+\sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)} d_{G_1}(xv)\\ &+\sum_{x,y\in N_{G_0}(u)\setminus\{v\}}(d_{G_1}(xu)+d_{G_1}(uy)-1)+\sum_{x\in N_{G_0}(u)\setminus\{v\}}(d_{G_1}(xu)+d_{G_1}(uv)-1)\\ &+\sum_{x\in N_{G_0}(v)\setminus\{u\}} d_{G_1}(xv)+d_{G_1}(vv_1)+2d_{G_1}(uu_1)+d_{G_1}(uu_1)d_{G_1}(uv)-10\\ &\geq\ d_{G_1}(vv_1)+2d_{G_1}(uu_1)+d_{G_1}(uu_1)d_{G_1}(uv)-10\\ &>\ 0,\quad \text{since}\ \ d_{G_1}(vv_1)\geq 3,\ \ d_{G_1}(uu_1)\geq 3 \text{ and } d_{G_1}(uv)\geq 3.\end{aligned}$$

Proof is complete for $r,t\geq 3$.

Case 2: When $r=1$ and $t=3$, then from the definition of $EM_2(G)$, we have

Subcase 2.1: If vertex u and v are not adjacent, then we have two subcases.

Subcase 2.1.1: If $N_{G_0}(u)\cap N_{G_0}(v)=\emptyset$, then from definition of $EM_2(G)$, we have

$$\begin{aligned}EM_2(G_1) = &\sum_{e_i\sim e_j,e_i,e_j\in E(G_0\setminus\{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j)+\sum_{x\in N_{G_0}(u)}\ \sum_{y\in N_{G_0}(x)\setminus\{u\}} d_{G_1}(ux)d_{G_1}(xy)\\ &+\sum_{x,y\in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uy)+\sum_{x\in N_{G_0}(v)}\ \sum_{y\in N_{G_0}(x)\setminus\{v\}} d_{G_1}(vx)d_{G_1}(xy)\\ &+\sum_{x,y\in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vy)+\sum_{x\in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uu_1)\\ &+\sum_{x\in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vv_1)+2d_{G_1}(vv_1)+2,\end{aligned}$$

and

$$\begin{aligned}EM_2(G_2) = &\sum_{e_i\sim e_j,e_i,e_j\in E(G_0\setminus\{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j)+\sum_{x\in N_{G_0}(u)}\ \sum_{y\in N_{G_0}(x)\setminus\{u\}}(d_{G_1}(ux)-1)d_{G_1}(xy)\\ &+\sum_{x,y\in N_{G_0}(u)}(d_{G_1}(xu)-1)(d_{G_1}(uy)-1)+\sum_{x\in N_{G_0}(v)}\ \sum_{y\in N_{G_0}(x)\setminus\{v\}} d_{G_1}(vx)d_{G_1}(xy)\\ &+\sum_{x,y\in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vy)+\sum_{x\in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vv_1)+2d_{G_1}(vv_1)+6,\end{aligned}$$

then,

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) &> d_{G_1}(ux)d_{G_1}(uu_1) - 4 \\ &> 0, \quad \text{since} \;\; d_{G_1}(uu_1) \geq 3 \text{ and } d_{G_1}(ux) \geq 3. \end{aligned}$$

Subcase 2.1.2: If $N_{G_0}(u) \cap N_{G_0}(v) \neq \emptyset$, then from definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) = &\sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u) \setminus N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux)d_{G_1}(xy) \\ &+ \sum_{x,y \in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uy) + \sum_{x \in N_{G_0}(v) \setminus N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx)d_{G_1}(xy) \\ &+ \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vy) + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} d_{G_1}(ux)d_{G_1}(xv) \\ &+ \sum_{x \in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uu_1) + \sum_{x \in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vv_1) \\ &+ 2d_{G_1}(vv_1) + 2, \end{aligned}$$

and

$$\begin{aligned} EM_2(G_2) = &\sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u) \setminus N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} (d_{G_1}(ux) - 1)d_{G_1}(xy) \\ &+ \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) - 1)(d_{G_1}(uy) - 1) + \sum_{x \in N_{G_0}(v) \setminus N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx)d_{G_1}(xy) \\ &+ \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vy) + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} (d_{G_1}(ux) - 1)d_{G_1}(xv) \\ &+ \sum_{x \in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vv_1) + 2d_{G_1}(vv_1) + 6, \end{aligned}$$

then,

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) &\geq d_{G_1}(ux)d_{G_1}(uu_1) - 4 \\ &> 0, \quad \text{since} \;\; d_{G_1}(uu_1) \geq 3 \text{ and } d_{G_1}(ux) \geq 3. \end{aligned}$$

Subcase 2.2 If vertex u and v are adjacent then from definition of $EM_2(G)$, we have

$$
\begin{aligned}
EM_2(G_1) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0\backslash\{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x\in N_{G_0}(u)\backslash\{v,N_{G_0}(v)\}} \sum_{y\in N_{G_0}(x)\backslash\{u\}} d_{G_1}(ux)d_{G_1}(xy) \\
& + \sum_{x,y\in N_{G_0}(u)\backslash\{v\}} d_{G_1}(xu)d_{G_1}(uy) + \sum_{x\in N_{G_0}(v)\backslash\{u,N_{G_0}(u)\}} \sum_{y\in N_{G_0}(x)\backslash\{v\}} d_{G_1}(vx)d_{G_1}(xy) \\
& + \sum_{x,y\in N_{G_0}(v)\backslash\{u\}} d_{G_1}(xv)d_{G_1}(vy) + \sum_{x\in N_{G_0}(u)\backslash\{v\}} d_{G_1}(xu)d_{G_1}(uu_1) \\
& + \sum_{x\in N_{G_0}(v)\backslash\{u\}} d_{G_1}(xv)d_{G_1}(vv_1) + \sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)} d_{G_1}(uy)d_{G_1}(yv) \\
& + \sum_{x\in N_{G_0}(u)\backslash\{v\}} d_{G_1}(xu)d_{G_1}(uv) + \sum_{x\in N_{G_0}(v)\backslash\{u\}} d_{G_1}(xv)d_{G_1}(vu) \\
& + d_{G_1}(uu_1)d_{G_1}(uv) + d_{G_1}(vv_1)d_{G_1}(uv) + 2d_{G_1}(vv_1) + 2,
\end{aligned}
$$

