Chapter-3

Finite Deformation Formulation

3.1 Introduction

To analyze the behavior of a material it is required to develop a constitutive model or use existing model which is capable to calculate/predict the behavior of that material. It is very necessary to understand the basic concepts of continuum mechanics before developing a new model or using existing one. After full understanding about the fundamentals of continuum mechanics a numerical technique is required to solve the equations of equilibrium which are obtained from continuum mechanics approach. In the present work, finite element method is used to solve the equations of equilibrium. In this chapter we introduce the basics of continuum mechanics, which includes kinematics and balance laws, used in the present finite deformation formulation. After explaining the various terms and equations related to continuum mechanics, finite strain descriptions of the governing/field equations have been provided based on a Lagrangian formulation and using convected coordinates. In order to make a generalization of physical laws in continuum mechanics one needs to use the powerful tool of tensor calculus. There are two aspects of tensor calculus that are of practical and fundamental importance: tensor notation and tensor invariance. Tensor notation is of great practical importance, since it simplifies handling of complex equation systems and presents the equations in compact form to save space. The idea of tensor invariance is of both practical and fundamental importance. In continuum mechanics, any formulation of a physical law should be independent of coordinate system based on tensor invariance property. In general, scalar fields are referred to as tensor fields of rank (or order) zero whereas vector fields are called tensor fields of rank one. Tensors are physical entities with components that are the coefficients of a linear relationship between vectors. In most continuum mechanics problems the term tensor is used mainly for second order tensor such as stresses, strains etc.

3.2 Continuum Mechanics

The mathematical description of deformation and related stress is known as continuum mechanics. In continuum mechanics the fundamental assumptions inscribed in the name is that the materials are assumed to be continuous. Fourth order constitutive tensor is encountered frequently in continuum mechanics problems. The mathematical description of the deformation of a continuous body follows one of the two approaches: (i) the material description or (ii) the spatial description. The material description is also known as the Lagrangian description whereas the spatial description is known as the Eulerian description. In the material description, the motion of the body is referred to a reference configuration and this reference configuration is often selected to be the initial configuration. Hence, in the Lagrangian description, we express the current coordinates in terms of the reference coordinates. On the other hand, the motion is referred to the current configuration in the special description and is expressed with respect to the current position. In solid mechanics, the Eulerian description is less useful because the current configuration is unknown and therefore we follow the Lagrangian description, Reddy (2013), Bower (2009), Jog (2015).

3.2.1 Kinematics of Deformation

We know that kinematics refers to the results obtained concerning the nature of a continuum with no reference to the dynamics of the continuum. Accordingly, kinematics refers to those results which can be obtained solely from geometrical considerations, without having any reference to the force acting on the continuum. In this section we explain the motion of particle, deformation gradient, different strain measures, the velocity gradient, different stress measures etc.

3.2.1.1 Motion

Let's consider that the continuum originally occupies a configuration in which a particle x occupies position x_i referred to a reference frame of right handed rectangular Cartesian axes (x_1, x_2, x_3) at a fixed origin O with orthogonal basis vectors. After the application of some external stimuli, the continuum changes its geometric shape and thus assumes a new configuration called the current or deformed configuration. Particle x now occupies position $\overline{x_i}$ in the deformed configuration. Here, in the Lagrangian description, the current coordinates are expressed in terms of the reference coordinates and the variation of a typical variable $\emptyset$ over the body is described with respect to the material coordinates x_i and time :

$$\emptyset = \emptyset(\overline{x_\mathrm{i}}(x_\mathrm{i}), t) = \emptyset(x_\mathrm{i}, t) \tag{3.1}$$

For a fixed value of x_i , $\emptyset(x_\mathrm{i}, t)$ gives the value of $\emptyset$ at time t associated with the fixed material particle whose position in the reference configuration is x_i. Thus a change in time t implies that the same material particle occupying position x_i has a different $\emptyset$. Fig. 3.1 shows the deformation of a fixed material volume with time.

