SOME INVESTIGATIONS ON NON-LINEAR WAVE PROPAGATION IN GASEOUS MEDIA

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By

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Chapter 6

Solution of Riemann problem for ideal polytropic dusty gas

6.1 Introduction

In recent years the solution of Riemann problem is widely used for the theoretical and numerical study of the system of conservation laws in ideal gasdynamics, non-ideal gasdynamics, magnetogasdynamics, reacting flows, shallow water theory etc. Riemann problem is an initial value problem for the one dimensional Euler equations supplemented by discontinuous initial data and its solution constitutes the basic building block for the construction of a solution to the general initial value problem (Glimm (1965)). The solution of the Riemann problem is composed of three waves, with always a contact discontinuity as the middle one while the other two are indifferently rarefaction or shock wave. If both external waves are rarefaction then it might occur to the formation of a vacuum region between two parts of the gas receding from each other. The Riemann problem for the ideal gas does not admit a solution in closed form. This has led several authors such as Godunov (1959, 1976), Chorin (1976), Smoller (1969), Gottlieb and Groth (1988), Quartapelle (2003) and Toro (1997) to develop iterative solution schemes to determine the different waves issuing from an initial discontinuity in the flow field variables. Two methods were first proposed by Godunov (1959, 1976), one based on a fixed point scheme and the other based on a higher order Newton's iterative scheme, with a tangent parabola instead of a straight line. An experimental and numerical investigation of shock wave attenuation was studied by

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Berger et al. (2015) in which they have focused on the dependency of the shock wave attenuation on a wide span of barrier geometries. In the case of Euler equations, the Riemann problem corresponds to the shock tube problem and for a detailed discussion of this, the reader is suggested to the book by Courant and Friedrich (1948). Lax (1963) solved the Riemann problem for the case when the initial data consisting of constant initial states U_l^* and U_r^* are such that $\|U_l^* - U_r^*\|$ is sufficiently small; here U^* is the vector of conserved variable with U_l^* to the left of $x = 0$ and U_r^* to the right of $x = 0$ separated by a discontinuity at $x = 0$. Exact solution of the Riemann problem has been studied by Godunov (1959), Chorin (1976) and Giacomazzo and Rezzolla (2000). The special solution of Euler equations in which one of the Riemann invariants remains constant throughout the flow field is called a simple wave. In simple wave solution, wave breaks and the solution has to be complemented by the introduction of shock wave. When the shock strength is small (i.e. weak shock) and even moderate, jumps in entropy and the Riemann invariants are surprisingly small, see Whitham (1974). Exact solution of weak shock waves in gasdynamics was studied by Singh et al. (2011) for planar and nonplanar flows. The Riemann problem and elementary wave interactions of isentropic system in magnetogasdynamics were studied by Raja Shekhar and Sharma (2010, 2012) and Liu et al. (2013). Mentrelli et al. (2008) studied thoroughly the problem of interaction of waves originated from Riemann problem in an Euler fluid. The solution of Riemann problem in ideal and non-ideal isentropic magnetogasdynamics was studied by Sahadeb Kuila et al. see (2014, 2016). Further solution of Riemann problem in magnetogasdynamics was studied by Singh and Singh (2014) in which they solved the Riemann problem analytically without any restriction on initial states.

Dusty gas is a mixture of gas and small solid particles, where the volume of solid particles should not be more than 5% of the total volume of the gas. The study of Riemann problem in dusty gas is of great interest due to its wide application in gas dynamics (G. Rudinger (1980)). Miura and Glass (1983) studied theoretically the problem of propagation of shock wave through a dusty-air layer. Pai et al. (1980) studied the similarity solution of strong shock wave propagation in a mixture of a gas and dust particles. Further Chadha et al. (2014) studied the self similar solutions and converging shocks in a non-ideal gas with dust particles using Lie group transformation. Gupta et al. (2016) have used a direct approach to analyze the solution of the Riemann problem for a dusty gas flow. The main motivation of the present chapter is to study the Riemann problem for the unsteady one-dimensional motion of ideal polytropic gas with dust particles without any restriction on the initial states. Here we have derived the explicit expression for shock waves, rarefaction waves and contact discontinuities in terms of fluid flow parameters (density, velocity, and pressure). The existence and uniqueness of the solution of Riemann problem in a dusty gas is discussed. And also those cases are discussed which gives information about the existence of shock waves or simple waves for a 1-family and for a 3-family of curves in a dusty gas. Also the effect of dust particles on the density and velocity profiles, for the case of shock wave and rarefaction wave, is also discussed.

