CHPATER- II

BASIC IDEAS OF FRACTIONAL CALCULUS

INTRODUCTION

The traditional integral and derivative are, to say the least, a staple for the technology professional, essential as a means of understanding and working with natural and artificial systems. Fractional Calculus is a field of mathematic study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. This chapter presents the basic idea about the non-integer calculus or Fractional Calculus. To say precisely those two names are incorrect, since the fractional order calculus is generalization of integer order calculus. In this line of theory irrational numbers are dealt with fractions and both the names completely fail to give a hint about complex numbers. Therefore a third name is emerged to address the issue i.e., “Generalized Calculus”. In this work the most commonly used name Fractional Calculus is employed and other names like Fractional Plants, Fraction systems, fractional orders, fractional controllers, fractional PIDs are used.

In case of fractional calculus the generalize notation of derivatives in which the differentiation order is not a integer number it may be negative number, a fraction, an irrational or even a imaginary or complex number. The general expression of derivatives is $\frac{d^{\alpha}}{dt^{\alpha}} f(\mathrm{t})$ where α is a real number. This chapter is organized as follows; mathematical background of integer order calculus is given in 2.1 section. In section 2.2, the Riemann-Liouville definition of fractional order integral and fractional order derivative has been presented. Caputo definition of real order calculus is provided in section 2.3. Section 2.4 demonstrates the Grünwald – Letnikoff definition of fractional calculus. Riemann-Liouville definition of complex order calculus is presented in section 2.5. Section 2.6 gives the Laplace transform of Riemann-Liouville definition, Caputo definition and Grünwald – Letnikoff defination. Finally, section 2.7 presents the conclusion of the chapter.

2.1 INTEGER ORDER CALCULUS

Let us start with defining the integer order calculus or normal derivatives. D denotes the functional operator (i.e., $\frac{d}{dt}$ is equal to D).

$$D f(t) = \lim_{h\to 0} \frac{f(t) - f(t-h)}{h} \tag{2.1}$$

For second order derivatives

$$D^2 f(t) = \lim_{h\to 0} \frac{D^1 f(t) - D^1 f(t-h)}{h}$$

$$D^2 f(t) = \lim_{h\to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2} \tag{2.2}$$

For Third order derivatives

$$D^3 f(t) = \lim_{h\to 0} \frac{D^2 f(t) - D^2 f(t-h)}{h}$$

$$D^3 f(t) = \lim_{h\to 0} \frac{f(t) - 3f(t-h) + 3f(t+2h) - f(t-3h)}{h^3} \tag{2.3}$$

For α^{th} order derivative

Lemma 2.1

The α^{th} order derivative is defined by

$$D^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\alpha} (-1)^k \binom{\alpha}{k} f(t-kh) \quad , \quad \alpha \in \mathrm{N} \tag{2.4}$$

Where $\binom{\alpha}{k} = \frac{n(n-1)(n-2)....(n-k+1)}{k!}$

Proof: This is proven by mathematical induction

$$D\left[D^{\alpha} f(t)\right] = \frac{1}{h}\lim_{h\to 0}\frac{\sum_{k=0}^{\alpha}(-1)^k\binom{\alpha}{k} f(t-kh)}{h^n} - \frac{\sum_{k=0}^{\alpha}(-1)^k\binom{\alpha}{k} f(t-kh-h)}{h^n}$$

$$D[D^{\alpha} f(\mathrm{t})] = \frac{1}{h^{\alpha+1}}\lim_{h\to 0}\sum_{k=0}^{\alpha}(-1)^k\binom{\alpha}{k} f(t-kh) - \sum_{k=1}^{\alpha-1}(-1)^{k-1}\binom{\alpha}{k-1} f(t-kh) \tag{2.5}$$

By using Pascal's triangle property

$$DD^{n} f(t) = \lim_{h\to 0}\frac{(-1)^0\binom{\alpha}{0} f(t-0h) + \sum_{k=1}^{\alpha}\left[(-1)^k\binom{\alpha+1}{k} f(t-kh)\right] + \ldots + (-1)^{\alpha+1}\binom{\alpha}{\alpha} f(t-(\alpha+1)h)}{h^{\alpha+1}} \tag{2.6}$$

Where $\binom{\alpha}{0} = 1 = \binom{\alpha+1}{0} = 1 = \binom{\alpha+1}{0}$ and $\binom{\alpha}{\alpha} = 1 = \binom{\alpha+1}{\alpha+1}$