When $\emptyset$ is known in the material description,$\emptyset = \emptyset(x_\mathrm{i}, t)$, its total time derivative , D/Dt, is simply the partial derivative with respect to time because the material coordinates x_i do not change with time:

$$\frac{D}{Dt}[\emptyset(x_\mathrm{i}, t)] \equiv \frac{\partial}{\partial t}[\emptyset(x_\mathrm{i}, t)] = \frac{\partial \emptyset}{\partial t} \tag{3.2}$$

The displacement of the particle here is defined by

$$u_{\mathrm{i}} = \bar{x}_{\mathrm{i}} - x_{\mathrm{i}} \tag{3.3}$$

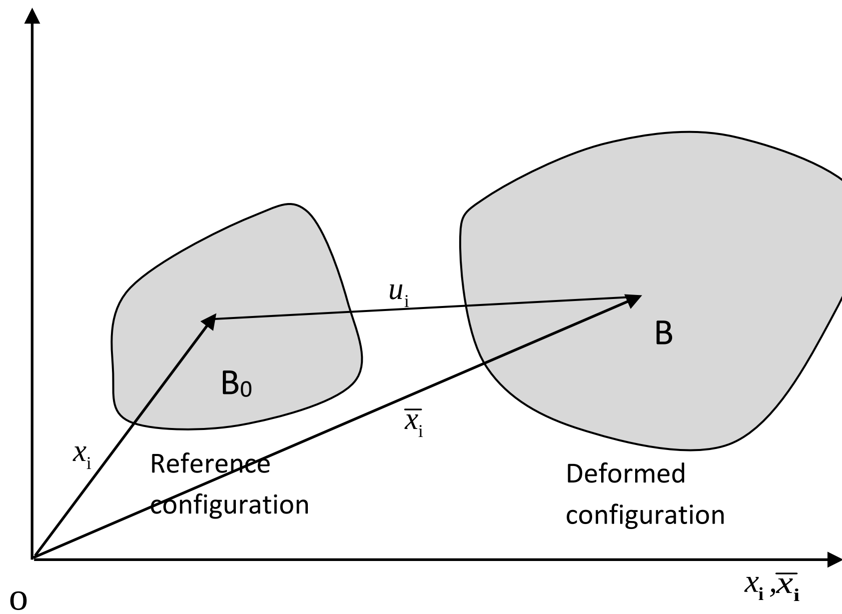

Figure 3.1: Reference and deformed configurations

3.2.1.2 The Deformation Gradient

In deformation analysis, one of the key quantities is the deformation gradient denoted by **F** or F_{ij}which provides the relationship between a material line dx_{i} before deformation and the line $d\bar{x}_{\mathrm{i}}$, consisting of the same material as dx_{i} after deformation. It is defined as follows:

$$d\bar{x}_{\mathrm{i}} = F_{\mathrm{ij}} dx_{\mathrm{j}} \tag{3.4}$$

Where

$$F_{\mathrm{ij}} = \frac{\partial \bar{x}_{\mathrm{i}}}{\partial x_{\mathrm{j}}} \tag{3.5}$$

More explicitly we have

$$F_{\mathrm{ij}} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial \bar{x}_1}{\partial x_1} & \frac{\partial \bar{x}_1}{\partial x_2} & \frac{\partial \bar{x}_1}{\partial x_3} \\ \frac{\partial \bar{x}_2}{\partial x_1} & \frac{\partial \bar{x}_2}{\partial x_2} & \frac{\partial \bar{x}_2}{\partial x_3} \\ \frac{\partial \bar{x}_3}{\partial x_1} & \frac{\partial \bar{x}_3}{\partial x_2} & \frac{\partial \bar{x}_3}{\partial x_3} \end{bmatrix} \tag{3.6}$$

The indices with bar refer to the current (spatial) coordinates and indices without bar refer to the reference (material) coordinates. The determinant of F_{ij} is called the Jacobian of the motion i. e.

$$J = \det(F_{\mathrm{ij}}) \tag{3.7}$$

We can also define, in a similar way, volume and surface elements in the reference and deformed configurations. If an element of volume of the body in the reference configuration is dV and the corresponding volume element in the current configuration is $d\bar{V}$, then we can write

$$d\bar{V} = \det(F_{\mathrm{ij}})\, dV = JdV \tag{3.8}$$

In the same way, if an element of area of the body in the reference configuration is dS and the corresponding area element in the current configuration is $d\bar{S}$, then we can write using Nanson's relation

$$\bar{n}_{\mathrm{i}} d\bar{S} = J n_{\mathrm{j}} F_{\mathrm{ji}}^{-1} dS \tag{3.9}$$

Where n_{j} is the positive unit normal to the surface in the reference configuration and $\bar{n}_{\mathrm{i}}$ being the outward unit normal to the surface in the current configuration.

