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Dr. Debashis Khan	**Prof. S.K.Panda**
(Supervisor)	**(Co-Supervisor)**
Associate Professor	**Professor**
Dept. of Mechanical Engineering	**Dept. of Mechanical Engineering**
Indian Institute of Technology (BHU)	**Indian Institute of Technology (BHU)**
Varanasi-221005	**Varanasi-221005**

DECLARATION BY THE CANDIDATE

I, **Shushant Singh**, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervisions of **Dr. Debashis Khan** and **Prof. S.K. Panda** from July 2013 to July 2018, at the Department of Mechanical Engineering, Indian Institute of Technology (BHU) Varanasi. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not wilfully copied any other's work, paragraphs, text, data, results, etc., reported in journals, books, magazines, reports, dissertations, theses, etc., or available at websites and have not included them in this thesis and have not cited as my own work.

Date:

Place: IIT (BHU) Varanasi **(Shushant Singh)**

CERTIFICATE BY THE SUPERVISORS

It is certified that the above statement made by the student is correct to the best of our knowledge.

Dr. Debashis Khan
(Supervisor)
Associate Professor

Prof. S.K. Panda
(Co-Supervisor)
Professor

Prof. S.K. Sinha
Head of Department
Dept. of Mechanical Engineering
Indian Institute of Technology (BHU)
Varanasi-221005

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DEDICATION

This thesis is dedicated to

Lord ***Sri Krishna***

ॐ पूर्णमदः पूर्णमिदं पूर्णात्पुर्णमुदच्यते।
पूर्णस्य पूर्णमादाय पूर्णमेवावशिष्यते ||

Sri Isopanishad

The Personality of Godhead is perfect and complete. And because He is completely perfect, all emanations from Him, such as this phenomenal world, are perfectly equipped as a complete whole. Whatever is produced of the complete whole is also complete by itself. And because He is the Complete Whole, even though so many complete units emanate from Him, He remains the complete balance.

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ABSTRACT

Considering the potential applications of relatively new materials like toughened structural polymers, metallic foams, plastics, transformation toughened ceramics, vertically aligned carbon nanotubes (VACNTs) etc and their limited exploration till now, it appears that there is a need to investigate more about the behavior of such materials under a wide range of loadings. Even though the effect of plastic dilatancy is neglected in classical plasticity theory, the above materials exhibit plastic volume changes and/or pressure-sensitive flow strength. In a recent study, it has been observed that the deformation of the entangled arrays of carbon nanotubes or VACNTs follow elastic-viscoplastic constitutive relation which incorporates plastic compressibility, plastic non-normality and a hardening-softening-hardening type hardness function. These VACNTs have prospective uses in a variety of applications like viscoelastic energy absorption, compliant thermal interfaces, biomimetic dry adhesives etc and hence it is useful to develop a predictive framework for the mechanical behavior of VACNTs under a wide range of loadings. In this thesis work, finite element finite deformation quasistatic mode I plane strain small scale yielding analysis of crack tip blunting and near crack tip fields was carried out for plastically compressible solids exhibiting a variety of uniaxial stress – strain responses. In particular solids with hardening-softening-hardening responses as can occur for foams and VACNTs have been considered. The novelty of this model includes unique characteristics as mentioned earlier like the hardening-softening-hardening material response, strain rate-dependence, and plastically compressible solids with plastic non-normality.

As for localization studies it needs a finite strain description, using FORTRAN a finite element finite deformation code has been developed in this work for the simulation purpose. A convected coordinate Lagrangian formulation of the field equations was used. Quasistatic deformation conditions have been assumed and the equilibrium equations were expressed through the virtual work principle. The plane strain calculations were carried out for a semicircular region with a blunt notch. Quadrilateral elements each comprised of four crossed constant strain triangular elements have been used for mesh generation. Such elements with a proper aspect ratio and orientation are extensively used to replicate localized deformation pattern at finite strains. The initial part of the investigation refers to crack tip blunting and field quantities analyses under monotonic load while in the next part studies were conducted for fatigue loading to find the key results of crack tip blunting and fields. Even though most of the results presented are for plastic normality condition, however for comparison purpose some results have also been illustrated for constitutive relations exhibit plastic non-normality. While to simulate fatigue crack growth by means of finite elements, different techniques have been proposed over the years, however, in this work the crack growth modelling strategy employed was crack tip blunting/ resharpening mechanism where it is assumed that the crack tip blunts during the maximum load and resharpening of the crack tip takes place under minimum load. The simulations attempt to explain some of the salient features, like crack tip opening displacement, crack tip advancement, plastic zone shape and size, equivalent plastic strain distribution, equivalent stress, and distribution of hydrostatic stress at near crack region. The influences of plastic compressibility, material softening, cyclic stress intensity factor range, load ratio, number of fatigue load cycles on the near tip deformation and stress-strain fields were studied.