and

$$
\begin{aligned}
EM_2(G_2) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0\backslash\{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x,y\in N_{G_0}(v)\backslash\{u\}} d_{G_1}(xv)d_{G_1}(vy) \\
& + \sum_{x,y\in N_{G_0}(u)\backslash\{v\}} (d_{G_1}(xu)-1)(d_{G_1}(uy)-1) + \sum_{x\in N_{G_0}(v)\backslash\{u\}} d_{G_1}(xv)d_{G_1}(vv_1) \\
& + \sum_{x\in N_{G_0}(u)\backslash\{v,N_{G_0}(v)\}} \sum_{y\in N_{G_0}(x)\backslash\{u\}} (d_{G_1}(ux)-1)d_{G_1}(xy) \\
& + \sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)} (d_{G_1}(ux)-1)d_{G_1}(xv) + \sum_{x\in N_{G_0}(u)\backslash\{v\}} (d_{G_1}(xu)-1)(d_{G_1}(uv)-1) \\
& + \sum_{x\in N_{G_0}(v)\backslash\{u,N_{G_0}(u)\}} \sum_{y\in N_{G_0}(x)\backslash\{v\}} d_{G_1}(vx)d_{G_1}(xy) + \sum_{x\in N_{G_0}(v)\backslash\{u\}} d_{G_1}(xv)(d_{G_1}(vu)-1) \\
& + d_{G_1}(vv_1)(d_{G_1}(uv)-1) + 2d_{G_1}(vv_1) + 6,
\end{aligned}
$$

then

$$
\begin{aligned}
EM_2(G_1) - EM_2(G_2) & \geq d_{G_1}(ux)d_{G_1}(uu_1) - 4 \\
& > 0, \quad \text{since} \quad d_{G_1}(uu_1) \geq 3 \text{ and } d_{G_1}(ux) \geq 3.
\end{aligned}
$$

Proof is complete for $r = 1,\ t = 3$.

Case 3: When $r = 2$ and $t = 3$, then from the definition of $EM_2(G)$, we have

Subcase 3.1: If vertex u and v are not adjacent, then we have two subcases.

Subcase 3.1.1: If $N_{G_0}(u) \cap N_{G_0}(v) = \emptyset$, then from definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u)} d_{G_1}(xu) d_{G_1}(uy) + \sum_{x \in N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vy) + \sum_{x \in N_{G_0}(u)} d_{G_1}(xu) d_{G_1}(uu_1) \\ & + \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vv_1) + d_{G_1}(uu_1) + 2d_{G_1}(vv_1) + 2, \end{aligned}$$

and

$$\begin{aligned} EM_2(G_2) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} (d_{G_1}(ux) - 1) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) - 1)(d_{G_1}(uy) - 1) + \sum_{x \in N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vy) + \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vv_1) + 2d_{G_1}(vv_1) + 10, \end{aligned}$$

then,

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) & \geq d_{G_1}(ux) d_{G_1}(uu_1) + d_{G_1}(uu_1) - 8 \\ & > 0, \quad \text{since} \quad d_{G_1}(uu_1) \geq 3 \text{ and } d_{G_1}(ux) \geq 3. \end{aligned}$$

Subcase 3.1.2: If $N_{G_0}(u) \cap N_{G_0}(v) \neq \emptyset$, then from definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u) \setminus N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u)} d_{G_1}(xu) d_{G_1}(uy) + \sum_{x \in N_{G_0}(v) \setminus N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vy) + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} d_{G_1}(ux) d_{G_1}(xv) \\ & + \sum_{x \in N_{G_0}(u)} d_{G_1}(xu) d_{G_1}(uu_1) + \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vv_1) + d_{G_1}(uu_1) \\ & + 2d_{G_1}(vv_1) + 2, \end{aligned}$$

and

$$\begin{aligned} EM_2(G_2) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u) \setminus N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} (d_{G_1}(ux) - 1) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) - 1)(d_{G_1}(uy) - 1) + \sum_{x \in N_{G_0}(v) \setminus N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vy) + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} (d_{G_1}(ux) - 1) d_{G_1}(xv) \\ & + \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vv_1) + 2 d_{G_1}(vv_1) + 10, \end{aligned}$$

then,

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) & \geq d_{G_1}(ux) d_{G_1}(uu_1) + d_{G_1}(uu_1) - 8 \\ & > 0, \quad \text{since} \quad d_{G_1}(uu_1) \geq 3 \text{ and } d_{G_1}(ux) \geq 3. \end{aligned}$$

Subcase 3.2 If vertex u and v are adjacent then from definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u) \setminus \{v, N_{G_0}(v)\}} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u) \setminus \{v\}} d_{G_1}(xu) d_{G_1}(uy) + \sum_{x \in N_{G_0}(v) \setminus \{u, N_{G_0}(u)\}} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv) d_{G_1}(vy) + \sum_{x \in N_{G_0}(u) \setminus \{v\}} d_{G_1}(xu) d_{G_1}(uu_1) \\ & + \sum_{x \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv) d_{G_1}(vv_1) + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} d_{G_1}(uy) d_{G_1}(yv) \\ & + \sum_{x \in N_{G_0}(u) \setminus \{v\}} d_{G_1}(xu) d_{G_1}(uv) + \sum_{x \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv) d_{G_1}(vu) \\ & + d_{G_1}(uu_1) d_{G_1}(uv) + d_{G_1}(vv_1) d_{G_1}(uv) + d_{G_1}(uu_1) + 2 d_{G_1}(vv_1) + 2, \end{aligned}$$

and

$$\begin{aligned}
EM_2(G_2) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x,y \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv)d_{G_1}(vy) \\
& + \sum_{x,y \in N_{G_0}(u) \setminus \{v\}} (d_{G_1}(xu)-1)(d_{G_1}(uy)-1) + \sum_{x \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv)d_{G_1}(vv_1) \\
& + \sum_{x \in N_{G_0}(u) \setminus \{v, N_{G_0}(v)\}} \sum_{y \in N_{G_0}(x) \setminus \{u\}} (d_{G_1}(ux)-1)d_{G_1}(xy) \\
& + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} (d_{G_1}(ux)-1)d_{G_1}(xv) + \sum_{x \in N_{G_0}(u) \setminus \{v\}} (d_{G_1}(xu)-1)(d_{G_1}(uv)-1) \\
& + \sum_{x \in N_{G_0}(v) \setminus \{u, N_{G_0}(u)\}} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx)d_{G_1}(xy) + \sum_{x \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv)(d_{G_1}(vu)-1) \\
& + d_{G_1}(vv_1)(d_{G_1}(uv)-1) + 2d_{G_1}(vv_1) + 10,
\end{aligned}$$

then

$$\begin{aligned}
EM_2(G_1) - EM_2(G_2) &\geq d_{G_1}(ux)d_{G_1}(uu_1) + d_{G_1}(uu_1) - 8 \\
&> 0, \quad \text{since} \quad d_{G_1}(uu_1) \geq 3 \text{ and } d_{G_1}(ux) \geq 3.
\end{aligned}$$

Proof is complete for $r = 2,\ t = 3$.

Case 4: When $r = 1$ and $t = 2$, then from the definition of $EM_2(G)$, we have

Subcase 4.1: If vertex u and v are not adjacent, then we have two subcases.