3.2.1.3 Strain Measures

As deformation proceeds the particle gets strained and the relative position of the particle changes. There are various approaches to measure the strain of a particle. Due to the physical meaning and mathematical ease some strain measures are more common then others.

Lagrangian strain tensor is one of the most important classes of strain measurement. The difference in the squared lengths that occurs during the deformation of a body from the reference to the current configuration can be expressed relative to the original length as

$$(d\bar{l})^2 - (dl)^2 = 2dx_{\mathrm{i}}E_{\mathrm{ij}}dx_{\mathrm{j}} \tag{3.10}$$

Where E_{ij} is called the Lagrangian strain tensor, $d\bar{l}$ the length in current configuration and dl the length in reference configuration. In particular, in the rectangular Cartesian coordinate system, the components of E_{ij} are

$$E_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_{\mathrm{i}}}{\partial x_{\mathrm{j}}} + \frac{\partial u_{\mathrm{j}}}{\partial x_{\mathrm{i}}} + \frac{\partial u_{\mathrm{k}}}{\partial x_{\mathrm{i}}}\frac{\partial u_{\mathrm{k}}}{\partial x_{\mathrm{j}}}\right) \tag{3.11}$$

Another family of strain tensor is called Eulerian stain tensor. Here the change in the squared lengths that occurs during the deformation of a body from the initial to the current configuration can be expressed relative to the current length. We can write in this case

$$(d\bar{l})^2 - (dl)^2 = 2d\bar{x}_{\mathrm{i}}e_{\mathrm{ij}}d\bar{x}_{\mathrm{j}} \tag{3.12}$$

Where e_{ij} is called the Eulerian strain tensor. In particular, in the rectangular Cartesian coordinate system, the components of e_{ij} are

$$e_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_{\mathrm{i}}}{\partial \bar{x}_{\mathrm{j}}} + \frac{\partial u_{\mathrm{j}}}{\partial \bar{x}_{\mathrm{i}}} - \frac{\partial u_{\mathrm{k}}}{\partial \bar{x}_{\mathrm{i}}}\frac{\partial u_{\mathrm{k}}}{\partial \bar{x}_{\mathrm{j}}}\right) \tag{3.13}$$

3.2.1.4 The Velocity Gradient

We now consider velocity gradient which is the basic measure of deformation rate. This velocity gradient can be expressed in terms of the deformation gradient and its time derivative as

$$L_{\mathrm{ij}} = \dot{F}_{\mathrm{im}} \dot{F}_{\mathrm{mj}}^{-1} \quad (3.14)$$

Similar to the displacement gradient tensor we can also write the velocity gradient tensor L_{ij} as the sum of symmetric D_{ij} and skew symmetric W_{ij} tensors as

$$L_{\mathrm{ij}} = D_{\mathrm{ij}} + W_{\mathrm{ij}} \quad (3.15)$$

Where D_{ij} is called the rate of deformation tensor (or rate of stretch tensor) and this stretch rate quantifies the rate of stretching of a material fiber in the deformed solid. W_{ij} is called the vorticity tensor or spin tensor which can be shown to provide a measure of the average angular velocity of all material fibers passing through a material point.

In index notation we can write them as

$$D_{\mathrm{ij}} = \frac{1}{2}\left(L_{\mathrm{ij}} + L_{\mathrm{ji}}\right) \quad (3.16)$$

$$W_{\mathrm{ij}} = \frac{1}{2}\left(L_{\mathrm{ij}} - L_{\mathrm{ji}}\right) \quad (3.17)$$

3.2.1.5 Stress Measures

When the deformations are small there is no difference between deformed configuration and reference configuration and using Cauchy stress we can describe the action of surface forces. However, in case of large deformations, one has to refer to some reference configuration. The

Cauchy stress, σ_{ij}, the Kirchhoff stress, τ_{ij} , and the first Piola - Kirchhoff stress, t_{ij} are the three important stress measures. Out of these three stress measures, the Cauchy stress tensor is the most natural and physical measure of the state of stress at a point in the deformed configuration. It is most commonly used in spatial descriptions of problems in fluid mechanics. In order to use the Lagrangian description, which is common in solid mechanics, the equations of motion or equilibrium of a material body that are derived in the deformed configuration must be expressed in terms of the known reference configuration. In doing so we use the other measures of stress like Kirchhoff stress, first Piola – Kirchhoff etc. These stress measures come into view in a natural way as we transform volumes and areas from the deformed configuration to the reference configuration.

