2 Bianchi Type -II Inflationary Models with Stiff Matter and Decaying Cosmological Term*

2.1 Introduction

Inflation, the stage of accelerated expansion of the universe, has drawn attention of many workers. Guth (1981) proposed inflationary model in the context of Grand Unified Theory (GUT), which has been accepted as the model of the early universe. It is believed that the early universe evolved through some phase transitions, thereby yielding vacuum energy density which is at present is at least 118 orders of magnitudes smaller than the Planck time (Weinberg, 1989). Such a discrepancy between theoretical expectations and empirical observations constitutes a fundamental problem in the interface uniting astrophysics, particles physics and cosmology is the cosmological constant problem. The recent observational evidence for an accelerated phase of present universe, obtained from distant SNe Ia,(Perlmutter et al., 1997; Riess et al., 1998) gave strong support to search for alternative cosmologies. The state of affairs has stimulated interest in more general models containing an extra component describing dark energy, and simultaneously ac-

*Contents of this chapter have been Published in CHIN. PHYS. LETT. Vol. 31, No. 7 (2014) 070401(IOP) counting for the present accelerated phase of the universe. The observations suggest that the universe was previously decelerating which entered into on accelerating phase. It was decelerating when matter was dominant, but afterwards dark energy became dominant, the phase transition took place. The most-obvious theoretical candidate of dark energy is the cosmological term Λ. Some of the recent discussions on the cosmological constant problems and consequences on cosmology with a time -varying cosmological term have been investigated by Ratra and Peebles (1988); Dolgov (1983, 1990, 1997); Sahni and Starobinsky (2000) etc. A variable Λ term or a decaying vacuum energy density is on ingredient accounting for the accelerated phase of the present universe. Linde (1974) suggested that Λ term is a function of temperature and related to the spontaneous symmetry breaking, and therefore it could be a function of time.

Several ansatz have been proposed and well studied so far in which Λ term decays with time. Some authors have argued for the dependence $\Lambda \sim t^2$ keeping in mind the dimensional consideration in the spirits of quantum cosmology. Chen and Wu (1990) considered Λ varying as a^{-2} , where 'a' is the scale factor of the universe.Carvalho et al. (1992) generalized it by taking $\Lambda= \beta \left(\frac{\dot{a}}{a}\right)^2 + \frac{\alpha}{a^2}$, where α and β are adjustable dimensionless parameters on the basis of quantum field estimations in the curved expanding background, where a dot denotes time derivative. Schutzhold (2002a,b) proposed vacuum energy density proportional to Hubble parameter which leads to a vacuum energy density decaying as $\Lambda \simeq m^3 \left(\frac{\dot{a}}{a}\right)$, where m=150 Mev. However,not all vacuum decaying cosmological models, predict acceleration. Al-Rawaf and Taha (1996); Al-Rawaf (1998); Overdin and Cooperstock (1998) proposed cosmological models with decaying law $\Lambda=\beta \left(\frac{\ddot{a}}{a}\right)$. Arbab and Cosmo (2003a,b) have discussed cosmic acceleration with positive Λ and also analyzed the implication of the models based on this ansatz. Cunha et al. (2002) discussed the classical cosmological tests for large class of FRW type models driven by decaying vacuum energy density.

Spatially homogeneous and anisotropic Bianchi type-II space-times play fundamental role in constructing cosmological models suitable for describing the early stages of evolution of the universe. Asseo and Sol (1987) have emphasized the importance of such space-times. A locally rotationally symmetric (LRS) Bianchi type-II space-time has already been considered by a number of cosmologists in different physical contexts. Lorentz (1980) discussed LRS Bianchi-type-II space-time in the presence of stiff-matter and electromagnetic field. Hajj-Bouttros (1986b) studied LRS Bianchi type-II model with perfect fluid by a generating technique and also constructed LRS Bianchi type-II perfect fluid models with an equation of state, which is a function of time. Shanti and Rao (1991) studied such space-times in Barber's self creation theory of gravitation. Venkateswarlu and Reddy (1991) obtained cosmological solutions for stiff fluid models in the presence of an electromagnetic field.Coley and Wainright (1991) studied LRS Bianchi type-II models in two-fluid cosmology. Singh and Kumar (2007) presented inflationary cosmological models of Bianchi type -II with constant deceleration parameter. Pradhan et al. (2008) have shown that Bianchi type -II models with stiff fluid and decaying law $\Lambda = \beta \left(\frac{\ddot{a}}{a}\right)$ are compatible with recent observations on accelerated universe. Verma et al. (2011)presented anisotropic stiff-fluid model of Bianchi type -II with above decaying law of Λ and negative deceleration parameter. Tiwari et al. (2012) studied LRS Bianchi type-II stiff-fluid cosmological model with varying Λ term. Bali and Swati (2012) investigated Bianchi type-II inflationary universe with massless scalar field and time varying Λ.

