Chapter 6

On fuzzy topology generated by fuzzy relations

6.1 Introduction

In this chapter, we have introduced a fuzzy topology generated by a fuzzy relation, as a generalization of the corresponding concept given by Smithson[102]. We have then obtained sufficient conditions under which this generated fuzzy topology will satisfy separation axioms, fuzzy T_0, fuzzy T_1 and fuzzy T_2. Further, we have introduced 'finite intersection property(F.I.P.)' in a fuzzy topological space and then obtained a characterization of (Lowen's) fuzzy compactness in terms of F.I.P. Using this result, we have obtained a sufficient condition under which a fuzzy topology generated by a fuzzy relation, becomes fuzzy compact.

Now we give some definitions and results which will be used in this chapter.

Definition 6.1. Let X be a non empty set, $A \subseteq X$ and $\mathcal{R}$ be a fuzzy relation on X. Then A is called an $\mathcal{R}$-*antiset* of X if no two distinct elements of A are $\mathcal{R}$-related, where two elements x, y are said to be $\mathcal{R}-$*related* if $\mathcal{R}(x, y) > 0$.

Definition 6.2. Let X be a non empty set, $A \subseteq X$ and $\mathcal{R}$ be a fuzzy relation on X. Then fuzzy sets L_A and R_A on X, are defined as follows:

$$L_A(y) = \sup_{a \in A} \mathcal{R}(y, a),$$
$$R_A(y) = \sup_{a \in A} \mathcal{R}(a, y),$$

for each $y \in X$.

Smithson[102] introduced and studied a topology on a set X, induced by a relation $\mathcal{R}$. Here we generalize this concept in the fuzzy setting.

Definition 6.3. Let $\mathcal{R}$ be a fuzzy relation on X and $\mathcal{A}$ be a collection of $\mathcal{R}$-antisets of X. Then $\tau_{\mathcal{R},\mathcal{A}}$ is the fuzzy topology on X generated by taking

$$\mathcal{S} = \{L_A\}_{A\in\mathcal{A}} \cup \{R_A\}_{A\in\mathcal{A}} \cup \{\alpha_X : \alpha \in [0,1]\},$$

as a subbase for the fuzzy closed sets in X i.e., every fuzzy closed set in X can be written as an intersection of finite unions of members of $\mathcal{S}$.

We mention here that in [77], for a given fuzzy relation $\mathcal{R}$ on X, the authors have introduced and studied the fuzzy topology on X generated by $\{L_{\{x\}}, R_{\{x\}}\}_{x\in X}$ considered as a subbase for the fuzzy open sets in X. This is a generalization of the corresponding concept given in [62].

Example 6.1. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{a, b\}$, which is given as follows:*

$\mathcal{R}$	a	b
a	*0.1*	*0*
b	*0*	*0.3*

and $\mathcal{A} = \{\{a\}, \{a, b\}\}$. Then the fuzzy topology $\tau_{\mathcal{R},\mathcal{A}}$ is generated by the following subbase $\mathcal{S}$ for the fuzzy closed sets in X:

$$\mathcal{S} = \{L_{\{a\}}, L_{\{a,b\}}, R_{\{a\}}, R_{\{a,b\}}\} \cup \{\alpha_X : \alpha \in [0,1]\},$$

where $L_{\{a\}}, L_{\{a,b\}}, R_{\{a\}}, R_{\{a,b\}}$ are given by:

$$L_{\{a\}} = \frac{0.1}{a} + \frac{0}{b}, \quad L_{\{a,b\}} = \frac{0.1}{a} + \frac{0.3}{b}, \quad R_{\{a\}} = \frac{0.1}{a} + \frac{0}{b}, \quad R_{\{a,b\}} = \frac{0.1}{a} + \frac{0.3}{b}.$$

So, the collection $\mathcal{H}$ of fuzzy closed sets in X consists of arbitrary intersections of finite unions of members of $\mathcal{S}$ and hence $\tau_{\mathcal{R},\mathcal{A}}$ is the set obtained by taking complements of members of $\mathcal{H}$.