Subcase 4.1.1: If $N_{G_0}(u) \cap N_{G_0}(v) = \emptyset$, then from definition of $EM_2(G)$, we have

$$\begin{aligned}
EM_2(G_1) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux)d_{G_1}(xy) \\
& + \sum_{x,y \in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uy) + \sum_{x \in N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx)d_{G_1}(xy) \\
& + \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vy) + \sum_{x \in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uu_1) \\
& + \sum_{x \in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vv_1) + d_{G_1}(vv_1),
\end{aligned}$$

and

$$\begin{aligned} EM_2(G_2) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} (d_{G_1}(ux) - 1) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) - 1)(d_{G_1}(uy) - 1) + \sum_{x \in N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vy) + \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vv_1) + 2d_{G_1}(vv_1) + 2, \end{aligned}$$

then,

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) & \geq d_{G_1}(ux) d_{G_1}(uu_1) - 2 \\ & > 0, \quad \text{since} \quad d_{G_1}(uu_1) \geq 3 \text{ and } d_{G_1}(ux) \geq 3. \end{aligned}$$

Subcase 4.1.2: If $N_{G_0}(u) \cap N_{G_0}(v) \neq \emptyset$, then from definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u) \setminus N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u)} d_{G_1}(xu) d_{G_1}(uy) + \sum_{x \in N_{G_0}(v) \setminus N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vy) + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} d_{G_1}(ux) d_{G_1}(xv) \\ & + \sum_{x \in N_{G_0}(u)} d_{G_1}(xu) d_{G_1}(uu_1) + \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vv_1) + d_{G_1}(vv_1), \end{aligned}$$

and

$$\begin{aligned} EM_2(G_2) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i) d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u) \setminus N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} (d_{G_1}(ux) - 1) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) - 1)(d_{G_1}(uy) - 1) + \sum_{x \in N_{G_0}(v) \setminus N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx) d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vy) + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} (d_{G_1}(ux) - 1) d_{G_1}(xv) \\ & + \sum_{x \in N_{G_0}(v)} d_{G_1}(xv) d_{G_1}(vv_1) + 2d_{G_1}(vv_1) + 2, \end{aligned}$$

then,

$$\begin{aligned}EM_2(G_1) - EM_2(G_2) &\geq d_{G_1}(ux)d_{G_1}(uu_1) - 2\\ &> 0, \quad \text{since} \quad d_{G_1}(uu_1) \geq 3 \text{ and } d_{G_1}(ux) \geq 3.\end{aligned}$$

Subcase 4.2 If vertex u and v are adjacent then from definition of $EM_2(G)$, we have

$$\begin{aligned}EM_2(G_1) = &\sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u) \setminus \{v, N_{G_0}(v)\}} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux)d_{G_1}(xy)\\
&+ \sum_{x,y \in N_{G_0}(u) \setminus \{v\}} d_{G_1}(xu)d_{G_1}(uy) + \sum_{x \in N_{G_0}(v) \setminus \{u, N_{G_0}(u)\}} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx)d_{G_1}(xy)\\
&+ \sum_{x,y \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv)d_{G_1}(vy) + \sum_{x \in N_{G_0}(u) \setminus \{v\}} d_{G_1}(xu)d_{G_1}(uu_1)\\
&+ \sum_{x \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv)d_{G_1}(vv_1) + \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} d_{G_1}(uy)d_{G_1}(yv)\\
&+ \sum_{x \in N_{G_0}(u) \setminus \{v\}} d_{G_1}(xu)d_{G_1}(uv) + \sum_{x \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv)d_{G_1}(vu)\\
&+ d_{G_1}(uu_1)d_{G_1}(uv) + d_{G_1}(vv_1)d_{G_1}(uv) + d_{G_1}(vv_1),\end{aligned}$$

and

$$\begin{aligned}EM_2(G_2) = &\sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x,y \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv)d_{G_1}(vy)\\
&+ \sum_{x,y \in N_{G_0}(u) \setminus \{v\}} (d_{G_1}(xu) - 1)(d_{G_1}(uy) - 1) + \sum_{x \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv)d_{G_1}(vv_1)\\
&+ \sum_{x \in N_{G_0}(u) \setminus \{v, N_{G_0}(v)\}} \sum_{y \in N_{G_0}(x) \setminus \{u\}} (d_{G_1}(ux) - 1)d_{G_1}(xy)\\
&+ \sum_{x \in N_{G_0}(u) \cap N_{G_0}(v)} (d_{G_1}(ux) - 1)d_{G_1}(xv) + \sum_{x \in N_{G_0}(u) \setminus \{v\}} (d_{G_1}(xu) - 1)(d_{G_1}(uv) - 1)\\
&+ \sum_{x \in N_{G_0}(v) \setminus \{u, N_{G_0}(u)\}} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx)d_{G_1}(xy) + \sum_{x \in N_{G_0}(v) \setminus \{u\}} d_{G_1}(xv)(d_{G_1}(vu) - 1)\\
&+ d_{G_1}(vv_1)(d_{G_1}(uv) - 1) + 2d_{G_1}(vv_1) + 2,\end{aligned}$$

then

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) &\geq d_{G_1}(ux)d_{G_1}(uu_1) - 2 \\ &> 0, \quad \text{since } \; d_{G_1}(uu_1) \geq 3 \text{ and } d_{G_1}(ux) \geq 3. \end{aligned}$$

Proof is complete for $r = 1, \; t = 2$.

Case 5: When $r = 2$ and $t = 2$, then from the definition of $EM_2(G)$, we have

Subcase 5.1: If vertex u and v are not adjacent, then we have two subcases.

Subcase 5.1.1: If $N_{G_0}(u) \cap N_{G_0}(v) = \emptyset$, then from definition of $EM_2(G)$, we have

$$\begin{aligned} EM_2(G_1) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux)d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uy) + \sum_{x \in N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx)d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vy) + \sum_{x \in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uu_1) \\ & + \sum_{x \in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vv_1) + d_{G_1}(uu_1) + d_{G_1}(vv_1), \end{aligned}$$

and

$$\begin{aligned} EM_2(G_2) = & \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} (d_{G_1}(ux) - 1)d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(u)} (d_{G_1}(xu) - 1)(d_{G_1}(uy) - 1) + \sum_{x \in N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{v\}} d_{G_1}(vx)d_{G_1}(xy) \\ & + \sum_{x,y \in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vy) + \sum_{x \in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vv_1) + 2d_{G_1}(vv_1) + 6, \end{aligned}$$

then,

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) &\geq d_{G_1}(ux)d_{G_1}(uu_1) - 6 \\ &> 0, \quad \text{since } \; d_{G_1}(uu_1) \geq 3 \text{ and } d_{G_1}(ux) \geq 3. \end{aligned}$$