In this chapter, we present Einstein's field equations for LRS Bianchi type-II space-time in the presence of a perfect fluid and time -dependent cosmological term. We obtain exact solutions of the flied equations by applying a special law of variation of the Hubble's parameter, proposed by Berman (1983), which leads to a negative deceleration parameter. We also assume the decay law of cosmological term of the forms (i) $\Lambda =$

$\beta\left(\frac{\ddot{a}}{a}\right)+\frac{\alpha}{a^2}$ and (ii) $\Lambda=\frac{\alpha}{a^2}$. The solutions correspond to cosmological models of the accelerating universe.

2.2 Metric and Field Equations

In an orthogonal frame, the metric for LRS Bianchi type-II is given by Lorentz (1980)

$$ds^2=\eta_{ij}\theta^i\theta^j, \quad \eta_{ij}=diag(-1,1,1,1) \tag{2.1}$$

where the cartan bases θ^i are given by

$$\theta^0=dt, \quad \theta^1=S(t)w^1, \quad \theta^2=R(t)w^2, \quad \theta^3=R(t)\omega^3. \tag{2.2}$$

Here R(t) and S(t) are metric functions. Taking (x,y,z) as local coordinates, the invariant basis ω^i is given by

$$\omega^1=dy+xdz, \quad \omega^2=dz, \quad \omega^3=dx. \tag{2.3}$$

Einstein's field equations are

$$R_{ij}-\frac{1}{2}g_{ij}R+\Lambda g_{ij}=-8\pi T_{ij} \tag{2.4}$$

In comoving coordinates, Einstein's field equations (1.8) lead to the following set of three independent equations

$$2\frac{\dot{R}\dot{S}}{RS}+\frac{\dot{R}^2}{R^2}-\frac{S^2}{4R^4}=8\pi\rho+\Lambda, \tag{2.5}$$

$$2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}-\frac{3}{4}\frac{S^2}{R^4}=-8\pi p+\Lambda, \tag{2.6}$$

$$\frac{\ddot{S}}{S}+\frac{\ddot{R}}{R}+\frac{\dot{R}\dot{S}}{RS}+\frac{S^2}{4R^4}=-8\pi p+\Lambda. \tag{2.7}$$

The energy conservation equation $T^{ij}_{;j}=0$ leads to

$$\dot{\rho}+(\rho+p)\left(2\frac{\dot{R}}{R}+\frac{\dot{S}}{S}\right)=-\frac{\dot{\Lambda}}{8\pi}. \tag{2.8}$$

Equations (2.5)- (2.7) are coupled system of highly non-linear differential equations. In order to obtain physically realistic solution for application in cosmology and astrophysics, we normally assume a form for matter content or relation between metric functions . The solutions to the field equations may also be generated by applying the law of variation for Hubble's parameter, initially proposed by Berman (1983) for FRW models, which yields a constant value of deceleration parameter. Berman and Gomide (1988); Johri and Desikan (1994); Reddy et al. (2006); Singh and Baghel (2009); Ram et al. (2009) etc. have studied cosmological models with constant deceleration parameter.

We define the deceleration parameter as

$$q=-\frac{a\ddot{a}}{\dot{a}^2}=constant. \tag{2.9}$$

The sign of q indicates that whether the model accelerates or not. The positive sign corresponds to a standard decelerating model whereas the negative sign $-1\leqslant q<0$ indicates acceleration and q=0 corresponds to expansion of universe with constant velocity. For an accelerating model of the universe, we take the constant as negative. The solution of Eq. (2.9) is then given by

$$a=(c_1t+c_2)^{\frac{1}{1+q}} \tag{2.10}$$

where c_1 and c_2 are integration constants. Eq.(2.10) implies that the condition of expansion is 1+q> 0.

