3 Anisotropic Bianchi type-III Bulk Viscous Fluid Cosmological Model in Lyra Geometry*

3.1 Introduction

After Einstein (1916) proposed his theory of general relativity, which provided a geometrical description of gravitation, many physicists attempted to generalize the idea of geometrizing the gravitation to include a geometrical description of electromagnetism. One of the first attempts was made by Wely (1918) who proposed a more general theory by formulating a new kind of gauge theory involving metric tensor to geometrize gravitation and electromagnetism. But Weyl theory was criticized due to non-integrability of length of vector under parallel displacement. Later, Lyra (1951) suggested a modification of Riemannian geometry by introducing a gauge function into the structureless manifold which removed the non-integrability condition. This modified geometry is known as Lyra geometry. Subsequently, Sen (1957) formulated a new scalar-tensor theory of gravitation and constructed an analogue of the Einstein's field equations based on Lyra geometry. He investigated that the static model with finite density in Lyra manifold is

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similar to the static model in Einstein's general relativity. Halford (1970) has shown that the constant displacement vector field in Lyra geometry plays the role of cosmological constant in general relativity. He has also shown that the scalar-tensor treatment based in Lyra geometry predicts the same effects, within observational limits, as in Einstein's theory (Halford, 1972).

Soleng (1987) has investigated cosmological models based on Lyra geometry and has shown that the constant guage vector field either includes a creation field and be identical to Hoyle's creation cosmology (Hoyle, 1948; Hoyle and Narlikar, 1964a,b) or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term. In the later case, solutions are identical to the general relativistic cosmologies with a cosmological term.

The cosmological models based on Lyra geometry with constant and time-dependent displacement vector fields have been investigated by a number of authors viz. Beesham (1988); Singh and Singh (1993); Chakraborty and Ghosh (2000); Rahaman et al. (2002, 2003, 2005); Pradhan and Vishwakarma (2004); Pradhan et al. (2006); Pradhan (2009a,b); Pradhan et al. (2001); Ram and Singh (1992); Ram et al. (2010); Mohanty et al. (2006); Bali and Chandnani (2009) etc.

Bali and Chandnani (2008) have investigated a Bianchi type-III bulk viscous dust filled universe in Lyra geometry under certain physical assumptions. Recently, Yadav and Yadav (2011) have presented Bianchi type-III bulk viscous and barotropic perfect fluid Cosmological models in Lyra's geometry with the assumption that the coefficient of viscosity of dissipative fluid is a power function of the energy density.

In this chapter, we investigate a Bianchi type-III universe filled with a bulk viscous fluid within the framework of Lyra geometry with time-dependent displacement vector field without assuming the barotropic equation of state for the matter field. The organization of the chapter is as follows: In Section 3.2, we present the metric and Einstein's field equations. In Section 3.3, we deal with the solution of the field equations. We first show that the field equations are solvable for any arbitrary scale function. Thereafter we obtain exact solutions of the field equation by assuming (i) a power-law form of a scale factor and (ii) the bulk viscosity coefficient is directly proportional to the energy density of the matter. In Section 3.4, we discuss the physical and dynamical behaviors of the universe. Section 3.5 summarizes the main results.

3.2 Metric and Field Equations

The diagonal form of the Bianchi type-III space-time is considered in the form

$$ds^2 = dt^2 - A^2dx^2 - B^2e^{-2\alpha x}dy^2 - C^2dz^2 \tag{3.1}$$

where α is a constant and A, B, C are functions of cosmic time t. For the metric (3.1) the field equations (1.17) together with eqs.(1.9) and (1.10) in comoving coordinates lead to the following set of equations:

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -\overline{p} - \frac{3}{4}\beta^2, \tag{3.2}$$

$$\frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{C}\dot{A}}{CA} = -\overline{p} - \frac{3}{4}\beta^2, \tag{3.3}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{\alpha^2}{A^2} = -\overline{p} - \frac{3}{4}\beta^2, \tag{3.4}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} - \frac{\alpha^2}{A^2} = \rho + \frac{3}{4}\beta^2, \tag{3.5}$$

$$\alpha(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}) = 0 \tag{3.6}$$

where an overhead dot denotes differentiation with respect to time t. Here we have used the geometrized unit in which $8\pi G = 1$, $c = 1$.