Subcase 5.1.2: If $N_{G_0}(u) \cap N_{G_0}(v) \neq \emptyset$, then from definition of $EM_2(G)$, we have

$$EM_2(G_1) = \sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x \in N_{G_0}(u) \setminus N_{G_0}(v)} \sum_{y \in N_{G_0}(x) \setminus \{u\}} d_{G_1}(ux)d_{G_1}(xy)$$

$$+\sum_{x,y\in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uy)+\sum_{x\in N_{G_0}(v)\setminus N_{G_0}(u)}\ \sum_{y\in N_{G_0}(x)\setminus\{v\}} d_{G_1}(vx)d_{G_1}(xy)$$
$$+\sum_{x,y\in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vy)+\sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)} d_{G_1}(ux)d_{G_1}(xv)$$
$$+\sum_{x\in N_{G_0}(u)} d_{G_1}(xu)d_{G_1}(uu_1)+\sum_{x\in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vv_1)+d_{G_1}(uu_1)$$
$$+d_{G_1}(vv_1),$$

and

$$EM_2(G_2)=\sum_{e_i\sim e_j,e_i,e_j\in E(G_0\setminus\{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j)+\sum_{x\in N_{G_0}(u)\setminus N_{G_0}(v)}\ \sum_{y\in N_{G_0}(x)\setminus\{u\}} (d_{G_1}(ux)-1)d_{G_1}(xy)$$
$$+\sum_{x,y\in N_{G_0}(u)} (d_{G_1}(xu)-1)(d_{G_1}(uy)-1)+\sum_{x\in N_{G_0}(v)\setminus N_{G_0}(u)}\ \sum_{y\in N_{G_0}(x)\setminus\{v\}} d_{G_1}(vx)d_{G_1}(xy)$$
$$+\sum_{x,y\in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vy)+\sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)} (d_{G_1}(ux)-1)d_{G_1}(xv)$$
$$+\sum_{x\in N_{G_0}(v)} d_{G_1}(xv)d_{G_1}(vv_1)+2d_{G_1}(vv_1)+6,$$

then,

$$EM_2(G_1)-EM_2(G_2)\geq d_{G_1}(ux)d_{G_1}(uu_1)-6$$
$$>\ 0,\quad \text{since}\ \ d_{G_1}(uu_1)\geq 3 \text{ and } d_{G_1}(ux)\geq 3.$$

Subcase 5.2 If vertex u and v are adjacent then from definition of $EM_2(G)$, we have

$$EM_2(G_1)=\sum_{e_i\sim e_j,e_i,e_j\in E(G_0\setminus\{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j)+\sum_{x\in N_{G_0}(u)\setminus\{v,N_{G_0}(v)\}}\ \sum_{y\in N_{G_0}(x)\setminus\{u\}} d_{G_1}(ux)d_{G_1}(xy)$$
$$+\sum_{x,y\in N_{G_0}(u)\setminus\{v\}} d_{G_1}(xu)d_{G_1}(uy)+\sum_{x\in N_{G_0}(v)\setminus\{u,N_{G_0}(u)\}}\ \sum_{y\in N_{G_0}(x)\setminus\{v\}} d_{G_1}(vx)d_{G_1}(xy)$$
$$+\sum_{x,y\in N_{G_0}(v)\setminus\{u\}} d_{G_1}(xv)d_{G_1}(vy)+\sum_{x\in N_{G_0}(u)\setminus\{v\}} d_{G_1}(xu)d_{G_1}(uu_1)$$
$$+\sum_{x\in N_{G_0}(v)\setminus\{u\}} d_{G_1}(xv)d_{G_1}(vv_1)+\sum_{x\in N_{G_0}(u)\cap N_{G_0}(v)} d_{G_1}(uy)d_{G_1}(yv)$$
$$+\sum_{x\in N_{G_0}(u)\setminus\{v\}} d_{G_1}(xu)d_{G_1}(uv)+\sum_{x\in N_{G_0}(v)\setminus\{u\}} d_{G_1}(xv)d_{G_1}(vu)$$

$$+ d_{G_1}(uu_1)d_{G_1}(uv) + d_{G_1}(vv_1)d_{G_1}(uv) + d_{G_1}(uu_1) + d_{G_1}(vv_1),$$

and

$$\begin{aligned}EM_2(G_2) = &\sum_{e_i \sim e_j, e_i, e_j \in E(G_0 \setminus \{u,v\})} d_{G_1}(e_i)d_{G_1}(e_j) + \sum_{x,y \in N_{G_0}(v)\setminus\{u\}} d_{G_1}(xv)d_{G_1}(vy)\\ &+ \sum_{x,y \in N_{G_0}(u)\setminus\{v\}} (d_{G_1}(xu)-1)(d_{G_1}(uy)-1) + \sum_{x \in N_{G_0}(v)\setminus\{u\}} d_{G_1}(xv)d_{G_1}(vv_1)\\ &+ \sum_{x \in N_{G_0}(u)\setminus\{v, N_{G_0}(v)\}} \sum_{y \in N_{G_0}(x)\setminus\{u\}} (d_{G_1}(ux)-1)d_{G_1}(xy)\\ &+ \sum_{x \in N_{G_0}(u)\cap N_{G_0}(v)} (d_{G_1}(ux)-1)d_{G_1}(xv) + \sum_{x \in N_{G_0}(u)\setminus\{v\}} (d_{G_1}(xu)-1)(d_{G_1}(uv)-1)\\ &+ \sum_{x \in N_{G_0}(v)\setminus\{u, N_{G_0}(u)\}} \sum_{y \in N_{G_0}(x)\setminus\{v\}} d_{G_1}(vx)d_{G_1}(xy) + \sum_{x \in N_{G_0}(v)\setminus\{u\}} d_{G_1}(xv)(d_{G_1}(vu)-1)\\ &+ d_{G_1}(vv_1)(d_{G_1}(uv)-1) + 2d_{G_1}(vv_1) + 6,\end{aligned}$$

then

$$\begin{aligned}EM_2(G_1) - EM_2(G_2) &\geq d_{G_1}(ux)d_{G_1}(uu_1) - 6\\ &> 0, \text{ since } d_{G_1}(uu_1) \geq 3 \text{ and } d_{G_1}(ux) \geq 3.\end{aligned}$$

Proof is complete for $r = 2,\ t = 2$. □

Remark 2.9. By suitable and repeated application of transformations D and E to any unicyclic graph on n vertices and having girth k, we can obtain a $(k, n-k)$-tadpole graph. We denote a $(k, n-k)$-tadpole graph by C_n^k.

Note: A $(p, q)-$tadpole graph is a special type of unicyclic graph obtained by attaching a path graph on q vertices to a cycle graph on p (at least 3) vertices with a bridge.