In order to obtain physically viable solutions of Eqs. (2.5)-(2.7), we consider the case when space-time is filled with stiff-matter (ρ=p). In this case Eqs. (2.5) and (2.7) give

$$\frac{\ddot{R}}{R}+\frac{\ddot{S}}{S}+\frac{3\dot{R}\dot{S}}{RS}+\frac{\dot{R}^2}{R^2}=2\Lambda. \tag{2.11}$$

From Eqs. (2.6) and (2.7), we obtain the condition for isotropy of pressure as

$$\frac{\ddot{R}}{R}-\frac{\ddot{S}}{S}+\frac{\dot{R}^2}{R^2}-\frac{\dot{R}\dot{S}}{RS}-\frac{S^2}{R^4}=0. \tag{2.12}$$

Integration of Eq. (2.11) leads to

$$R^2\dot{S}+R\dot{R}S=\int 2\Lambda(R^2S)dt+h \tag{2.13}$$

where h is an integration constant.

2.3 Model I with Λ=$\beta\left(\frac{\ddot{a}}{a}\right)+\frac{\alpha}{a^2}$

Inserting in Eq. (2.13) the phenomenological decay law of the form and integrating, we obtain

$$R^2\dot{S}+R\dot{R}S=\frac{2\beta qc_1(c_1t+c_2)^{\frac{2-q}{1+q}}}{(1+q)(q-2)}+\frac{2\alpha(1+q)(c_1t+c_2)^{\frac{2+q}{1+q}}}{c_1(q+2)}+h. \tag{2.14}$$

Dividing by R^2 S, we get

$$\frac{\dot{R}}{R}+\frac{\dot{S}}{S}=\frac{2\beta q c_1(c_1t+c_2)^{-1}}{(1+q)(q-2)}+\frac{2\alpha(1+q)(c_1t+c_2)^{\frac{q-1}{1+q}}}{c_1(q+2)}+h(c_1t+c_2)^{\frac{-3}{1+q}}. \tag{2.15}$$

Eq. (2.15), on integration, provides

$$RS=m(c_1t+c_2)^{\frac{2\beta q}{(1+q)(q-2)}}\exp\left[\frac{\alpha(1+q)^2(c_1t+c_2)^{\frac{2q}{1+q}}}{c_1^2(2+q)}+\frac{h(1+q)(c_1t+c_2)^{\frac{q-2}{1+q}}}{c_1(q-2)}\right] \tag{2.16}$$

where m is an integration constant. By using $a^3=R^2S$ and from Eq (2.10), we also have

$$R^2S=(c_1t+c_2)^{\frac{3}{1+q}}. \tag{2.17}$$

Solving Eqs. (2.16) and (2.17), we obtain the expressions of the scale factors R and S as

$$R=\frac{1}{m}(c_1t+c_2)^{\frac{6-3q+2\beta q}{(1+q)(2-q)}}\exp\left[\frac{-\alpha(1+q)^2(c_1t+c_2)^{\frac{2q}{1+q}}}{c_1^2q(2+q)}+\frac{h(1+q)(c_1t+c_2)^{\frac{q-2}{1+q}}}{c_1(2-q)}\right], \tag{2.18}$$

$$S=m^2(c_1t+c_2)^{\frac{3q-6-4\beta q}{(1+q)(2-q)}}\exp\left[\frac{2\alpha(1+q)^2(c_1t+c_2)^{\frac{2q}{1+q}}}{c_1^2q(2+q)}-\frac{2h(1+q)(c_1t+c_2)^{\frac{q-2}{1+q}}}{c_1(2-q)}\right], \tag{2.19}$$

The cosmological term $\Lambda(t)$ has the value given by

$$\Lambda=-\frac{\beta q c_1^2}{(1+q)^2(c_1t+c_2)^2}+\frac{\alpha}{(c_1t+c_2)^{\frac{2}{1+q}}}. \tag{2.20}$$

Thus, the cosmological term is a decreasing function of time and approaching a small positive value at present epoch which is corroborated by consequences from recent supernovae Ia observations (Perlmutter et al., 1997; Riess et al., 1998). These observations on magnitude and red-shift of type Ia supernovae suggested that our universe may be accelerating one with induced cosmological density through Λ term.

The metric (1) R(t) and S(t), given by Eqs. (2.18) and (2.19), represents an exact stiff fluid LRS Bianchi type-II cosmological model with negative constant deceleration parameter and time-decaying cosmological term $\Lambda(t)$ given by (2.20). Obviously this is an accelerating model of the universe.The solution of the scale factors has a combination of power-law term and the exponential term in the product form . We observe that the solutions are inflationary in nature. So, initially the exponential term may be more significant and it is possible to have inflationary scenario during the evolution of the universe. Thus, the space-time may be dominated by vacuum energy.