Therefore

$$DD^{n} f(t) = \lim_{h\to 0}\frac{\sum_{k=0}^{\alpha+1}(-1)^k\binom{\alpha+1}{k} f(t-kh)}{h^{\alpha-1}} = D^{\alpha+1} f(t) \tag{2.7}$$

Likewise it can be proved for indefinite integration represented by ${}_cI_t^{\alpha}$ the indexes being the limits of the integration

$${}_cI_x^{\alpha} f(t) = \begin{cases} \int_c^t f(t)\,dt & , \; if \quad \alpha = 1 \\ \int_c^t {}_cI_t^{\alpha-1} f(t)\,dt & , \; if \quad \alpha \in \square \wedge \alpha > 1 \end{cases} \tag{2.8}$$

It is known fact that indefinite integration and differentiation are inverse operations

Since

$$D^{\alpha}{}_{c}I_{t}^{\alpha}f(t)=f(t) \tag{2.9}$$

Let's define

$$_{c}D_{x}^{\alpha}f(t)={}_{c}I_{t}^{-\alpha}f(t),\alpha\in\mathbb{R}^{-} \tag{2.10}$$

For α=0

$$D^{0}f(x)=f(x) \tag{2.11}$$

For first order Equation (2.9) will be

$$_{c}I_{t}^{1}D^{1}f(t)=f(t)-f(c)=f(t)-f(t-c)^{0}f(c) \tag{2.12}$$

For second order Equation (2.9) will be

$$_{c}I_{t}^{2}f(t)=f(t)-f(c)-(t-c)Df(c)$$

$$_{c}I_{t}^{2}D^{2}=f(t)-(t-c)^{0}f(c)-(t-c)Df(c) \tag{2.13}$$

For third order Equation (2.9) becomes

$$_{c}I_{t}^{3}D^{3}=f(t)-(t-c)^{0}f(c)-(t-c)Df(c)-\frac{(t-c)^{2}}{2}D^{2}f(c) \tag{2.14}$$

For α^{th} order Equation (2.9) becomes

$$_{c}I_{t}^{\alpha}D^{\alpha}f(t)=f(t)-\sum_{i=0}^{\alpha-1}\frac{(t-c)^{i}}{i!}D^{i}f(c) \tag{2.15}$$

The order of the exponent D^{α} defines as

$$ {}_{c}D_{t}^{-\alpha}\,{}_{c}D_{x}^{\alpha} f(t) = \begin{cases} f(t), \alpha \in \mathbb{Z}_0^- \\ f(t) - \sum_{k=0}^{\alpha-1} \frac{(t-c)^{\alpha-k-1}}{(\alpha-k-1)!} D^{\alpha-k-1} f(c), \quad \alpha \in N \end{cases} \qquad (2.16) $$

2.2 RIEMANN-LIOUVILLE DEFINITION OF FRACTIONAL CALCULUS

Riemann-Liouville generalizes the fractional operator D. which is most famous definition in the literature.

2.2.1 FRACTIONAL ORDER INTEGRAL

According to the Riemann-Liouville approach to fractional calculus the notion of fractional integral of order n> 0 is a natural consequence of the well known formula (usually attributed to Cauchy), that reduces the calculation of the n−fold primitive of a function f(t) to a single integral of convolution type. In our notation the Cauchy formula reads,

$$ I_c^{\alpha} = {}_{c}D_{t}^{-\alpha} = \int_c^t \frac{(t-x)}{(\alpha-1)!} f(x)dx, \quad \alpha \in \mathbb{N} \qquad (2.17) $$

Where N is the set of positive integers. From this definition we note that ${}_{c}D_{t}^{-\alpha}$ vanishes at t = 0 with its derivatives of order 1, 2,…,n − 1 .

For convention we require that f(t) and henceforth ${}_{c}D_{t}^{-\alpha}$ be a causal function, i.e. Identically vanishing for t < 0 .In a natural way one is led to extend the above formula from positive integer values of the index to any positive real values by using the Gamma function. Indeed, noting that (n − 1)! = Γ (n), and introducing the arbitrary positive real number α,

Riemann-Liouville fractional order integral is defined as

$$I_c^{\,n} f(t) = \frac{1}{\Gamma(n)} \int_c^t (t-\tau)^{n-1} f(\tau) d\tau, \quad \text{t>c, n} \in \Re^+ \tag{2.18}$$

Where $\mathbb{R}^+$ is the set of positive real numbers.