The energy-conservation equation (1.7) leads to

$$\dot{\rho} + \frac{3}{2}\beta\dot{\beta} + (\rho + \overline{p} + \frac{3}{2}\beta^2)(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}) = 0. \tag{3.7}$$

This equation can be separated into two equations

$$\dot{\rho} + (\rho + \overline{p})(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}) = 0. \tag{3.8}$$

$$\beta\dot{\beta} + \beta^2(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}) = 0. \tag{3.9}$$

Bali and Chandnani (2009)

An important observational quantity in cosmology is the deceleration parameter q which is defined in (1.31)

3.3 Exact Solutions

We now solve the field equations (3.2)-(3.5) by using the method developed by Mazumdar (1994)and further used by Verma and Ram (2010).

From Eq.(3.6), we obtain

$$A = kB \tag{3.10}$$

where k is an integration constant. Without loss of generality, we take k=1. Using

Eq.(3.10), Eqs. (3.2)-(3.9) reduce to

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}=-\overline{p}-\frac{3}{4}\beta^2, \tag{3.11}$$

$$\frac{2\ddot{B}}{B}+\frac{\dot{B}^2}{B^2}-\frac{\alpha^2}{B^2}=-\overline{p}-\frac{3}{4}\beta^2, \tag{3.12}$$

$$\frac{\dot{B}^2}{B^2}+\frac{2\dot{B}\dot{C}}{BC}-\frac{\alpha^2}{B^2}=\rho+\frac{3}{4}\beta^2, \tag{3.13}$$

$$\dot{\rho}+(\rho+\overline{p})(\frac{2\dot{B}}{B}+\frac{\dot{C}}{C})=0, \tag{3.14}$$

$$\frac{\dot{\beta}}{\beta}+(\frac{2\dot{B}}{B}+\frac{\dot{C}}{C})=0. \tag{3.15}$$

From Eqs.(3.11) and (3.12), we get

$$\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}+\frac{\dot{B}^2}{B^2}-\frac{\dot{B}\dot{C}}{BC}-\frac{\alpha^2}{B^2}=0. \tag{3.16}$$

This is an equation involving two unknown functions B and C which will admit solution if one of them is a known function of t. To get a physically realistic model, Bali and Chandnani (2008); Yadav and Yadav (2011) have assumed a supplementary condition $B=C^n$ between the metric potentials B and C. This condition is based on the physical assumption that the shear scalar σ is proportional to the expansion scalar θ. In order to obtain a simple but physically realistic solution, we make the mathematical assumption that

$$C=t^n \tag{3.17}$$

where n is positive real number. For this we first show that Eq.(3.16) is solvable for an arbitrary choice of C. Multiplying Eq.(3.16) by B^2C, we get

$$BC\ddot{B}-B\dot{B}\dot{C}+C\dot{B}^2-B^2\ddot{C}=\alpha^2C. \tag{3.18}$$

This can be written in the form

$$\frac{d}{dt}\left(-B^2\dot{C} + BC\dot{B}\right) = \alpha^2 C. \tag{3.19}$$

The first integral of Eq.(3.19) is

$$-B^2\dot{C} + BC\dot{B} = \alpha^2\left(\int C dt + k_1\right) \tag{3.20}$$

where k_1 is a constant of integration. We can write Eq.(3.20) in the form

$$\frac{d}{dt}(B^2) - \frac{2\dot{C}}{C}B^2 = F(t) \tag{3.21}$$

where

$$F(t) = \frac{2\alpha^2}{C}\left(\int C dt + k_1\right). \tag{3.22}$$

The general solution of eq.(3.21) is given by

$$B^2 = C^2\left(\int \frac{F(t)}{C^2} dt + k_2\right) \tag{3.23}$$

where k_2 is a constant of integration. Note that in this case the solution of Einstein's field equations reduces to the integration of Eqs. (3.23) if C is explicitly known function of t. Making use Eq.(3.17) in Eqs.(3.22) and (3.23), we obtain the general solution B^2 of the form

$$B^2 = \frac{\alpha^2 t^2}{1-n^2} + \frac{2k_1\alpha^2 t^{1-n}}{1-3n} + k_2 t^{2n}. \tag{3.24}$$

For further discussion of the solutions, we take $k_1 = k_2 = 0$. Then

$$B^2 = \frac{\alpha^2 t^2}{1-n^2}, \qquad n \neq 1. \tag{3.25}$$

Hence, the metric of our solutions can be written in the form

$$ds^2 = dt^2 - \frac{\alpha^2 t^2}{1-n^2}(dx^2 + e^{-2\alpha x}dy^2) - t^{2n}dz^2 \tag{3.26}$$

where $0 < n < 1$.