The directional Hubble's factors H_1, H_2 and H_3 in the directions of x, y and z are given by

$$H_1 = H_2 = \frac{c_1(6-3q+2\beta q)}{(1+q)(2-q)(c_1 t + c_2)} + \frac{h}{(c_1 t + c_2)^{\frac{3}{1+q}}} - \frac{2\alpha(1+q)}{(2+q)(c_1 t + c_2)^{\frac{1-q}{1+q}}}, \tag{2.21}$$

$$H_3 = \frac{c_1(3q-6-4\beta q)}{(1+q)(2-q)(c_1 t + c_2)} - \frac{2h}{(c_1 t + c_2)^{\frac{3}{1+q}}} + \frac{4\alpha(1+q)}{(2+q)(c_1 t + c_2)^{\frac{1-q}{1+q}}}, \tag{2.22}$$

$$H = \frac{c_1}{1+q}(c_1 t + c_2)^{-1}. \tag{2.23}$$

Fig. (2.1) depicts the behavior of Hubble parameter with cosmic time t

The expansion scalar θ and shear scalar σ are given by

$$\theta = \frac{3c_1}{1+q}(c_1 t + c_2)^{-1}, \tag{2.24}$$

$$\sigma = \frac{1}{\sqrt{3}}\left[\frac{c_1(4+2\beta q-2q)}{(1+q)(2-q)(c_1 t + c_2)} + \frac{3h}{(c_1 t + c_2)^{\frac{3}{1+q}}} - \frac{2\alpha(1+q)}{(2+q)(c_1 t + c_2)^{\frac{1-q}{1+q}}}\right]. \tag{2.25}$$

The anisotropic parameter A_m

$$A_m=\frac{1}{3(2-q)^2}\left[2(6-3q+2\beta q)^2+(3q-6-4\beta q)^2\right]+\frac{20\alpha^2(1+q)^2}{(2+q)^2(c_1t+c_2)^{\frac{2(1-q)}{1+q}}}+\frac{5h^2}{(c_1t+c_2)^{\frac{6}{1+q}}}+\frac{(36\alpha qc_1-72\alpha c_1-40\alpha\beta qc_1)}{(4-q^2)(c_1t+c_2)^{\frac{2}{1+q}}}+\frac{(36hc_1-18hc_1q+20\beta qhc_1)}{(1+q)(2+q)(c_1t+c_2)^{\frac{4+q}{1+q}}}+\frac{8\alpha h(1+q)}{(2+q)(c_1t+c_2)^{\frac{4-q}{1+q}}}-2. \tag{2.26}$$

By the use of Eqs. (2.12) ,(2.18) and (2.19) in Eq. (2.5), we obtain the expression for the energy density ρ as

$$8\pi\rho=\frac{M_1}{(c_1t+c_2)^2}-\frac{M_2}{(c_1t+c_2)^{\frac{2}{1+q}}}+\frac{M_3}{(c_1t+c_2)^{\frac{4+q}{1+q}}}-\frac{3h^2}{(c_1t+c_2)^{\frac{6}{1+q}}}-\frac{12\alpha^2(1+q)^2}{(2+q)^2(c_1t+c_2)^{\frac{2(1-q)}{1+q}}}+\frac{12\alpha h(1+\alpha)}{(2+q)(c_1t+c_2)^{\frac{4-q}{1+q}}}. \tag{2.27}$$

Fig. (2.2) shows to variation of energy density ρ with time in model 1

where M_1, M_3 are constants in terms of β, q whereas M_2 involves β, q and α. The expressions of these constants are not needed in further discussions.

We observe that the spatial volume is zero at $t=t_0$ where $t_0=-\frac{c_2}{c_1}$ and expands with time t. One of the scale factors R(t) vanishes while the other one S(t) diverges at $t=t_0$ which means that the model has a cigar-type initial singularity at $t=t_0$. At this epoch the energy density ρ becomes infinite. The scalars of expansion and shear are infinite at $t=t_0$. These indicate that the universe starts evolving with zero volume at $t=t_0$ and expands with time. The model is well behaved for $t<\infty$. The expansion scalar θ tends to zero as $t\to\infty$ which shows that the universe is expanding with the increase of time.The shear scalar σ is non-zero for t> t_0 and tends to zero as t$\to\infty$. The energy density is zero for large time and so the model would essentially gives an empty universe for large time. We also find that $\frac{\sigma^2}{\theta}$ does not tend to zero as t$\to\infty$, which means that the model does not approach isotropy for large time. When $\alpha=0$, we obtain the model presented by Verma et al. (2011).