Transformation F: Let G_0 be a graph containing a path of length at least 2, say $< u, x_1, v >$. Let G_1 be the graph obtained from G_0 by attaching a path $< x_1, x_2, \ldots, x_k >, (k \geq 1)$ at x_1. Let $G_2 := G_1 - \{x_1v\} + \{x_kv\}$; see Figure 2.6. We say that G_2 is obtained from G_1 by **transformation F**.

Lemma 2.10. *If G_2 is obtained from G_1 using transformation F, then $EM_2(G_1) > EM_2(G_2)$.*

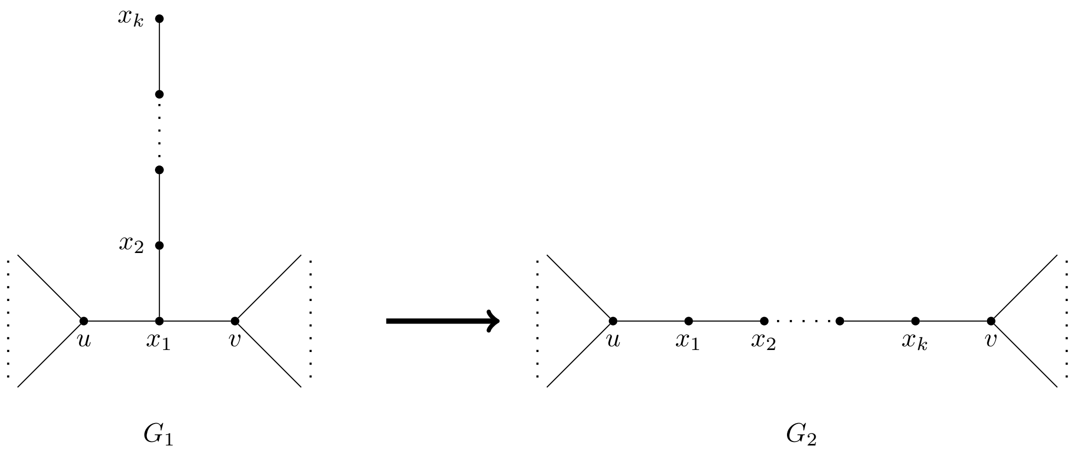
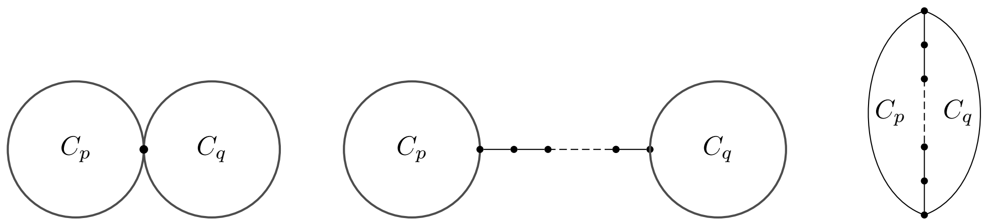


FIGURE 2.6: Representation of transformation F

Proof: From the definition of $EM_2(G)$ and using the fact that $d_{G_2}(ux_1) = d_{G_1}(ux_1) - 1$, $d_{G_2}(x_kv) = d_{G_1}(x_1v) - 1$, we have

$$\begin{aligned} EM_2(G_1) - EM_2(G_2) &\geq d_{G_1}(ux_1)d_{G_1}(x_1v) + 2d_{G_1}(ux_1) + 2d_{G_1}(x_1v) \\ &\quad - 2d_{G_2}(ux_1) - 2d_{G_2}(x_kv) \\ &= d_{G_1}(ux_1)d_{G_1}(x_1v) + 4 > 0. \end{aligned}$$

Remark 2.11. Similarly, repeated application of the transformations D and E to a bicyclic graph will result in such a bicyclic graph where a path is attached to one of the graphs in Figure 2.7 and finally applying transformation F to the resultant graph will result in exactly one of the graphs in Figure 2.7.



(a) $D_n^1(p,q)$ (b) $D_n^2(p,q,l)$ (c) $D_n^3(p,q,l)$

FIGURE 2.7: Bicyclic graphs which has no pendant vertices

2.3 Bounds of Second Reformulated Zagreb Index

In this section, we compute the upper and lower bounds for trees, unicyclic, bicyclic and tricyclic graphs using the six transformations discussed in sections 2 and 3.

2.3.1 Trees

It is well known that the path graph is the only tree on 3 vertices and $EM_2(P_3) = 1$. For trees on at least 4 vertices we have the following theorem.

Theorem 2.12. *If* $T \neq \{P_n, S_n\}$ *is a tree on* $n \geq 4$ *vertices, then* $4n-12 = EM_2(P_n) < EM_2(T) < EM_2(S_n) = \frac{1}{2}(n-1)(n-2)^3$.

Proof: Since any tree T can be transformed into a star S_n by using the tranformations A and C, it follows immediately from Lemma 2.1 and Lemma 2.3, $EM_2(S_n) > EM_2(T)$.
Similarly, for the lower bound, we can see that the result follows directly from Lemma 2.6 and Remark 2.7.
By direct computation using the definition of the second reformulated Zagreb index, it follows that $EM_2(P_n) = 4n - 12$ and $EM_2(S_n) = \frac{1}{2}(n-1)(n-2)^3$ and hence the theorem holds.

From the above theorem, it is immediate that among the trees on n vertices, the path graph attains the least value and the star graph attains the greatest value for the second reformulated Zagreb index.

2.3.2 Unicyclic Graphs

Before finding the upper bounds for the unicyclic graph, let us fix some notation. Let U_n^k denotes the unicyclic graph obtained from the cycle C_k of girth k by attaching $n-k$ pendent vertices to any one vertex on C_k. Let C_n^k denotes the tadpole graph with girth k, that is, the cycle of girth k is connected to a path P_{n-k} with a bridge.

Theorem 2.13. *If* G *is a unicyclic graph on* n *vertices and girth* k*, then* $EM_2(G) \leq EM_2(U_n^k)$*. Equality holds if and only if* $G \simeq U_n^k$

Proof: Proof follows from Lemma 2.1 and Lemma 2.3 as any unicyclic graph can be transformed into U_n^k by using transformations A and C.

Corollary 2.14. *For fixed $n \geq 4$ and girth $4 \leq k \leq n$, $EM_2(U_n^k) \leq EM_2(U_n^{k-1})$.*

Proof: From the definition of $EM_2(G)$, we have upon simplification

$$EM_2(U_n^k) = \frac{1}{2}(n-k+1)^2[(n-k)^2 + 3(n-k+2)] + (4n-3)$$

Therefore,

$$\begin{aligned} EM_2(U_n^k) - EM_2(U_n^{k-1}) =& -\frac{1}{2}(n-k)(n-k+1)(4n-4k+5) \\ & -6(n-k+2)(n-k+1) - 2(n-k+2) - 1 \\ \leq& 0. \end{aligned}$$

Hence the proof.