Definition 6.4. Let (X, τ) be a fuzzy topological space. Then (X, τ) is said to be

1. *fuzzy T_0* if for each $x, y \in X$ such that $x \neq y$, there exists a fuzzy closed set U in X such that $U(x) \neq U(y)$;

2. *fuzzy T_1* if for each $x, y \in X$ such that $x \neq y$, there exist two fuzzy closed sets U, V such that $U(x) = 1$, $U(y) = 0$, $V(x) = 0$, $V(y) = 1$;

3. *fuzzy T_2* if for each pair of distinct fuzzy points x_r, y_s in X, there exist two fuzzy closed sets U, V such that $r > U(x), s > V(y)$ and $U \cup V = X$.

We remark here that (1), (2), (3) given above are equivalent to the definitions of fuzzy T_0, fuzzy T_1 and fuzzy T_2 given in [71], [107], [106], respectively.

Proposition 6.5. *[110] Let (X, τ) be a fuzzy topological space. Then the following statements are equivalent:*

1. *(X, τ) is fuzzy T_1.*

2. *$\{x\}$ is fuzzy closed, $\forall x \in X$.*

6.2 Separation axioms

In this section, we prove some results in the fuzzy setting, which are counterparts of the corresponding results given in [102]. In our discussion here, we shall assume that $\mathcal{R}$ is a fuzzy relation on X and $\mathcal{A}$ is a collection of $\mathcal{R}$-antisets.

Proposition 6.6. *Let for each $x, y \in X$ such that $x \neq y$, there exist $z \in X$ such that $\mathcal{R}(x, z) \neq \mathcal{R}(y, z)$ (or $\mathcal{R}(z, x) \neq \mathcal{R}(z, y)$)and $\mathcal{A}$ contains singletons. Then $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_0.*

Proof. Let $x, y \in X$ such that $x \neq y$. Then by our assumption, there exists $z \in X$ such that $\mathcal{R}(x, z) \neq \mathcal{R}(y, z)$ which implies that $L_{\{z\}}(x) \neq L_{\{z\}}(y)$. Since $L_{\{z\}}$ is fuzzy closed in $\tau_{\mathcal{R},\mathcal{A}}$ and is such that $L_{\{z\}}(x) \neq L_{\{z\}}(y)$, so $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_0.

Similarly, we can proceed for the case when $\mathcal{R}(z, x) \neq \mathcal{R}(z, y)$. □

Example 6.2. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{a, b\}$, which is given as follows:*

$\mathcal{R}$	a	b
a	*0.7*	*0.3*
b	*0.6*	*0.5*

and $\mathcal{A} = \{\{a\}, \{b\}\}$. Then the fuzzy topology $\tau_{\mathcal{R},\mathcal{A}}$ is generated by the following subbase $\mathcal{S}$ for the fuzzy closed sets in X:

$$\mathcal{S} = \{L_{\{a\}}, L_{\{b\}}, R_{\{a\}}, R_{\{b\}}\} \cup \{\alpha_X : \alpha \in [0,1]\},$$

where $L_{\{a\}}, L_{\{b\}}, R_{\{a\}}, R_{\{b\}}$ are given by:

$$L_{\{a\}} = \frac{0.7}{a} + \frac{0.6}{b}, \quad L_{\{b\}} = \frac{0.3}{a} + \frac{0.5}{b}, \quad R_{\{a\}} = \frac{0.7}{a} + \frac{0.3}{b}, \quad R_{\{b\}} = \frac{0.6}{a} + \frac{0.5}{b}.$$

Note that $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_0 since for $a, b \in X$, $a \neq b$, there exists a fuzzy closed set $U = L_{\{a\}}$ in X such that $U(a) \neq U(b)$.

Proposition 6.7. *Let for each $x \in X$ and $y \in X \setminus \{x\}$, there exist $z \in X$ such that $\mathcal{R}(x,z) = 1$ (or $\mathcal{R}(z,x) = 1$) and $\mathcal{R}(y,z) = 0$(or $\mathcal{R}(z,y) = 0$). Then $\mathcal{A}$ contains singletons implies that $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_1.*

Proof. In view of Proposition 6.5, we show that $\{x\}$ is fuzzy closed. Let $y_r \in X \setminus \{x\}$. Then $x \neq y$. So according to our assumption, there exists $z \in X$ such that $\mathcal{R}(x,z) = 1$ and $\mathcal{R}(y,z) = 0$, which implies that $L_{\{z\}}(x) = 1$ and $L_{\{z\}}(y) = 0$. Therefore, there exists $L^c_{\{z\}} \in \tau_{\mathcal{R},\mathcal{A}}$ such that $y_r \in L^c_{\{z\}} \subseteq X \setminus \{x\}$, which implies that $\{x\}$ is fuzzy closed.