The Cauchy stress tensor completely characterizes the internal forces acting in a deformed solid. It is also known as true stress in elementary strength of material courses and defined as

$$t_{\mathrm{i}} = \sigma_{\mathrm{ij}} n_{\mathrm{j}} \qquad (3.18)$$

Where, t_{i} is known as surface traction and n_{j} is its associated normal vector. The Cauchy stress is having two parts one is hydrostatic part and second one is deviatoric term

The second stress measure is the Kirchhoff stress tensor, denoted by τ_{ij} and defined as

$$\tau_{\mathrm{ij}} = J\sigma_{\mathrm{ij}} \qquad (3.19)$$

Kirchhoff stress can also be split into two parts as true (Cauchy) stress, one is deviatoric and other is hydrostatic (spherical) part. In many equations, the Cauchy stress appears together with the Jacobian and the use of τ_{ij} simplifies formula.

Here mentioned last stress measure is the first Piola - Kirchhoff stress tensor which is also known as nominal stress and denoted by t_{ij}. This can be expressed as

$$t_{\mathrm{ji}} = F_{\mathrm{jk}}^{-1} J \sigma_{\mathrm{ik}} = F_{\mathrm{jk}}^{-1} \tau_{\mathrm{ik}} \qquad (3.20)$$

In general, the first Piola - Kirchhoff stress tensor is unsymmetric even when the Cauchy stress tensor is symmetric.

3.2.1.6 The Rate Viewpoint

The equations describing finite deformation of elastic-viscoplastic solids may be derived in rate form. That is, attention is focused not upon field quantities such as stress and strain but rather upon their rates of change with respect to time.

In analysis of infinitesimal deformation stress and strain tensors as well as all governing equations are referred to a single configuration of the body. Either deformed or undeformed states may be employed as they are by assumption indistinguishable from one another. Thus time derivatives of field quantities reflect only changes in component magnitudes with respect to an invariant frame of reference. When the deformation is regarded as finite, however, deformed and undeformed configurations must be distinguished. Time derivatives of the field quantities such as stress and strain must reflect changes in the fundamental reference frame provided by the deforming configuration of the body.

The development of the constitutive and equilibrium equations is predicated upon the character of certain tensorial measures of stress and strain and their time rates of change. Subsequently the field equations are derived, boundary and initial conditions developed and full velocity problem is assembled.

3.2.2 Fundamental/ Balance Laws

The fundamental laws are the same for all materials in contrast to constitutive equations that are different for each material. There are three basic fundamental laws which include the conservation of mass, conservation of linear momentum, conservation of angular momentum. Each of the balance laws is a general statement and can be used to calculate the specific response of a particular material body. The quantities and definitions mentioned above are known to be the essential mathematical relations of deformation.

3.2.2.1 Conservation of Mass

According to the law of conservation of mass, total mass of any part of a body does not change in any motion. In Lagrangian description of motion, the mathematical form of this law is

$$\int_V \rho dV = \int_{\bar{V}} \bar{\rho}\, d\bar{V} \tag{3.21}$$

Where ρ and $\bar{\rho}$ are the densities in the reference and current configurations, respectively.

3.2.2.2 Conservation of Linear Momentum

The principle of balance of linear momentum is also known as Newton's second law of motion. This principle, when applied to a set of particles (or rigid bodies) can be stated as follows: The time rate of change of linear momentum of a collection of particles equals the net force exerted on the collection. Mathematically, it can be written as

$$\oint_{\bar{S}} \bar{T}_{\mathrm{i}}\, d\bar{S} + \int_{\bar{V}} \bar{b}_{\mathrm{i}}\, d\bar{V} = \int_{\bar{V}} \bar{\rho} \frac{\partial^2 u_{\mathrm{i}}}{\partial t^2}\, d\bar{V} \tag{3.22}$$

Where, $d\bar{S}$ is an element of surface area in the current configuration and $\bar{T}_{\mathrm{i}}$ is the force per unit current area and

$$\bar{T}_{\mathrm{i}} = \sigma_{\mathrm{ij}}\bar{n}_{\mathrm{j}} \tag{3.23}$$

Where, $\bar{n}_{\mathrm{j}}$ is the normal to $d\bar{S}$ and σ_{ij} is the Cauchy stress.