Theorem 2.15. *U_n^3 is the unique graph with the largest $EM_2(G)$ among all the unicyclic graphs.*

Proof: From Theorem 2.13 and Corollary 2.14, the result follows immediately. The value of $EM_2(U_n^3)$ can be directly computed and is given by

$$EM_2(U_n^3) = \frac{1}{2}(n-3)(n-4)(n-2)^2 + (n-1)(2n^2 - 9n + 15).$$

Theorem 2.16. *Let G be a unicyclic graph on n vertices. If $G \neq \{C_n, C_n^k\}$, then $EM_2(G) > EM_2(C_n^k) > EM_2(C_n)$, where C_n^k is a tadpole graph.*

Proof: From Remark 2.9, Lemma 2.6, Lemma 2.8, and Lemma 2.10, the result follows immediately and we have $EM_2(G) > EM_2(C_n^k) > EM_2(C_n)$.

Theorem 2.17. *The value of $EM_2(C_n) = 4n$ and* $EM_2(C_n^k) = \begin{cases} 4n+17, & n = k+1; \\ 4n+22, & n = k+2; \\ 4n+23, & n \geq k+3 \end{cases}$

Proof: Proof follows by direct computation.

2.3.3 Bicyclic Graphs

Before proving the bounds of bicyclic graph, we define some notations. Let $\mathbb{B}(n, n+1)$ be the set of bicyclic graphs with n vertices and $n+1$ edges. Any bicyclic graph belongs to any one of the following collection of bicyclic graphs.

1. Let $\mathbb{B}_n^1(p,q) \subset \mathbb{B}(n, n+1)$ be the set of bicyclic graphs on n vertices, where $p+q-1 \leq n$, such that if $G \in \mathbb{B}_n^1(p,q)$, then two cycles C_p and C_q are attached by a common vertex in G. Let $D_n^1(p,q) \subset \mathbb{B}_n^1(p,q)$ be the set of graphs having exactly $n = p+q-1$ vertices, that is, if $G \in D_n^1(p,q)$, then G has no any pendant vertex, see Figure 2.7(a).

2. Let $\mathbb{B}_n^2(p,q,l) \subset \mathbb{B}(n, n+1)$ be the set of bicyclic graphs on n vertices, where $p+q+l \leq n+1$, such that if $G \in \mathbb{B}_n^2(p,q,l)$, then the two cycles C_p and C_q are connected by a path of length $l \geq 1$ in G. Let $D_n^2(p,q,l) \subset \mathbb{B}_n^2(p,q,l)$ be the set of graphs having exactly $n+1 = p+q+l$ vertices, that is, if $G \in D_n^2(p,q,l)$, then G has no pendant vertex, see Figure 2.7(b).

3. Let $\mathbb{B}_n^3(p,q,l) \subset \mathbb{B}(n, n+1)$ be the set of bicyclic graphs on n vertices, where $p+q-l \leq n+1$, such that if $G \in \mathbb{B}_n^3(p,q,l)$, then the two cycles C_p and C_q have a common path of length $l \geq 1$ in G. Let $D_n^3(p,q,l) \subset \mathbb{B}_n^3(p,q,l)$ be the set of graphs having exactly $n+1 = p+q-l$ vertices, that is, if $G \in D_n^3(p,q,l)$, then G has no pendant vertex, see Figure 2.7(c).

Theorem 2.18. *F_n^3 is the unique graph which has the largest value of $EM_2(G)$ among all bicyclic graph.*

Proof: Let $G \in \mathbb{B}(n, n+1)$ be a bicyclic graph. G lies in one of the classes of bicyclic graphs $\mathbb{B}_n^1(p,q)$, $\mathbb{B}_n^2(p,q,l)$, $\mathbb{B}_n^3(p,q,l)$.

Suppose $G \in \mathbb{B}_n^1(p,q)$ or $G \in \mathbb{B}_n^2(p,q,l)$: By repeated and suitable application of the operations A, B and C on graph G, G is transformed into one of the graphs of the form F_n^1 or F_n^2 as shown in Figure 2.8(a) or 2.8(b), respectively. Hence by Lemma 2.1, Lemma 2.2 and Lemma 2.3, we have $EM_2(G) \leq EM_2(F_n^1)$ or $EM_2(G) \leq EM_2(F_n^2)$, respectively.

Similarly, if $G \in \mathbb{B}_n^3(p,q)$ and upon repeated application of the transformations A, B and C suitably we can convert the graph G into one of the graphs of the form F_n^3 or F_n^4 as shown in Figure 2.8(c), 2.8(d), respectively. Hence by Lemma 2.1, Lemma 2.2 and Lemma 2.3, we have $EM_2(G) \leq EM_2(F_n^3)$ or $EM_2(G) \leq EM_2(F_n^4)$.

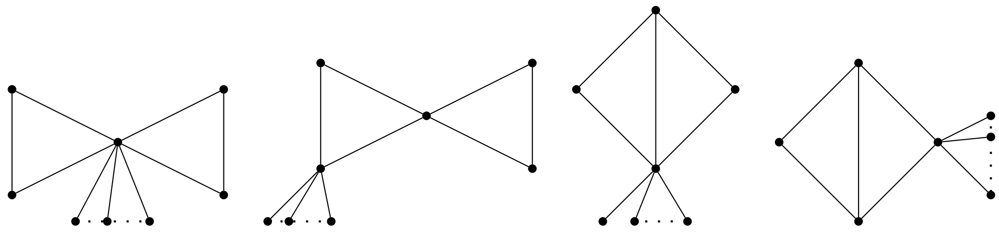


FIGURE 2.8: Some bicyclic graphs which are using in upper bounds

As a resultant, we only need to compare the values of EM_2 for graphs in F_n^1, F_n^2, F_n^3 and F_n^4. By direct computation, we obtain the following.

$$\begin{aligned} EM_2(F_n^1) =& \frac{1}{2}\{n^4 - 7n^3 + 26n^2 - 36n + 36\}, \\ EM_2(F_n^2) =& \frac{1}{2}\{n^4 - 15n^3 + 92n^2 - 240n + 406\}, \\ EM_2(F_n^3) =& \frac{1}{2}\{n^4 - 7n^3 + 26n^2 - 28n + 56\}, \\ EM_2(F_n^4) =& \frac{1}{2}\{n^4 - 11n^3 + 53n^2 - 101n + 172\}. \end{aligned}$$

Thus, by comparing the above values, we get

$$EM_2(G) \leq \frac{1}{2}\{n^4 - 7n^3 + 26n^2 - 28n + 56\} = EM_2(F_n^3)$$

Theorem 2.19. *If G is a bicyclic graph on $n \geq 8$ vertices, then $EM_2(G) \geq 4n + 58$. Equality holds if and only if either $G \in D_n^2(p, q, l)$ for $l \geq 3$ or $G \in D_n^3(p, q, l)$ for $p \geq 6, q \geq 6, l \geq 3$.*

Proof: From our observation in Remark 2.11, it is immediate that any bicyclic graph G on n vertices can be transformed into a graph in one of the classes $D_n^1(p, q)$, $D_n^2(p, q, l)$, or $D_n^3(p, q, l)$ on n vertices using repeated application of transformations D, E and F. Hence by Lemmas 2.8 and Lemma 2.10, the smallest value of $EM_2(G)$ is attained by the graphs in one of the above classes.