The other case can be handled similarly. □

Corollary 6.8. *Let $\mathcal{R}$ be a fuzzy relation on X which is reflexive and antisymmetric. Then $\mathcal{A}$ contains singletons implies that $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_1.*

Proof. Let $x, y \in X$ such that $x \neq y$. Then by the antisymmetry of $\mathcal{R}$, either $\mathcal{R}(x,y) = 0$ or $\mathcal{R}(y,x) = 0$. Also by the reflexivity of $\mathcal{R}$, $\mathcal{R}(x,x) = 1$, for each $x \in X$. In both the cases, if we set $z = x$, then by Proposition 6.7, $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_1. □

Definition 6.9. A collection $\mathcal{A}$ of $\mathcal{R}$-antisets is called *separating* if for each $x \in X$ and $y \in X \setminus \{x\}$, there exists $A \in \mathcal{A}$ such that $L_A(x) = 1$ and $L_A(y) = 0$ or $R_A(x) = 1$ and $R_A(y) = 0$.

Proposition 6.10. *If $\mathcal{A}$ is separating, then $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_1.*

Proof. Let $x, y \in X$ such that $x \neq y$. Then $y \in X \setminus \{x\}$ and so by our assumption, there exists $A \in \mathcal{A}$ such that $L_A(x) = 1$ and $L_A(y) = 0$ or $R_A(x) = 1$ and

$R_A(y) = 0$. Similarly, for $x \in X \setminus \{y\}$, there exists $A' \in \mathcal{A}$ such that $L_{A'}(y) = 1$ and $L_{A'}(x) = 0$ or $R_{A'}(y) = 1$ and $R_{A'}(x) = 0$. Since $L_A, L_{A'}, R_A, R_{A'}$ are fuzzy closed sets in X, therefore $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_1. □

Example 6.3. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{a, b\}$, which is given as follows:*

$\mathcal{R}$	a	b	c
a	*1*	*0.3*	*0.4*
b	*0*	*1*	*0*
c	*0*	*0.5*	*1*

and $\mathcal{A} = \{\{a\}, \{b\}, \{c\}\}$. Then the fuzzy topology $\tau_{\mathcal{R},\mathcal{A}}$ is generated by the following subbase $\mathcal{S}$ for the fuzzy closed sets in X:

$$\mathcal{S} = \{L_{\{a\}}, L_{\{b\}}, L_{\{c\}}, R_{\{a\}}, R_{\{b\}}, R_{\{c\}}\} \cup \{\alpha_X : \alpha \in [0, 1]\},$$

where $L_{\{a\}}, L_{\{b\}}, L_{\{c\}}, R_{\{a\}}, R_{\{b\}}, R_{\{c\}}$ are given by:

$$L_{\{a\}} = \frac{1}{a} + \frac{0}{b} + \frac{0}{c}, \quad L_{\{b\}} = \frac{0.3}{a} + \frac{1}{b} + \frac{0.5}{c}, \quad L_{\{c\}} = \frac{0.4}{a} + \frac{0}{b} + \frac{1}{c},$$
$$R_{\{a\}} = \frac{1}{a} + \frac{0.3}{b} + \frac{0.4}{c}, \quad R_{\{b\}} = \frac{0}{a} + \frac{1}{b} + \frac{0}{c}, \quad R_{\{c\}} = \frac{0}{a} + \frac{0.5}{b} + \frac{1}{c}.$$

Since, $L_{\{a\}} = \{a\}, R_{\{b\}} = \{b\}$ and $L_{\{c\}} \cap R_{\{c\}} = \{c\}$ are fuzzy closed set in X, so in view of the Proposition 6.5, $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_1.