3.2.2.3 Conservation of Angular Momentum

According to the law of balance of angular momentum the resultant moment applied on a body must equal the rate of change of angular momentum of that material body. Assuming no distributed couple, it implies

$$\sigma_{\mathrm{ij}} = \sigma_{\mathrm{ji}} \tag{3.24}$$

3.3 Quasi-static Deformation Histories

Now, using equation (3.9) in equation (3.22) balance of liner momentum (with no body forces) can be written as

$$\oint_{\mathrm{S}} J\sigma_{\mathrm{ik}}\, n_{\mathrm{j}} F_{\mathrm{jk}}^{-1} dS = \int_{\mathrm{V}} \rho\, \frac{\partial^2 u_{\mathrm{i}}}{\partial t^2}\, dV \tag{3.25}$$

The use of equation (3.20) in (3.25) yields

$$\oint_{\mathrm{S}} T_{\mathrm{i}}\, dS = \oint_{\mathrm{S}} t_{\mathrm{ji}}\, n_{\mathrm{j}} dS = \int_{\mathrm{V}} \rho\, \frac{\partial^2 u_{\mathrm{i}}}{\partial t^2}\, dV \tag{3.26}$$

The relation (3.26) is simply the expression of equilibrium written in the reference configuration. Using Green’s theorem, equation (3.25) can be written in Cartesian coordinates as

$$\int_V \frac{\partial t_{ij}}{\partial x_j} dV = \int_V \rho \frac{\partial^2 u_i}{\partial t^2} dV \tag{3.27}$$

So that

$$\frac{\partial t_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{3.28}$$

In quasi-static deformation load is applied very slowly and the structure deform also very slowly. Therefore due to very low strain rate, inertia force is very small and can be neglected. So if we consider a loading where inertial effects are negligible so that equation (3.28) becomes

$$\frac{\partial t_{ij}}{\partial x_j} = 0 \tag{3.29}$$

We can start from equation (3.29) and go through our virtual work procedure to obtain

$$\int_V t_{ji}\, \delta u_{i,j} dV = \oint_S T_i\, \delta u_i dS \tag{3.30}$$

Now we can expand equation (3.30) about the current state to obtain

$$\int_V \dot{t}_{ji}\, \delta \dot{u}_{i,j} dV = \oint_S \dot{T}_i\, \delta \dot{u}_i dS - \frac{1}{dt}\left[\int_V t_{ji}\, \delta u_{i,j} dV - \oint_S T_i\, \delta u_i dS\right] \tag{3.31}$$

We will consider circumstances where we can write the constitutive relation in the form

$$\dot{t}_{ij} = K_{ijkl}\dot{F}_{kl} = K_{ijkl}\dot{u}_{k,l} \tag{3.32}$$

A key step in the generalization to finite deformations of what we have done so far is the relation between K_{ijkl} and the moduli that “naturally” arise in finite deformation constitutive formulations. In particular, a range of constitutive relations of interest can be written as

$$\hat{\tau}_{\mathrm{ij}} = C_{\mathrm{ijkl}} D_{\mathrm{kl}} \tag{3.33}$$

Where $\hat{\tau}_{\mathrm{ij}}$ is the Jaumann derivative of Kirchhoff stress and it can also be expressed by

$$\hat{\tau}_{\mathrm{ij}} = \frac{D\tau_{\mathrm{ij}}}{Dt} - W_{\mathrm{im}}\tau_{\mathrm{mj}} - W_{\mathrm{jm}}\tau_{\mathrm{mi}} \tag{3.34}$$

Thus the problem is to determine the relation between C_{ijkl} and K_{ijkl} .