Numerical results obtained from the quasistatic mode I plane strain analysis demonstrate that plastic compressibility is found to give an increased crack opening displacement for a given value of the applied loading. The plastic zone shape and size are found to depend on the plastic compressibility. Even though, material softening does not have a significant effect on the plastic zone size and shape, however, the near crack tip stress and deformation fields depend sensitively on whether or not material softening occurs. The combination of softening or softening-hardening material response and plastic compressibility leads to major deviation in the near crack tip stress and deformation fields from those that prevail for a hardening material. Plastic compressibility coupled with a softening or softening hardening material response leads to localized deformation in front of the initial crack tip, which in turn affects the shape of the blunted crack tip. The present numerical calculations show that the convergence of the cyclic trajectories of CTOD to stable self similar loops and plastic crack growth depend significantly on cyclic stress intensity factor range, load ratio, number of fatigue load cycles.

Keywords: Finite Deformation, Mode I crack, Plasticity, Compressible Solids, Material Softening, Monotonic Loading, Fatigue Loading, Finite Element Method.

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LIST OF ABBREVIATIONS AND SYMBOLS

FEM	Finite Element Method
BEM	Boundary Element Method
LEFM	Elastic Fracture Mechanic
EPFM	Elastic Plastic Fracture Mechanics
CTOD	Crack Tip Opening Displacement
VACNT	Vertically Aligned Carbon Nano Tube
PFM	Probabilistic fracture mechanics
SSY	Small Scale Yielding
HRR	Hutchinson, Rice and Rosengren
X-FEM	Extended Finite Element Method
CZM	Cohesive Zone Model
FDM	Finite Difference Method
σ_{ij}	Stress Tensor
$\sigma_{\mathrm{xx}}, \sigma_{\mathrm{yy}}, \sigma_{\mathrm{zz}}$	Stress Components in x, y and z direction
ν	Poisson ratio
μ	Shear modulus
K	Stress Intensity Factor
r_{t}	Radius
$u_{\mathrm{xx}}, u_{\mathrm{yy}}, u_{\mathrm{zz}}$	Displacement in x, y and z direction
K_{IC}	Mode I Critical Stress Intensity Factor
U	Potential Energy
V	Volume of the Body
W	Strain Energy Density
S_{t}	Part of the body Subjected to Traction
E	Young’s Modulus
G	Energy Release Rate

Γ	Contour Enclosing the Crack Tip
n	Unit Outward Normal on Γ
$\bar{T}$	Surface Traction
T_{i}	Traction Vector
r_{y}	Irwin Plastic Zone Correction
d	Crack Tip Opening Displacement
a	Half Crack Length
J_{app}	Applied *J* integral
F_{ij}	Deformation Gradient
e_{ij}	Eulerian Strain Tensor
D_{ij}	Deformation Tensor
L_{ij}	Velocity Gradient
W_{ij}	Velocity tensor or spin tensor
ρ	Density
$\hat{\tau}_{\mathrm{ij}}$	Jaumann rate of Kirchhoff stress
$\dot{\varepsilon}_{\mathrm{p}}$	Plastic Strain Rate
σ_{e}	Effective Equivalent Stress
δ_{ij}	Kronecker Delta
p_{ij}	Deviatoric Kirchhoff Plastic Stress Tensor
L_{ijkl}	Tensor of elastic Moduli
$\dot{\varepsilon}_{0}$	Reference Strain Rate
m	Rate Hardening Exponent
σ_{0}	Reference Stress
α_{p}	Plastic Compressibility

Symbols not listed here, are defined as they appear in text.