Definition 6.11. A collection $\mathcal{A}$ of $\mathcal{R}$-antisets *completely separates points* of X if for each $x, y \in X$ such that $x \neq y$, there exist $B_1, B_2, ..., B_k$, where for each i, $B_i = L_A$ or R_A, $A \in \mathcal{A}$ such that $X = \bigcup_{i=1}^{k} B_i$ and $B_i(x) > 0$ implies that $B_i(y) = 0$.

Theorem 6.12. *If $\mathcal{A}$ completely separates points of X, then $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_2.*

Proof. Let x_r and y_s be two distinct fuzzy points in X. Then $x \neq y$. Since $\mathcal{A}$ completely separates points of X, so there exist $B_1, B_2, ..., B_k$, where for each i, $B_i = L_A$ or R_A, $A \in \mathcal{A}$ such that

$$X = \bigcup_{i=1}^{k} B_i \tag{6.1}$$

and $B_i(x) > 0$ implies that $B_i(y) = 0$. Since $X = \bigcup_{i=1}^{k} B_i$, so for $x, y \in X$, there exist i_1 and i_2 such that $B_{i_1}(x) = 1$ and $B_{i_2}(y) = 1$. Next, since $B_{i_1}(x) = 1 > 0$, so $B_{i_1}(y) = 0$. Let $X_1 = \cup\{B_i | B_i(y) = 0\}$ and $X_2 = \cup\{B_i | B_i(y) > 0\}$. Here $X_1(y) = 0, X_2(x) = 0$ and in view of (6.1), we have $X_1 \cup X_2 = X$. Note that X_1 and X_2 are fuzzy closed sets in X such that $r > X_2(x)$, $s > X_1(y)$ and $X_1 \cup X_2 = X$. Therefore, $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_2. □

Example 6.4. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{a, b, c\}$, which is given as follows:*

$\mathcal{R}$	a	b	c
a	*1*	*0*	*0.3*
b	*0.7*	*1*	*0*
c	*0*	*0.8*	*1*

and $\mathcal{A} = \{\{a\}, \{b\}, \{c\}\}$. Then the fuzzy topology $\tau_{\mathcal{R},\mathcal{A}}$ is generated by the following subbase $\mathcal{S}$ for the fuzzy closed sets in X:

$$\mathcal{S} = \{L_{\{a\}}, L_{\{b\}}, L_{\{c\}}, R_{\{a\}}, R_{\{b\}}, R_{\{c\}}\} \cup \{\alpha_X : \alpha \in [0, 1]\},$$

where $L_{\{a\}}, L_{\{b\}}, L_{\{c\}}, R_{\{a\}}, R_{\{b\}}, R_{\{c\}}$ are given by:

$$L_{\{a\}} = \frac{1}{a} + \frac{0.7}{b} + \frac{0}{c}, \quad L_{\{b\}} = \frac{0}{a} + \frac{1}{b} + \frac{0.8}{c}, \quad L_{\{c\}} = \frac{0.3}{a} + \frac{0}{b} + \frac{1}{c},$$
$$R_{\{a\}} = \frac{1}{a} + \frac{0}{b} + \frac{0.3}{c}, \quad R_{\{b\}} = \frac{0.7}{a} + \frac{1}{b} + \frac{0}{c}, \quad R_{\{c\}} = \frac{0}{a} + \frac{0.8}{b} + \frac{1}{c}.$$

Since for the fuzzy points $x_r, y_s \in X$, there exist two fuzzy closed sets $U = L_{\{b\}} \cup R_{\{c\}}$ and $V = L_{\{c\}} \cup R_{\{a\}}$ in X such that $r > U(x)$, $s > V(y)$ and $U \cup V = X$, for the fuzzy points $y_r, z_s \in X$, there exist two fuzzy closed sets $U = L_{\{c\}} \cup R_{\{a\}}$ and $V = L_{\{a\}} \cup R_{\{b\}}$ in X such that $r > U(y)$, $s > V(z)$ and $U \cup V = X$ and for the fuzzy points $z_r, x_s \in X$, there exist two fuzzy closed sets $U = L_{\{a\}} \cup R_{\{b\}}$ and $V = L_{\{b\}} \cup R_{\{c\}}$ in X such that $r > U(z)$, $s > V(x)$ and $U \cup V = X$, so $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_2.

6.3 Finite intersection property and fuzzy compactness

In this section, we introduce 'finite intersection property' in fuzzy topological spaces and find a sufficient condition under which a fuzzy topology generated by a fuzzy relation, becomes fuzzy compact.