3.3.1 Conventional Formulation

Differentiating equation (3.20) with respect to time gives

$$\dot{t}_{\mathrm{ji}} = \dot{F}_{\mathrm{jk}}^{-1}\tau_{\mathrm{ki}} + F_{\mathrm{jk}}^{-1}\dot{\tau}_{\mathrm{ki}} \tag{3.35}$$

But

$$\dot{F}_{\mathrm{jk}}^{-1} = -F_{\mathrm{jm}}^{-1}\dot{F}_{\mathrm{mn}}F_{\mathrm{nk}}^{-1} \tag{3.36}$$

Hence,

$$\dot{t}_{\mathrm{ji}} = -F_{\mathrm{jm}}^{-1}\dot{F}_{\mathrm{mn}}F_{\mathrm{nk}}^{-1}\tau_{\mathrm{ki}} + F_{\mathrm{jk}}^{-1}\left(C_{\mathrm{kimn}}\dot{F}_{\mathrm{mp}}F_{\mathrm{pn}}^{-1} + W_{\mathrm{km}}\tau_{\mathrm{mi}} + W_{\mathrm{im}}\tau_{\mathrm{mk}}\right) \tag{3.37}$$

Upon using equation (3.17)

$$\dot{t}_{\mathrm{ji}} = K_{\mathrm{ijnm}}\dot{F}_{\mathrm{mn}} \tag{3.38}$$

Where

$$K_{\mathrm{ijnm}} = F_{\mathrm{jk}}^{-1}C_{\mathrm{kimq}}F_{\mathrm{nq}}^{-1} + \frac{1}{2}F_{\mathrm{jm}}^{-1}F_{\mathrm{np}}^{-1}\tau_{\mathrm{pi}} - \frac{1}{2}F_{\mathrm{jk}}^{-1}F_{\mathrm{nk}}^{-1}\tau_{\mathrm{mi}}$$

$$+\frac{1}{2}F_{\mathrm{jk}}^{-1}F_{\mathrm{np}}^{-1}\tau_{\mathrm{pm}} - \frac{1}{2}F_{\mathrm{jk}}^{-1}F_{\mathrm{ni}}^{-1}\tau_{\mathrm{mk}} - F_{\mathrm{jm}}^{-1}F_{\mathrm{nk}}^{-1}\tau_{\mathrm{ki}} \tag{3.39}$$

In an updated Lagrangian formulation, the current configuration is taken as reference so that $F_{ij} = \delta_{ij}$ and equation (3.39) simplifies to

$$K_{ijnm} = C_{jimn} + \frac{1}{2}\delta_{jm}\tau_{ni} - \frac{1}{2}\delta_{jn}\tau_{mi} + \frac{1}{2}\delta_{jk}\tau_{nm} - \frac{1}{2}\delta_{ni}\tau_{mj} - \delta_{jm}\tau_{ni} \tag{3.40}$$

3.3.2 Convected Coordinate Formulation

A computationally convenient way is to use a convected coordinate representation of the governing equations. In this formulation, a convected coordinate net is introduced which can be visualized as being inscribed on the body in the reference configuration and deforming with the material. Since the convected coordinate net can undergo a general transformation it will not necessarily remain orthogonal and therefore general tensor notation needs to be used.

Assume that the reference coordinate is Cartesian and we denote its base vectors by $\mathbf{g}_i$ so that

$$d\mathbf{x} = dy^{i}\mathbf{g}_{i} \tag{3.41}$$

The convected coordinates y^{i} serve as particle labels and the displacement vector and deformation gradient are considered as functions of the convected coordinates and of time.

In the current configuration

$$d\bar{\mathbf{x}} = dy^{i}\bar{\mathbf{g}}_{i} \tag{3.42}$$

So that

$$\bar{\mathbf{g}}_{i} = \frac{\partial \bar{\mathrm{x}}}{\partial y^{i}} = \mathbf{F}.\mathbf{g}_{i} = \left(\delta_{i}^{j} + u_{,i}^{j}\right)\mathbf{g}_{j} \tag{3.43}$$

Where, assuming a Cartesian reference frame, $u_{,i}^{j}$ denotes the partial derivatives of the displacements with respect to y^{i}.

Metric tensors are introduced via

$$g_{\mathrm{ij}} = \mathbf{g}_{\mathrm{i}}.\mathbf{g}_{\mathrm{j}} \tag{3.44}$$

$$\bar{g}_{\mathrm{ij}} = \bar{\mathbf{g}}_{\mathrm{i}}.\bar{\mathbf{g}}_{\mathrm{j}} \tag{3.45}$$

with inverses $\mathbf{g}^{\mathrm{ij}}$ and $\bar{\mathbf{g}}^{\mathrm{ij}}$respectively.