When we deal with dynamic systems it is usual that $f(t)$ be casual function of t, and so in what follows the definition for the fractional order integral to be used is

$$I^{\,n} f(t) = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, \quad \text{t>0, n} \in \Re^+ \tag{2.19}$$

These Rimann-Liouville fractional operators will fulfill some important properties

- The R-L fractional derivative of order α is defined as the left inverse (but not the right inverse) of the R-L fractional integral

$$(D^{\alpha} I^{\,\alpha} f)(x) = f(x), \quad \alpha > 0 \tag{2.20}$$

- Semi group property of the fractional integration operators

$$(I^{\,\alpha} I^{\,\beta} f)(x) = (I^{\alpha+\beta} f)(x), \quad \alpha, \beta > 0 \tag{2.21}$$

2.2.2 FRACTIONAL ORDER DERIVATIVE

After the notion of fractional integral, that of fractional derivative of order (n> 0) becomes a natural requirement and one is attempted to substitute α with –α in the above formulas. However, this generalization needs some care in order to guarantee the convergence of the integrals and preserve the well-known properties of the ordinary derivative of integer order.

Denoting the derivative operator of order n ∈ N by D^n, and the identity operator by I, we can verify that

$$I^{\,n} D^{\,n} = I, \; I^{\,n} D^{\,n} \neq I, \quad n \in N \tag{2.22}$$

In other words, the operator D^n is only a left inverse of the operator I^n. in fact we can deduce that

$$I^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{t^k}{k!}, \quad t>0 \tag{2.23}$$

Where $f^{(k)}(.)$ is the k^{th} order derivative of the function $f(.)$. Consequently, it must be verified whether D^α is a left inverse of I^α or not. For this purpose, introducing the positive integer *m* so that $m-1 < \alpha < m$, the Riemann-Liouville definition for the fractional order derivative of the order $\alpha \in R^+$ has the following form

$${}_R D^\alpha f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}}\right] \tag{2.24}$$

Where $m-1 < \alpha < m,\ m \in N$

The complete Riemann-Liouville definition foe function $f(t)$ is given by following equation

$${}_c D_x^\alpha = \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t \dfrac{f(\tau)}{(t-\tau)^{\alpha-m+1}}\right], & \alpha<0 \\ f(\alpha), \text{if } \alpha=0 \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, & t>0,\ \alpha>0,\ n\in R^+ \end{cases} \tag{2.25}$$

Note 1: it is important to stress that operator D is local only if the order is a natural number, that is to say, for usual derivative only, in all other cases integration limits are needed, even if the order is positive

Note 2: Just as when the orders are integer, the following equality, fulfill for the Riemann-Liouville definitions

$${}_c D_x^\alpha\, {}_c D_x^{-\alpha} = f(x), \quad \alpha \geq 0 \tag{2.26}$$

3.4 CAPUTO DEFINITION OF FRACTIONAL CALCULUS

The Riemann-Liouville approach has its limitations that are it requires the fractional order initial conditions for the solution of an initial value problem. In order to overcome this problem Caputo derived another form of the fractional derivative to represent the continuous viewpoint of Riemann-Liouville. In his definition he proved that it is possible to represent the initial value problem in terms of integer order derivatives for the initial values. The Caputo fractional derivative of order α is:

$$ {}_a^cD_t^{\alpha} = \frac{1}{\Gamma(\alpha - n)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \qquad (\text{n-1} < \alpha < \text{n}) \tag{2.27} $$

We came across such kind of approach in mathematical modeling of the various physical systems, like heat conduction in materials with memory. Since these equations are encountered in combined condition, convection and radiation problems. Further the initial conditions for the Caputo derivatives are expressed in terms of initial values of integer order derivatives; it is known that for zero initial conditions the Riemann-liouville, Caputo fractional derivatives coincide. This concept permits a numerical solution of initial value problems for differential equations of non-integer order independently of the chosen definition of the fractional derivative. By making use of the advantage, in literature many authors resort to Caputo derivatives, to avoid the problem of initial values of fractional derivatives by treating only the case of zero initial conditions.