Definition 6.13. A family $\mathcal{F}$ of fuzzy sets is said to satisfy the *finite intersection property*$(F.I.P.)$ if for every $\alpha \in (0,1]$, there exists ϵ, $0 < \epsilon < \alpha$ such that for every finite subfamily $F_1, F_2, ..., F_n$ of $\mathcal{F}$, there exists $x \in X$ such that $(\bigcap_{i=1}^{n} F_i)(x) > 1 - \alpha + \epsilon$.

Note that the fuzzy set 0_X cannot be a member of a family satisfying finite intersection property.

Theorem 6.14. *Let (X, τ) be a fuzzy topological space. Then the following statements are equivalent:*

1. *If $\mathcal{F}$ is a family of fuzzy closed sets satisfying finite intersection property $(F.I.P.)$, then for each $\alpha \in (0,1]$, there exists $y \in X$ such that $(\bigcap_{F \in \mathcal{F}} F)(y) > 1 - \alpha$.*
2. *(X, τ) is fuzzy compact.*

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{G}$ be a family of fuzzy open sets in X such that

$$\begin{aligned} & \alpha_X \subseteq \bigcup_{G_i \in \mathcal{G}} G_i \text{ and } \alpha \in (0,1] \\ \Rightarrow \quad & \bigcap_{G_i \in \mathcal{G}} G_i^c \subseteq (1-\alpha)_X \\ \Rightarrow \quad & \nexists y \in X \text{ such that } (\bigcap_{G_i \in \mathcal{G}} G_i^c)(y) > 1 - \alpha. \end{aligned}$$

Therefore in view of (1), $\mathcal{F} = \{G_i^c : G_i \in \mathcal{G}\}$ does not satisfy F.I.P., so for each ϵ such that $0 < \epsilon < \alpha$, there exist $G_1^c, G_2^c, ..., G_n^c \in \mathcal{F}$ such that

$$\begin{aligned} & (\bigcap_{i=1}^{n} G_i^c)(x) \leq 1 - \alpha + \epsilon, \quad \text{for each } x \in X \\ \Rightarrow \quad & \bigcap_{i=1}^{n} G_i^c \subseteq (1 - \alpha + \epsilon)_X \\ \Rightarrow \quad & (\alpha - \epsilon)_X \subseteq \bigcup_{i=1}^{n} G_i \\ \Rightarrow \quad & \alpha_X \text{ is fuzzy compact.} \\ \Rightarrow \quad & (X, \tau) \text{ is fuzzy compact.} \end{aligned}$$

(2) $\Rightarrow$ (1) Conversely, assume that (X, τ) is fuzzy compact i.e., each α_X, $\alpha \in [0,1]$, is fuzzy compact. Let $\mathcal{F}$ be a family of fuzzy closed sets satisfying F.I.P. We have to show that for each $\alpha \in (0,1]$, there exists $x \in X$ such that

$$(\bigcap_{F\in\mathcal{F}} F)(x) > 1 - \alpha.$$

Assume the contrary, i.e, for $\alpha \in (0,1]$,

$$\begin{aligned} &(\bigcap_{F\in\mathcal{F}} F)(x) \leq 1-\alpha, \quad \forall x \in X \\ \Rightarrow\ &\bigcap_{F\in\mathcal{F}} F \subseteq (1-\alpha)_X \\ \Rightarrow\ &\alpha_X \subseteq \bigcup_{F\in\mathcal{F}} F^c \end{aligned}$$

This implies that $\{F^c : F \in \mathcal{F}\}$ is an open cover of α_X. Since α_X is fuzzy compact, so for each ϵ such that $0 < \epsilon < \alpha$, there exist $F_1^c, F_2^c, ..., F_n^c \in \mathcal{F}$ such that

$$\begin{aligned} &(\alpha-\epsilon)_X \subseteq \bigcup_{i=1}^{n} F_i^c \\ \Rightarrow\ &\bigcap_{i=1}^{n} F_i \subseteq (1-\alpha+\epsilon)_X \\ \Rightarrow\ &\nexists \text{ any } x \text{ such that } (\bigcap_{i=1}^{n} F_i)(x) > 1-\alpha+\epsilon, \end{aligned}$$

implying that $\mathcal{F}$ does not satisfy F.I.P., which is a contradiction. □

Definition 6.15. [61] Let $\mathcal{R}$ be a fuzzy partial ordering on X and $A \subseteq X$. Then the *fuzzy upper bound* for A is the fuzzy set denoted by $U(\mathcal{R}, A)$ and defined by

$$U(\mathcal{R}, A) = \bigcap_{x\in A} R_x, \text{ where } R_x(y) = \mathcal{R}(x, y), \text{ for each } y \in X.$$