The Lagrangian strain tensor is

$$E_{\mathrm{ij}} = \frac{1}{2}\left(\bar{g}_{\mathrm{ij}} - g_{\mathrm{ij}}\right) \tag{3.46}$$

Now, in terms of derivatives of displacement components

$$E_{\mathrm{ij}} = \frac{1}{2}(u_{\mathrm{i,j}} + u_{\mathrm{j,i}} + u_{,\mathrm{i}}^{\mathrm{k}} u_{\mathrm{k,j}}) \tag{3.47}$$

The Lagrangian strain rate is

$$\dot{E}_{\mathrm{ij}} = \frac{1}{2}(\dot{u}_{\mathrm{i,j}} + \dot{u}_{\mathrm{j,i}} + \dot{u}_{,\mathrm{i}}^{\mathrm{k}} u_{\mathrm{k,j}} + u_{,\mathrm{i}}^{\mathrm{k}} \dot{u}_{\mathrm{k,j}}) \tag{3.48}$$

Now we need a relation between $\dot{t}^{\mathrm{ij}}$ and $\hat{\tau}^{\mathrm{ij}}$. The equation (3.20) in tensor form is

$$\mathbf{t} = \mathbf{F}^{-1}.\boldsymbol{\tau} \tag{3.49}$$

and to get the component form we use

$$\mathbf{g}^{\mathrm{i}}.\mathbf{t}.\mathbf{g}^{\mathrm{j}} = \mathbf{g}^{\mathrm{i}}.\mathbf{F}^{-1}.\boldsymbol{\tau}.\mathbf{g}^{\mathrm{j}} = \bar{\mathbf{g}}_{\mathrm{i}}.\boldsymbol{\tau}.\mathbf{g}^{j} \tag{3.50}$$

to obtain

$$t^{\mathrm{ij}} = \tau^{\mathrm{ik}} F_{,\mathrm{k}}^{\mathrm{j}} \tag{3.51}$$

Now differentiating equation (3.51) with respect to time gives

$$\dot{t}^{\mathrm{ij}} = \dot{\tau}^{\mathrm{ik}} F_{,\mathrm{k}}^{\mathrm{j}} + \tau^{\mathrm{ik}} \dot{F}_{,\mathrm{k}}^{j} \tag{3.52}$$

Next, we need the relation between the Jaumann and convected stress rates which is

$$\hat{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}}^{\mathrm{c}} + \mathbf{D}.\boldsymbol{\tau} + \boldsymbol{\tau}.\mathbf{D} \tag{3.53}$$

We also make use of the fact that

$$\dot{E}_{\mathrm{ij}} = \mathbf{g}_{\mathrm{i}}.\mathbf{D}.\mathbf{g}_{\mathrm{j}} \tag{3.54}$$

The component form of equation (3.33) is

$$\hat{\tau}^{ij} = C^{\mathrm{ijkl}}\dot{E}_{\mathrm{kl}} \tag{3.55}$$

Define

$$L^{\mathrm{ijkl}} = C^{\mathrm{ijkl}} - \frac{1}{2}\left(\bar{g}^{\mathrm{ik}}\tau^{\mathrm{jl}} + \bar{g}^{\mathrm{jk}}\tau^{\mathrm{il}} + \bar{g}^{\mathrm{il}}\tau^{\mathrm{jk}} + \bar{g}^{\mathrm{jl}}\tau^{\mathrm{ik}}\right) \qquad /(3.56)$$

then we obtain

$$K^{\mathrm{ijkl}} = L^{\mathrm{piqk}}F^{l}_{,\mathrm{q}}F^{j}_{,\mathrm{p}} + \tau^{\mathrm{ik}}g^{\mathrm{jl}} \tag{3.57}$$

3.4 Concluding Remarks

In this chapter the essential continuum mechanics relations and the quasi-static convected coordinate formulation have been presented, which is useful in the derivation of the constitutive equation in chapter 4 and later. The purpose of the fundamental relations of solid mechanics is also important to describe the elastic and plastic material behavior in the present context. Further these key relations are also referred in different places as and when required in other chapters. Apart from these, two other important and fundamental topics necessary for the derivation presented in the next chapter are the virtual work principle and the divergence theorem. The virtual work principle may be viewed as an alternative statement of the equilibrium equation in mechanics and this principle has been explained at the appropriate places. The divergence theorem converts a volume integral into a surface integral or vice versa. This is also valid in two-dimensions where the volume and surface integrals are replaced by surface and line integrals, respectively.