6.2 Governing equations

Here we assume that the particles are spherical, of uniform size, incompressible and occupy less than 5% of the total volume, their specific heat is constant and the temperature is uniform within each particle, collisions between particles of different sizes are not considered. It is also assumed that the particles are uniformly distributed over the cross section of the duct, the size and distance between particles are small as compared with the cross sectional dimensions of the duct. The boundary layer effects and heat transfer with the duct walls are not considered, the particles are permanent i.e. no mass transfer takes place between the two phases. With the above assumptions, the governing equations describing a one dimensional planar flow of an ideal polytropic gas with dust particles may be written in the following form (Rudinger (1980), Miura and Glass (1983), Chadha (2014), Pai (1977))

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{6.2.1}$$

$$\rho(u_t + uu_x) + p_x = 0, \tag{6.2.2}$$

$$E_t + uE_x - \frac{p}{\rho^2}(\rho_t + u\rho_x) = 0, \tag{6.2.3}$$

where u is the velocity, ρ is the density, p is the pressure, t is the time and x is the spatial coordinate. The subscripts denote partial differentiation unless stated otherwise. The internal energy E per unit mass of the mixture is given by

$$E = \frac{(1-Z)p}{(\Gamma-1)\rho}. \tag{6.2.4}$$

Here, $Z = V_{sp}/V_g$ is the volume fraction and $k_p = m_{sp}/m_g$ is the mass fraction of the solid particles in the mixture where m_{sp} and V_{sp} are the total mass and volumetric extension of the solid particles respectively, V_g and m_g are the total volume and total mass of the mixture respectively, the Grüneisen coefficient $\Gamma = \gamma(1+\lambda\omega)/(1+\lambda\omega\gamma)$,with $\lambda = k_p/(1-k_p)$, $\omega = c_{sp}/c_p$, $\gamma = c_p/c_v$, where c_{sp} is the specific heat of the solid particles, c_p the specific heat of the gas at constant pressure and c_v the specific heat of the gas at constant volume. The entities Z and k_p are related via the expression $Z = \theta\rho$, where $\theta = k_p/\rho_{sp}$ with ρ_{sp} is the specific density of the solid particles.

For a polytropic dusty gas, the equation of state is

$$p = ke^{S/c_v}\left(\frac{\rho}{(1-\theta\rho)}\right)^{\Gamma}, \tag{6.2.5}$$

where k, c_v and γ are positive constants.

Using equation (6.2.4) in equation (6.2.3) we get

$$p_t + up_x + \frac{\Gamma p}{(1-\theta\rho)\rho}\rho u_x = 0. \tag{6.2.6}$$

Thus equations (6.2.1), (6.2.2) and (6.2.6) can be written as

$$\rho_t + u\rho_x + \rho u_x = 0 \;, \tag{6.2.7}$$

$$u_t + uu_x + \frac{1}{\rho}p_x = 0, \tag{6.2.8}$$

$$p_t + up_x + \frac{\Gamma p}{(1-\theta\rho)\rho}\rho u_x = 0 \;. \tag{6.2.9}$$

Equations (6.2.7)-(6.2.9) can be written in matrix form as

$$U_t + AU_x = 0, \tag{6.2.10}$$

where $U = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}$ and $A = \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho C^2 & u \end{bmatrix}$, with $C^2 = \dfrac{\Gamma p}{(1-\theta\rho)\rho}$.

The eigenvalues of the matrix A are

$$\lambda_1 = u - C,\ \lambda_2 = u \quad \text{and}\ \lambda_3 = u + C, \tag{6.2.11}$$

where C is the velocity of sound and is given as $C = \left(\Gamma p/\left((1-\theta\rho)\rho\right)\right)^{1/2}$ and the corresponding eigenvectors are

$$K^1 = \begin{bmatrix} -\rho/C \\ 1 \\ -\rho C \end{bmatrix},\quad K^2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad K^3 = \begin{bmatrix} \rho/C \\ 1 \\ \rho C \end{bmatrix}. \tag{6.2.12}$$

Since all the eigenvalues of the matrix A are real and distinct, hence the system of equations (6.2.10) is strictly hyperbolic.

6.3 Riemann problem and the generalized Riemann invariants

Equation (6.2.10) can be written in conserved form as

$$\frac{\partial U^*}{\partial t}+\frac{\partial F(U^*)}{\partial x}=0, \tag{6.3.1}$$

where $U^*=\left(\rho,\rho u,\rho\left(u^2/2+E\right)\right)^{tr}$, $F(U^*)=\left(\rho u, p+\rho u^2, \rho u\left(u^2/2+E\right)+pu\right)^{tr}$

with $E=(1-Z)p/\left((\Gamma-1)\rho\right)$.