Definition 6.16. A subset A of X is said to be *α-level bounded above*, $\alpha \in (0,1]$, if for every ϵ, $0 < \epsilon < \alpha$, there exists $y \in X$ such that

$$U(\mathcal{R}, A)(y) > 1 - \alpha + \epsilon.$$

We say that A is *fuzzy bounded above* if it is α-level bounded above, for each $\alpha \in (0,1]$.

Definition 6.17. A subset A of X is said to have *α-level least upper bound*, $\alpha \in (0,1]$, if for every ϵ, $0 < \epsilon < \alpha$, there exists $z \in X$ such that $U(\mathcal{R}, A)(z) > 1-\alpha+\epsilon$ and $\mathcal{R}(z, y) > 1 - \alpha + \epsilon$, $\forall y$ such that $\mathcal{R}(x, y) > 1 - \alpha$, for each $x \in A$.

Definition 6.18. A set X is said to be *α-level complete* if every $A \subseteq X$ which is α-level bounded above, has an α-level least upper bound.

We say that X is *fuzzy complete* if it is $\alpha-$level complete for every $\alpha \in (0, 1]$.

Proposition 6.19. *If X is α-level bounded above and $A \subseteq X$, then A is also α-level bounded above.*

Proof. Since X is α-level bounded above, so for every ϵ, $0 < \epsilon < \alpha$, there exists $y \in X$ such that

$$U(\mathcal{R}, X)(y) > 1 - \alpha + \epsilon.$$

We have to show that for every ϵ, $0 < \epsilon < \alpha$, there exists $y \in X$ such that $U(\mathcal{R}, A)(y) > 1 - \alpha + \epsilon$. Assume the contrary that for each $y \in X$,

$$\begin{aligned} & U(\mathcal{R}, A)(y) \leq 1 - \alpha + \epsilon \\ \Rightarrow\ & \inf_{a \in A} \mathcal{R}(a, y) \leq 1 - \alpha + \epsilon \\ \Rightarrow\ & U(\mathcal{R}, X)(y) = \inf_{x \in X} \mathcal{R}(x, y) \leq \inf_{a \in A} \mathcal{R}(a, y) \leq 1 - \alpha + \epsilon, \text{ for each } y \in Y \end{aligned}$$

which is a contradiction to the fact that X is α-level bounded above.

Theorem 6.20. *Let (X, τ) be a fuzzy topological space. Then the following statements are equivalent:*

1. *If $\mathcal{F}$ is a family of subbasic fuzzy closed sets satisfying finite intersection property (F.I.P.), then for each $\alpha \in (0, 1]$, there exists $y \in X$ such that $(\bigcap_{F \in \mathcal{F}} F)(y) > 1 - \alpha$.*
2. *For any subbase $\mathcal{S}$ of τ, if $\mathcal{G} \subseteq \mathcal{S}$ such that $\alpha_X \subseteq \bigcup_{f \in \mathcal{G}} f$, then for each ϵ, $0 < \epsilon < \alpha$, there exists a finite subset $\mathcal{G}_0$ of $\mathcal{G}$ such that $(\alpha - \epsilon)_X \subseteq \bigcup_{f \in \mathcal{G}_0} f$.*
3. *(X, τ) is fuzzy compact.*

Proof. The equivalence of (2) and (3) has already been proved in [69]. Here we prove (1) $\Leftrightarrow$ (2). For (1) $\Rightarrow$ (2), let $\mathcal{S}$ be a subbase for τ and $\mathcal{G} \subseteq \mathcal{S}$ such that

$$\begin{aligned}
& \alpha_X \subseteq \bigcup_{f\in\mathcal{G}} f \\
\Rightarrow\ & \bigcap_{f\in\mathcal{G}} f^c \subseteq (1-\alpha)_X \\
\Rightarrow\ & \nexists y \in X \text{ such that } (\bigcap_{f\in\mathcal{G}} f^c)(y) > 1-\alpha.
\end{aligned}$$