2.4 Model II with $\Lambda=\frac{\alpha}{a^2}$ and $\beta=0$

In this case the solutions for the scale factors R and S are given as

$$R=\frac{1}{m}(c_1t+c_2)^{\frac{3}{1+q}}\exp\left[\frac{-\alpha}{c_1^2}\frac{(1+q)^2}{q(q+2)}(c_1t+c_2)^{\frac{2q}{1+q}}-\frac{h(1+q)}{c_1(q-2)}(c_1t+c_2)^{\frac{q-2}{1+q}}\right], \quad (2.28)$$

$$S=m^2(c_1t+c_2)^{\frac{-3}{1+q}}\exp\left[\frac{2\alpha}{c_1^2}\frac{(1+q)^2}{q(q+2)}(c_1t+c_2)^{\frac{2q}{1+q}}+\frac{2h(1+q)}{c_1(q-2)}(c_1t+c_2)^{\frac{q-2}{1+q}}\right]. \quad (2.29)$$

The cosmological term has the value given by

$$\Lambda=\frac{\alpha}{(c_1t+c_2)^{\frac{2}{1+q}}}. \quad (2.30)$$

The directional Hubble's parameters H_1, H_2 and H_3 have values given by Eqs.(2.21), (2.22) and (2.23) with $\beta=0$

$$\begin{aligned}8\pi\rho=&\frac{12c_1^2(2q+1)}{(1+q)^2(c_1t+c_2)^2}+\frac{N_1}{(c_1t+c_2)^{\frac{2}{1+q}}}+\frac{N_2}{(c_1t+c_2)^{\frac{4+q}{1+q}}}+\frac{N_3h^2}{(c_1t+c_2)^{\frac{6}{1+q}}}\\&+\frac{N_4}{(c_1t+c_2)^{2\frac{q-1}{1+q}}}+\frac{N_5}{(c_1t+c_2)^{\frac{4+q}{1+q}}}+\frac{64\alpha h(q-2)}{c_1(q+2)(c_1t+c_2)^{\frac{4-q}{1+q}}}.\end{aligned} \quad (2.31)$$

The behavior of energy density of this model II is shown in the Fig. (2.3)

The physical and dynamical behaviors of this model is similar to that of the model with $\beta \neq 0$. where, N_1 ,N_2,N_3 ,N_4 and N_5 are constant involving q, h, α and β.

We now evaluate the relationship between the cosmological time t and BKL time τ. Such a relationship is of much significance, since the domain of the cosmological time is finite as the BKL time τ runs from minus infinity to infinity. Burd and Tavakol (1993); Wu (2010) have provided a new interpretation of zero Lyapunov exponents in BKL time for Mixmaster cosmology. Belinski and Khalatnikov (1976)have introduced a relationship between t and τ of the form

$$dt = V d\tau. \tag{2.32}$$

Substituting the value of V in (2.32) and integrating we obtain

$$\tau = \left(\frac{(1+q)(c_1 t + c_2)^{\frac{q-2}{1+q}}}{c_1(q-2)} \right). \tag{2.33}$$

2.5 Conclusion

In this chapter, we have analyzed Einstein's field equations for LRS Bianchi type-II space-time in the presence of a stiff fluid and time decaying cosmological terms of the forms (i) $\Lambda = \beta \left(\frac{\ddot{a}}{a}\right) + \frac{\alpha}{a^2}$ and (ii) $\Lambda = \frac{\alpha}{a^2}$. There are recent observational evidences for the accelerated state of the present universe. It is held that the deceleration parameter q was positive in the early phases of matter dominated era which becomes negative during the later stages of the evolution. Therefore to obtain exact solutions to the field equations, which would correspond to an accelerated universe, we have applied special law of variation of parameter, proposed by Berman (1983). Two classes of anisotropic cosmological models far the different forms of Λ are presented. The expressions for some important physical and kinematical parameters have been obtained and their behaviors are discussed. The solutions from the scale factors have combination of power law term and the exponential term in the product form. Initially the exponential term is more significance and it is possible to have inflationary scenario during the evolution of the universe. The interesting feature of the solution is that it is possible to exit from exponential scenario if β=0 and h=0. After some inflation time, the universe will continue to expand with power-law expansion. The cosmological models have singularity at the epoch t=$-\frac{c_2}{c_1}$ which expand indefinitely with acceleration expansion. As time t increases the physical parameters decrease and ultimately tend to zero for large time. Thus, the universe would give essentially empty space for large time. It is seen that $\frac{\sigma^2}{\theta}$ does not tend to zero as t $\rightarrow \infty$, which means that the anisotropy in the universe is maintained throughout the passage of time. As the deceleration parameter always negative, the present models represent accelerated expanding universe. The cosmological models presented in this paper are new an may be useful for better understanding of the inflationary scenario of universe.