As a resultant, we only need to compare the value of EM_2 for graphs in each of the class $D_n^1(p, q)$, $D_n^2(p, q, l)$, and $D_n^3(p, q, l)$. As the computations are straight forward, we present the values directly.

For a graph $H \in D_n^1(p, q)$, $EM_2(H) = 4n + 108$, $n \geq 5$.

For $H \in D_n^2(p, q, l)$, the cycles C_p and C_q are connected by a path of length $l \geq 1$. By direct caluclation, we have $EM_2(H) = 4n + 66$, $n \geq 6$ and $H \in D_n^2(p, q, 1)$.

$EM_2(H) = 4n + 59$, $n \geq 7$ and $H \in D_n^2(p, q, 2)$.

$EM_2(H) = 4n + 58$, $n \geq 8$ and $H \in D_n^2(p, q, l)$ for $l \geq 3$.

For $H \in D_n^3(p, q, l)$, recall in H, the cycles C_p and C_q have a common path of length $l \geq 1$.

1. If $n = 4$ or ($p = 3$, $q = 3$ and $l = 1$), then $EM_2(D_4^3(3, 3, 1)) = 84$.
2. If $H \in D_n^3(p, 3, 1)$ for $n \geq 5$, and $p \geq 4$, then $EM_2(H) = 4n + 67$.
3. If $H \in D_n^3(p, q, 1)$ for $n \geq 6$, $p \geq 4$, and $q \geq 4$, then $EM_2(H) = 4n + 66$.
4. If $n = 5$ or ($p = 4$, $q = 4$ and $l = 2$), then $EM_2(D_3(4, 4, 2)) = 81$.
5. If $H \in D_n^3(p, 4, 2)$ for $n \geq 6$ and $p \geq 5$, then $EM_2(H) = 4n + 60$.
6. If $H \in D_n^3(p, q, 2)$ for $n \geq 7$, $p \geq 5$ and $q \geq 5$, then $EM_2(H) = 4n + 59$.
7. If $H \in D_n^3(p, q, l)$ for $n \geq 8$, $p \geq 6$, $q \geq 6$ and $l \geq 3$, then $EM_2(H) = 4n + 58$.

Thus, from the above values we get the desired results.

2.3.4 Tricyclic Graphs

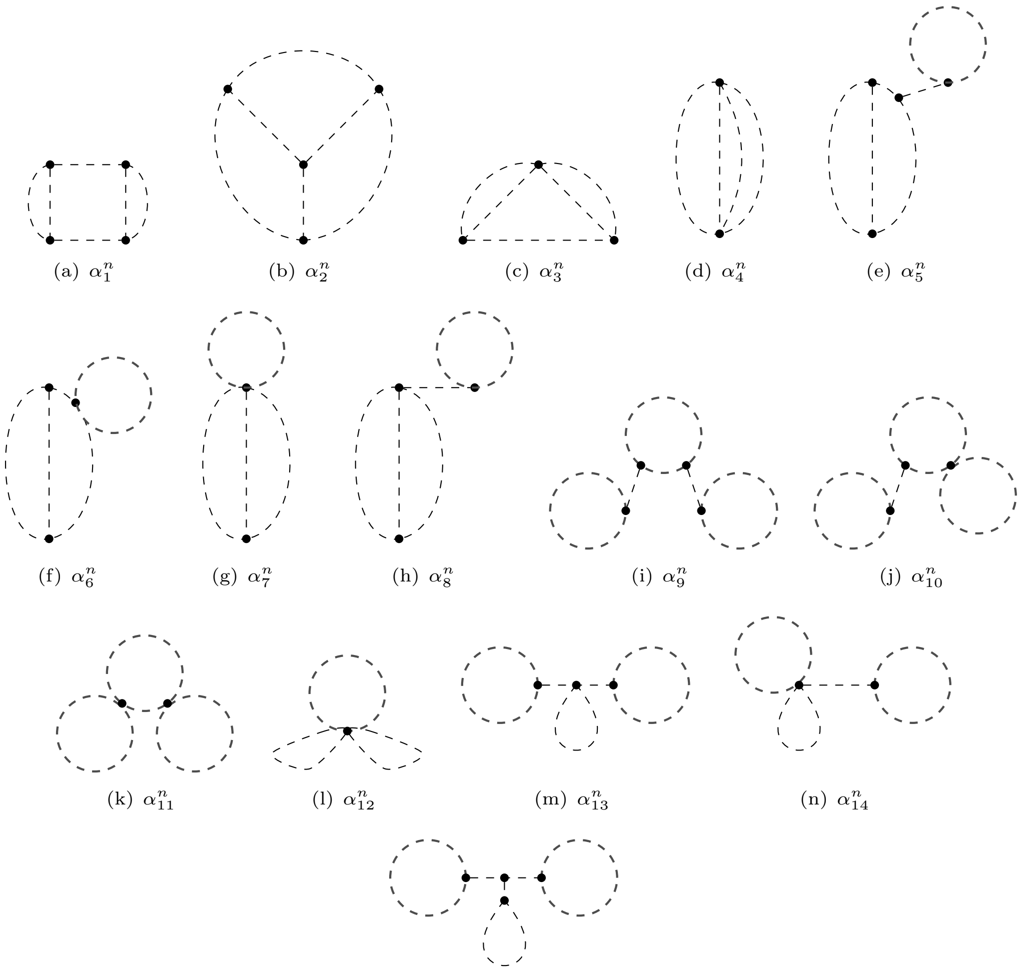
Before proving the bounds of tricyclic graphs, we will fix some notations that will be used in our proof. Let $\mathbb{X}^n$ be the set of tricyclic graphs of n vertices. We define two interesting sub collections of tricyclic graphs that are required.

1. Let $\mathbb{X}_0^n \subset \mathbb{X}^n$ be the set of tricyclic graphs of n vertices such that if $G \in \mathbb{X}_0^n$, then G has no pendent vertices. There are 15 possible collections of graphs in $\mathbb{X}_0^n$, whose representation is given in Figure 2.9.
2. Let $\mathbb{X}_1^n \subset \mathbb{X}_0^n$ be the set of tricyclic graphs such that if $G \in \mathbb{X}_1^n$, then G has maximum vertex degree three and any two vertices of degree three are connected by a path of length atleast two in G. Representation of graphs in $\mathbb{X}_1^n$ is given in Figure 2.10.