Therefore in view of (1), $\mathcal{F} = \{f^c : f \in \mathcal{G}\}$ does not satisfy F.I.P., so for each ϵ such that $0 < \epsilon < \alpha$, there exist $f_1^c, f_2^c, ..., f_n^c \in \mathcal{G}$ such that

$$\begin{aligned}
& (\bigcap_{i=1}^{n} f_i^c)(x) \leq 1-\alpha+\epsilon, \quad \text{for each } x \in X \\
\Rightarrow\ & \bigcap_{i=1}^{n} f_i^c \subseteq (1-\alpha+\epsilon)_X \\
\Rightarrow\ & (\alpha-\epsilon)_X \subseteq \bigcup_{i=1}^{n} f_i.
\end{aligned}$$

Conversely, assume that $\mathcal{F}$ be a family of subbasic fuzzy closed sets satisfying F.I.P. We have to show that for each $\alpha \in (0,1]$, there exists $x \in X$ such that

$$(\bigcap_{F\in\mathcal{F}} F)(x) > 1-\alpha.$$

Assume the contrary i.e, for $\alpha \in (0,1]$,

$$\begin{aligned}
& (\bigcap_{F\in\mathcal{F}} F)(x) \leq 1-\alpha, \quad \forall x \in X \\
\Rightarrow\ & \bigcap_{F\in\mathcal{F}} F \subseteq (1-\alpha)_X \\
\Rightarrow\ & \alpha_X \subseteq \bigcup_{F\in\mathcal{F}} F^c.
\end{aligned}$$

So according to our assumption, for each ϵ such that $0 < \epsilon < \alpha$, there exist $F_1^c, F_2^c, ..., F_n^c \in \mathcal{F}$ such that

$$\begin{aligned} &(\alpha - \epsilon)_X \subseteq \bigcup_{i=1}^{n} F_i^c \\ \Rightarrow\ &\bigcap_{i=1}^{n} F_i \subseteq (1 - \alpha + \epsilon)_X \\ \Rightarrow\ &\nexists \text{ any } x \text{ such that } (\bigcap_{i=1}^{n} F_i)(x) > 1 - \alpha + \epsilon, \end{aligned}$$

implying that $\mathcal{F}$ does not satisfy F.I.P., which is a contradiction. □

Theorem 6.21. *Let $\mathcal{R}$ be a fuzzy partial order. If X is fuzzy complete, fuzzy bounded above and $\mathcal{A}$ consists of singletons, then $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy compact.*

Proof. Let $\mathcal{F}$ be a family of subbasic fuzzy closed sets of X satisfying F.I.P. First, we show that there does not exist any δ_X, where $\delta \in (0, 1)$, belonging to $\mathcal{F}$. Since if we assume that $\mathcal{F}$ contains some δ_X, where $\delta \in (0, 1)$, then

$$\bigcap_{F \in \mathcal{F}_1} F \subseteq \delta_X,$$

where $\mathcal{F}_1$ is a finite subfamily of $\mathcal{F}$ containing δ_X, implies that

$$\bigcap_{F \in \mathcal{F}_1} F \subseteq (1 - \alpha_1)_X \subseteq (1 - \alpha_1 + \epsilon)_X, \quad \text{for } \alpha_1 \in (0, 1) \text{ such that } \delta = (1 - \alpha_1)$$
$$\text{and for each } \epsilon,\ 0 < \epsilon < \alpha_1$$

$\Rightarrow$ $\mathcal{F}$ does not satisfy F.I.P., which is a contradiction.

Next, if $\mathcal{F}$ contains only 1_X, then the proof of the theorem is trivial and if there exists some member of the form $L_{\{x\}}$ or $R_{\{x\}}$ other than 1_X in $\mathcal{F}$, then 1_X plays no role in the intersection $\bigcap_{F \in \mathcal{F}} F$, so it is sufficient to prove the theorem for $\mathcal{F}$ of the form $\{L_{\{x_\beta\}} | \beta \in \Omega_1\} \cup \{R_{\{x_\beta\}} | \beta \in \Omega_2\}$. Now, we have to show that for each $\alpha \in (0, 1]$, there exists $z \in X$ such that $(\bigcap_{F \in \mathcal{F}} F)(z) > 1 - \alpha$.