The complete definition is

$$ {}_cD_x^{\alpha} = \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{f(\tau)}{(t-\tau)^{\alpha-m+1}}\right], \ \alpha<0 \\ f(\alpha), \text{if } \ \alpha=0 \\ \dfrac{1}{\Gamma(\alpha-n)} \displaystyle\int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \ \alpha>0 \end{cases} \tag{2.28} $$

2.4 GRUNWALD-LETNIKOV DEFINITION OF FRACTIONAL CALCULUS

Here we generalize the operator D for non-integer order of differentiator and integrator according to definition presented by Grunwald and Letnikov

The definition is derived from the Equation (2.4) which gives the results as

$$f^{n}(\mathrm{x}) = D^{n} f(\mathrm{x}) = \lim_{h\to 0}\frac{1}{h^{n}}\sum_{m=0}^{n}(-1)^{m}\binom{n}{m} f(\mathrm{x}-\mathrm{mh})$$

$$\binom{n}{m} = \frac{n!}{m!(\mathrm{n}-\mathrm{m})!} \tag{2.29}$$

This can be replaced by Gamma function as $\frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-\mathrm{m}+1)}$ for non-integer n i.e., α therefore differentiation in fractional order is

$$_{a}D^{\alpha} f(\mathrm{x}) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{m=0}^{\left[\frac{x-a}{h}\right]}(-1)^{m}\frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-\mathrm{m}+1)} \tag{2.30}$$

For real order *n* i.e., *α*. Therefore differentiation in fractional order is:

$$_{a}D^{\alpha} f(\mathrm{x}) = \lim_{h\to 0}\frac{1}{h^{\alpha}}(-1)^{m}\frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-\mathrm{m}+1)} f(\mathrm{x}-\mathrm{mh}) \tag{2.31}$$

For negative *α* the process will be integration.

$$\binom{-n}{m} = \frac{-n(-\mathrm{n}-1)(-\mathrm{n}-2)....(-\mathrm{n}-\mathrm{m}+1)}{m!} = (-1)^{m}\frac{n(n+1)(\mathrm{n}+2)(\mathrm{n}+3)....(\mathrm{n}+\mathrm{m}-1)}{m!}$$

$$= (-1)^{m}\frac{(\mathrm{n}+\mathrm{m}-1)!}{m!(\mathrm{n}-1)!} \rightarrow (-1)^{m}\frac{(\alpha+1)!}{m!\Gamma(\alpha)} \tag{2.32}$$

Hence for real order integration the solution is

$$ {}_{a}D^{-\alpha} f(\mathrm{x}) = \lim_{h \to 0} h^{\alpha} \sum_{m=0}^{\left[\frac{x-a}{h}\right]} \frac{\Gamma(\alpha+\mathrm{m})}{m!\Gamma(\alpha)} f(\mathrm{x}-\mathrm{mh}) \tag{2.33} $$

The part $\left[\frac{x-a}{h}\right]$ is integer part. It is clear that the upper limit of the summation is the integer part of the fraction.

2.5 RIEMANN-LIOUVILLE FRACTIONAL DEFINITION FOR COMPLEX ORDER CALCULUS

Riemann-Liouville given definition of operator differential operator D for a complex order

$$ {}_{c}D_{x}^{v} f(x) = \begin{cases} \frac{1}{\Gamma(-z)} \int_{c}^{x} (x-\xi)^{-v-1} f(\xi) d\xi, if \operatorname{Re} z < 0 \\ {}_{c}D_{x}^{n} \left[{}_{c}D_{x}^{x-n} f(x) \right], n = \min\{k \in \square : k > \operatorname{Re} z\}, if \operatorname{Re} z > 0 \end{cases} \tag{2.34} $$

The integral of the previous formula is derivative if z is purely imaginary; so this case must be handled as follows:

$$ {}_{c}D_{x}^{v} f(x) = D^{1} {}_{c}D_{x}^{v-1} f(x), z \in \square : \operatorname{Re} Z = 0 \tag{2.35} $$

The expression above includes the identity operator found when z=0.

$$
{}_{c}D_{x}^{v} f(x) = \begin{cases} \frac{1}{\Gamma(-1)} \int_{c}^{x} (x-\xi)^{-x-1} f(d\xi), if \operatorname{Re} z < 0 \\ D^{1}{}_{c}D_{x}^{-1+x} f(x), ifz \in \square : \operatorname{Re} Z = 0 \\ f(x), ifz = 0 \\ D^{n}\left[{}_{c}D_{x}^{x-n}(x)\right], n = \min\{k \in \square : k > \operatorname{Re} z\}, if \operatorname{Re} z > 0 \end{cases} \tag{2.36}
$$

2.6 LAPLACE TRANSFORM OF FRACRIONAL CALCULUS

Operator D turns out to have Laplace transforms that follow rules quite similar to those valid for integer orders. The different definitions of the operator, however, cause slight differences in the initial conditions that show up in Laplace transform.