Theorem 2.20. *Let $G \in \mathbb{X}^n$ be a tricyclic graph of vertex n. Then*

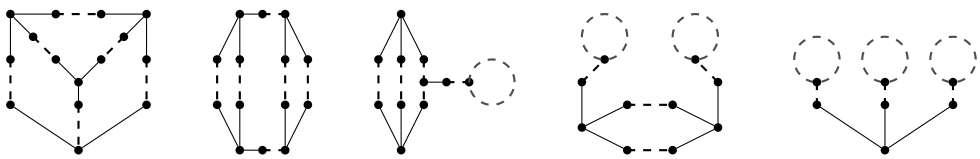
$$EM_2(G) \leq \frac{1}{2}(n^4 - 7n^3 + 30n^2 - 20n + 176),$$

where the equality holds if and only if $G \simeq \gamma_5^n$ or $G \simeq \gamma_6^n$, as in Figure 2.11.



(a) α_1^n (b) α_2^n (c) α_3^n (d) α_4^n (e) α_5^n (f) α_6^n (g) α_7^n (h) α_8^n (i) α_9^n (j) α_{10}^n (k) α_{11}^n (l) α_{12}^n (m) α_{13}^n (n) α_{14}^n (o) α_{15}^n

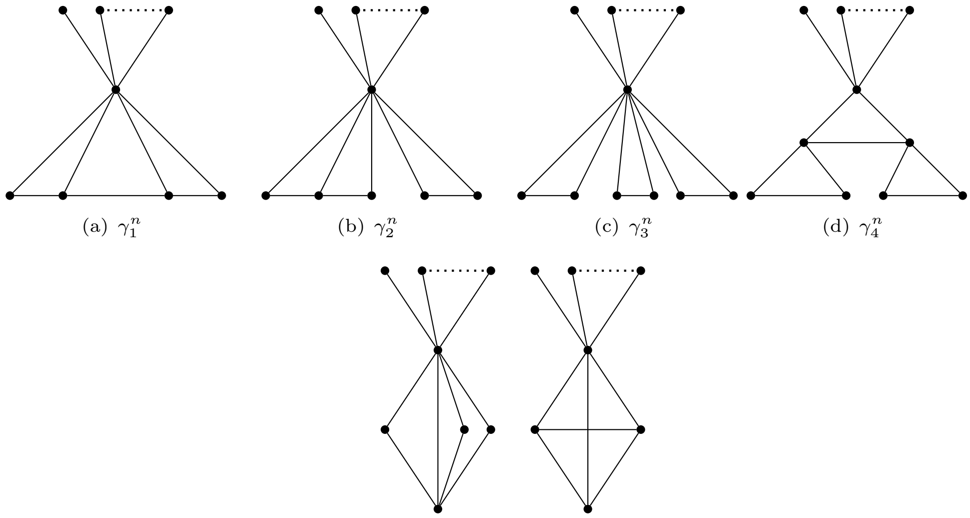
FIGURE 2.9: Representation of tricyclic graphs in $\mathbb{X}_0^n$ which has no pendant vertices.



(a) β_1^n (b) β_2^2 (c) β_3^n (d) β_4^n (e) β_5^n

FIGURE 2.10: Tricyclic graphs in $\mathbb{X}_1^n$

Proof: If, we are repeating transformations A, B and C, then, the tricyclic graph G can be converted into one of the six graphs shown in Figure 2.11. Then, from Lemma 2.1, Lemma 2.2 and Lemma 2.3, we have $EM_2(G) \leq EM_2(\gamma_i^n)$, $i = 1, \ldots, 6$. Hence, as before, we only need to



(a) γ_1^n (b) γ_2^n (c) γ_3^n (d) γ_4^n

(e) γ_5^n (f) γ_6^n

FIGURE 2.11: Tricyclic graphs which attains the upper bounds

compare the value of $EM_2(\gamma_i^n)$, $i = 1, \ldots, 6$. By direct computation we have

$$
\begin{aligned}
EM_2(\gamma_1^n) =& \frac{1}{2}(n^4 - 7n^3 + 30n^2 - 28n + 118),\\
EM_2(\gamma_2^n) =& \frac{1}{2}(n^4 - 7n^3 + 30n^2 - 36n + 82),\\
EM_2(\gamma_3^n) =& \frac{1}{2}(n^4 - 7n^3 + 30n^2 - 44n + 62),\\
EM_2(\gamma_4^n) =& \frac{1}{2}(n^4 - 23n^3 + 210n^2 - 844n + 1682),\\
EM_2(\gamma_5^n) =& \frac{1}{2}(n^4 - 7n^3 + 30n^2 - 20n + 176),\\
EM_2(\gamma_6^n) =& \frac{1}{2}(n^4 - 7n^3 + 30n^2 - 20n + 176).
\end{aligned}
$$

Thus, by comparing the above expressions, we get

$$EM_2(G) \leq \frac{1}{2}(n^4 - 7n^3 + 30n^2 - 20n + 176).$$

Hence we see that G attains the upper bound when G is isomorphic to a graph in γ_5^n or γ_6^n.

Theorem 2.21. *Let $G \in \mathbb{X}^n$ be a tricyclic graph of vertex n. Then $EM_2(G) \geq 4n + 116$. Equality holds if and only if $G \in \mathbb{X}_1^n$.*

Proof: By repeated application of the operations D, E and F to the tricyclic graph G, we can transform the graph G into one of the 15 tricyclic graphs as shown in Figure 2.9. From Lemma 2.6 , Lemma 2.8, and Lemma 2.10, we have $EM_2(G) \geq EM_2(\alpha_i^n)$, where $\alpha_i^n \in \mathbb{X}_0^n$, for $1 \leq i \leq 15$.

Among the graphs in $\mathbb{X}_0^n$, we observe that the five collection of graphs in $\mathbb{X}_1^n$ attain the lowest value of EM_2. By direct computation, we get

$$EM_2(\alpha_i) \geq EM_2(\beta_j^n) = 4n + 116,$$

where $\beta_j^n \in \mathbb{X}_1^n \subset \mathbb{X}_0^n$, $j = 1, \ldots, 5$.

2.4 Summary

In this chapter, we have presented a simple approach to find the upper and lower bounds of the second reformulated Zagreb index, $EM_2(G)$, by using six different graph transformations. We have proved that these operations significantly alter the value of reformulated Zagreb index. With the help of these transformations we have identified those graphs with cyclomatic number at most 3, namely trees, unicyclic, bicyclic and tricyclic graphs, which attain the upper and lower bounds of second reformulated Zagreb index.

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