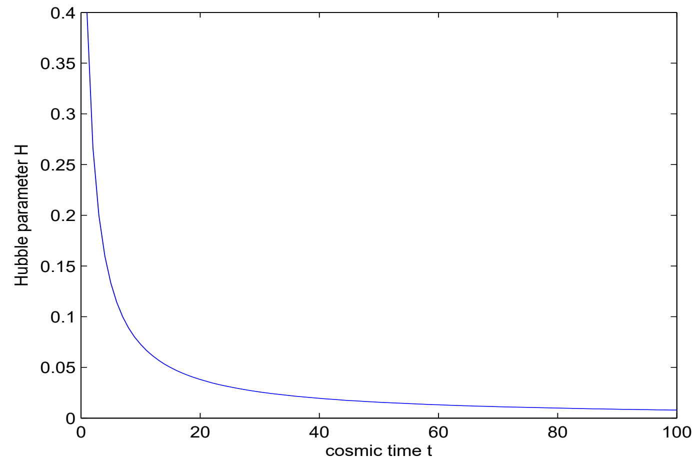

Figure 2.1: Hubble's parameter H verses cosmic time t, β=0.2; α=1; h=0.1

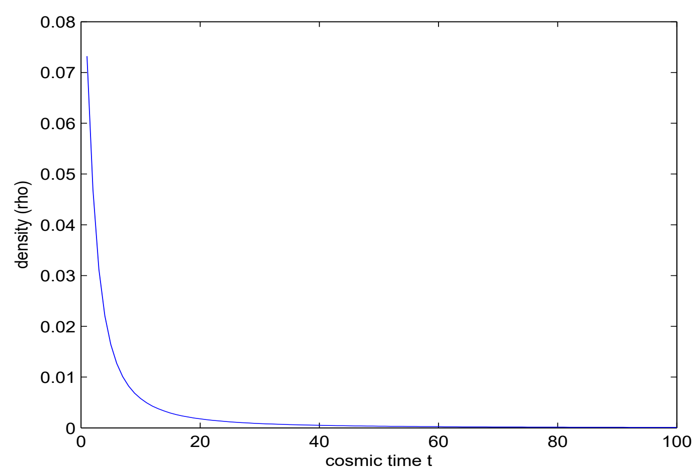

Figure 2.2: Density ρ verses cosmic time t for model 1 with β=0.2; α=1; h=0.1

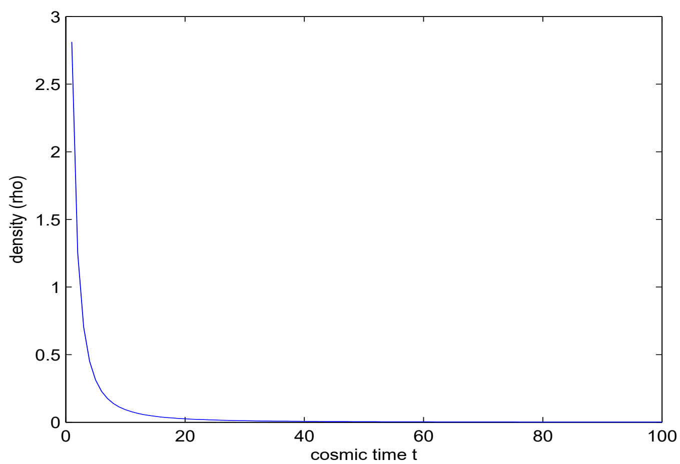

Figure 2.3: Density ρ verses cosmic time t for model 2 with β=0.2; α=1; h=0.1