2.6.1 THE LAPLACE TRANSFORM OF THE RIEMANN-LIOUVILLE DEFINITION

$$
L\left[{}_{0}D_{x}^{v} f(x)\right] = s^{v} F(s), v \le 0 \tag{2.37}
$$

$$
L\left[{}_{0}D_{x}^{v} f(x)\right] = s^{v} F(s) - \sum_{k-0}^{n-1} s^{k}{}_{0}D_{x}^{v-k-1} f(0), n-1 < v \le n \in \square \tag{2.38}
$$

The above definition is proved as follows

$$
L\left[{}_{0}D_{x}^{v} f(x)\right] = L\left[\frac{1}{\Gamma(-v)} \int_{0}^{x} (x-\xi)^{-v-1} f(\xi) d\xi\right] \tag{2.39}
$$

Using the Convolution theorem the Equation (2.39) becomes

$$
L\left[{}_{0}D_{x}^{v} f(x)\right] = \frac{1}{\Gamma(-v)} L\left[x^{-v-1}\right] L\left[f(x)\right] \tag{2.40}
$$

By applying the Laplace transform of power function the Equation (2.40) becomes

$$
L\left[{}_{0}D_{x}^{v} f(x)\right] = \frac{1}{\Gamma(-v)} s^{-1+v+1} \Gamma(-v-1+1) F(s) = s^{v} f(\ ) \tag{2.41}
$$

For positive orders the Equation (2.41) becomes

$$L\left[{}_0D_x^v f(x)\right] = L\left[D^n {}_0D_x^{v-n} f(x)\right] \tag{2.42}$$

By applying Laplace transform of integer derivative the Equation (2.42) becomes

$$L\left[{}_0D_x^v f(x)\right] = s^n s^{v-n} F(s) - \sum_{k-1}^{n} s^{n-k} D^{k-1} {}_0D_x^{v-n} f(0) =$$

$$= s^v F(s) - \sum_{k-0}^{n-1} s^{n-k-1} {}_0D_x^{v-n+k} f(0) = s^v F(s) - \sum_{k-0}^{n-1} s^k {}_0D_x^{v-k-1} f(0) \tag{2.43}$$

2.6.2 THE LAPLACE TRANSFORM OF THE CAPUTO DEFINITION

$$L\left[{}_0D_x^v f(x)\right] = s^v F(s), v \leq 0 \tag{2.44}$$

$$L\left[{}_0D_x^v f(x)\right] = s^v F(s) - \sum_{k-0}^{n-1} s^{v-k-1} D^k f(0), n-1 < v \leq n \in \square \tag{2.45}$$

The above definition is proved as follows

$$L\left[{}_0D_x^v f(x)\right] = L\left[{}_0D_x^{v-n} D^n f(x)\right] = s^{v-n}\left[s^n F(s) - \sum_{k-1}^{n} s^{n-k} D^{k-1} f(0)\right] =$$

$$= s^v F(s) - \sum_{k-0}^{n-1} s^{v-k-1} D^k f(o) \tag{2.46}$$

2.6.3 LAPLACE TRANSFORM OF THE GRÜNWALD - LETNIKOFF DEFINITION

$$L\left[{}_0D_x^v f(x)\right] = s^v F(s) \tag{2.47}$$

2.7 CONCLUSION

The aim of this chapter is to give a brief discussion about the fractional calculus. Fractional calculus having the history of more than 300 years but serious efforts has been dedicated to its study for past three decades. The basic concept regarding the fractional calculus has been presented. From literature it is evident that fractional calculus opens the mind to entirely new branch of thought. It bridges the gap of integer calculus, till now no one completely understands the integer calculus. The goal of this chapter is to expose the basic concepts of fractional calculus available in the literature. In the line of these, here definitions given by Rimann-Liouvile, Caputato, Gurvald-Lentinkov has been presented. Out of which Rimann-Lioville definition is most famous one and it is widely accepted in the research community.