3.4 Some Physical and Kinematical Properties of the Model

The model (3.26) represents an anisotropic Bianchi type-III cosmological universe filled with a bulk viscous fluid in the framework of Lyra geometry.
From Eqs.(3.15), (3.17) and (3.25), the gauge function β has the expression given by

$$\beta = \frac{c_1(1-n^2)}{\alpha^2}t^{-(n+2)} \tag{3.27}$$

where c_1 is an integration constant. Using Eqs.(3.17), (3.26) and (3.27) into Eqs. (3.11) and (3.13), we obtain

$$\overline{p} = -\left[\frac{n^2}{t^2} + \frac{3c_1^2(1-n^2)^2}{4\alpha^4 t^{2n+4}}\right], \tag{3.28}$$

$$\rho = \frac{n(n+2)}{t^2} - \frac{3c_1^2(1-n^2)^2}{4\alpha^4 t^{2n+4}}. \tag{3.29}$$

It is clear that, given $\xi(t)$, we can determine isotropic pressure p. In most of the investigations involving bulk viscosity, $\xi(t)$ is assumed to be simple power function of energy density (Weinberg, 1972):

$$\xi(t) = \xi_0 \rho^m \tag{3.30}$$

where ξ_0 and $m > 0$ are constants. The case m=1 has been considered by Murphy(1973) which corresponds to a radiating fluid. Here we take m=1, so that

$$\xi = \xi_0 \rho \tag{3.31}$$

From Eqs.(1.10), (3.29), (3.30) and (3.31), we obtain

$$p = \frac{\xi_0 n(n+2)^2}{t^3} - \frac{n^2}{t^2} - \frac{3c_1^2(1-n^2)^2}{4\alpha^4 t^{2n+4}}\left[1 + \frac{\xi_0(n+2)}{t}\right]. \tag{3.32}$$

Clearly the viscosity contributes significantly to the isotropic pressure of the fluid. The physical and kinematical parameters of the model (3.26) have the following expressions:

$$V = R^3 = \frac{\alpha^2}{1-n^2} t^{n+2}, \tag{3.33}$$

$$\theta = \frac{n+2}{t}, \tag{3.34}$$

$$\sigma = \frac{1-n}{\sqrt{3}t}, \tag{3.35}$$

$$H = \frac{n+2}{3t}, \tag{3.36}$$

$$q = \frac{1-n}{2+n}. \tag{3.37}$$

The deceleration parameter is positive which shows the decelerating behavior of the cosmological model. It is worthwhile to mention the work of Vishwakarma (2003) where he has shown that the decelerating model is also consistent with recent CMB observations by WNAP, as well as with the high redshift supernovae Ia data including 1997ff at Z=1.755.

We observe that the spatial volume V is zero at $t = 0$, and it increases with the cosmic time. This means that the model starts expanding with a big-bang at $t = 0$. All the physical and kinematical parameters ρ, p, θ and σ diverge at this initial singularity. The physical and kinematical parameters are well defined and are decreasing functions for $0 < t < \infty$, and ultimately tend to zero for large time. The gauge function $\beta(t)$ and bulk viscosity coefficient $\xi(t)$ are infinite at the beginning and gradually decrease as time increases and ultimately tend to zero at late times. Since $\frac{\sigma}{\theta} = \frac{1-n}{\sqrt{3}(n+2)} = \text{const}$, the anisotropy in the universe is maintained throughout the passage of time.

3.5 Conclusion

We have investigated an anisotropic Bianchi type-III cosmological model in the presence of a bulk viscous fluid within the framework of Lyra's geometry with time-dependent displacement vector. The model describes a expanding, shearing and decelerating universe with a big-bang singularity at $t = 0$. Since all the physical and kinematical parameters start off with extremely large values, which continue to decrease with the expansion of the universe and ultimately tend to zero for large time. As ρ tends to zero as t tends to infinity, the model would essentially gives an empty space-time for large time.