Let $A = \{x_\beta | \beta \in \Omega_2\} \subseteq X$. Since for $\alpha \in (0, 1]$, X is α-level bounded above, so by Proposition 6.19, A is also α-level bounded above. Therefore, by the fuzzy completeness of X, there exists an α-level least upper bound of A. So, for each ϵ

such that $0 < \epsilon < \alpha$, there exists an element $x_0 \in X$ such that

$$U(\mathcal{R}, A)(x_0) > 1 - \alpha + \epsilon \text{ and } \mathcal{R}(x_0, y) > 1 - \alpha + \epsilon, \forall\, y \in X$$
$$\text{such that}\, \mathcal{R}(x, y) > 1 - \alpha, \text{ for each}\, x \in A \tag{6.2}$$
$$\Rightarrow \inf_{x\in A} \mathcal{R}(x, x_0) > 1 - \alpha + \epsilon$$
$$\Rightarrow \mathcal{R}(x_\beta, x_0) > 1 - \alpha + \epsilon, \quad \text{for each}\, \beta \in \Omega_2$$
$$\Rightarrow R_{\{x_\beta\}}(x_0) > 1 - \alpha + \epsilon, \quad \text{for each}\, \beta \in \Omega_2. \tag{6.3}$$

Since for $\beta \in \Omega_2$, $\mathcal{B} = \{R_{\{x_\beta\}},\, L_{\{x_\gamma\}}\}$, where $\gamma \in \Omega_1$, is a finite subfamily of $\mathcal{F}$, so there exists ϵ_1, $0 < \epsilon_1 < \alpha$ and $y \in X$ such that

$$(R_{\{x_\beta\}} \cap L_{\{x_\gamma\}})(y) > 1 - \alpha + \epsilon_1$$
$$\Rightarrow \min\{R_{\{x_\beta\}}(y), L_{\{x_\gamma\}}(y)\} > 1 - \alpha + \epsilon_1$$
$$\Rightarrow \min\{\mathcal{R}(x_\beta, y), \mathcal{R}(y, x_\gamma)\} > 1 - \alpha + \epsilon_1$$
$$\Rightarrow \mathcal{R}(x_\beta, x_\gamma) > 1 - \alpha + \epsilon_1 \quad (\text{Using transitivity of } \mathcal{R})$$
$$\Rightarrow \mathcal{R}(x_\beta, x_\gamma) > 1 - \alpha.$$

Since the above inequality holds for each $\beta \in \Omega_2$, so we have

$$\Rightarrow \mathcal{R}(x_0, x_\gamma) > 1 - \alpha + \epsilon \quad (\text{Using 6.2 and putting } y = x_\gamma)$$
$$\Rightarrow L_{\{x_\gamma\}}(x_0) > 1 - \alpha + \epsilon.$$

The above inequality holds for every $\gamma \in \Omega_1$. So

$$L_{\{x_\beta\}}(x_0) > 1 - \alpha + \epsilon, \text{for each } \beta \in \Omega_1. \tag{6.4}$$

From (6.3) and (6.4), we get

$$\inf_{F\in\mathcal{F}} F(x_0) \geq 1 - \alpha + \epsilon > 1 - \alpha$$
$$\Rightarrow (\bigcap_{F\in\mathcal{F}} F)(x_0) > 1 - \alpha. \tag{6.5}$$

□

6.4 Conclusion

In this chapter, we have introduced fuzzy topologies generated by fuzzy relations, as a generalization of the corresponding concept given by Smithson[102]. We have then obtained sufficient conditions under which this generated fuzzy topology satisfies separation axioms, fuzzy T_0, fuzzy T_1 and fuzzy T_2. Further, we have introduced 'finite intersection property' in fuzzy topological spaces and obtained a characterization of Lowen's fuzzy compactness in terms of this property. Using this result, we have obtained a sufficient condition under which a fuzzy topology generated by a fuzzy relation, becomes fuzzy compact.