Since, in a hyperbolic system each characteristic field is either linearly degenerate or genuinely non-linear according as $\nabla\lambda_i K^i=0$ and $\nabla\lambda_i K^i\neq 0$ respectively. Clearly, from equation (6.2.10) first and third characteristic fields are genuinely non-linear and hence will be either a shock or rarefaction, while second characteristic field is linearly degenerate and hence will be contact discontinuity. Here we consider only the case when wave is associated with the characteristic field which is either a shock or rarefaction.

The Riemann problem for the system of equation (6.3.1) is an initial-value problem with data of the form

$$U^*(x,0)=U_0^*(x)=\begin{cases}U_l^* & ,x<0\\ U_r^* & ,x>0\end{cases}, \tag{6.3.2}$$

where U_l^* and U_r^* are left and right constant state as defined in equation (6.3.1) and $x=0$ is point of discontinuity. The exact solution of the Riemann problem (6.3.1) and (6.3.2) has three waves, which is associated with the eigenvalues $\lambda_1=u-C$, $\lambda_2=u$ and $\lambda_3=u+C$ as shown in fig. (1).We shall solve this problem for the class of functions consisting of constant states, separated by either shock or rarefaction waves.

Also, the pressure (p) is a function of density (ρ) and entropy (S) i.e.,

$$p = p(\rho, S)\,. \tag{6.3.3}$$

Now for the system of equation (6.3.1) the pair of Riemann invariants $\left(\Pi_1^j, \Pi_2^j\right)$ $j = 1, 3$ can be defined as follows:

Across $\lambda_1 = (u - C)$ characteristic field we have

$$\frac{d\rho}{-\rho / C} = \frac{du}{1} = \frac{dp}{-\rho C}\,. \tag{6.3.4}$$

From equation (6.3.3) and (6.3.4) we have

$$\Pi_1^1 = S \quad , \qquad \Pi_2^1 = u + \frac{2C}{(\Gamma - 1)}(1 - \theta\rho)\,. \tag{6.3.5}$$

Also across $\lambda_3 = (u + C)$ characteristic field we have

$$\frac{d\rho}{\rho / C} = \frac{du}{1} = \frac{dp}{\rho C}\,. \tag{6.3.6}$$

From equation (6.3.3) and (6.3.6) we have

$$\Pi_1^3 = S \quad , \qquad \Pi_2^3 = u - \frac{2C}{(\Gamma - 1)}(1 - \theta\rho). \tag{6.3.7}$$

The first and third Riemann invariants are given by equation (6.3.5) and (6.3.7).

Now the one parameter families of shock, simple waves, and contact discontinuities will be computed. Since the expressions which we shall obtain are explicit, therefore normalization condition is not required to make the analysis easier.

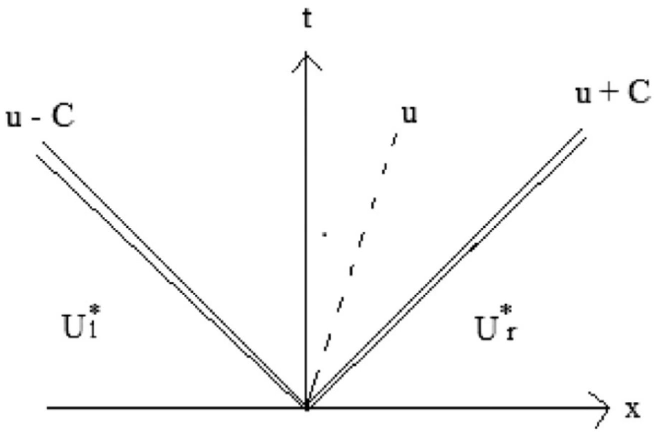

Fig.6.1 Structure of the solution of the Riemann problem in the x-t plane for 1-dimensonal Euler equations.

6.4 Shock waves

Shock waves are piecewise constant discontinuous solutions, satisfying Lax entropy condition, that propagate at a velocity σ dependent on the states existing on the two sides of the jump. The conservation variables must satisfy the Rankine-Hugoniot jump conditions. Let U_1^* and U_2^* denote the left and right states of either shock wave or rarefaction wave or contact discontinuity, i.e.

$$F\left(U_2^*\right)-F\left(U_1^*\right)=\sigma\left(U_2^*-U_1^*\right).$$

Thus we have the jump conditions for the system (6.3.1) as

$$\sigma[\rho]=[\rho u] \quad , \qquad (6.4.1)$$

$$\sigma[\rho u]=\left[p+\rho u^2\right], \qquad (6.4.2)$$

$$\sigma\left[\rho\left(u^2/2+E\right)\right]=\left[pu+\rho u\left(u^2/2+E\right)\right] , \qquad (6.4.3)$$

where σ is the shock velocity.

Now we shall rewrite above equations in a more convenient form by introducing the variables $v=u-\sigma$ and $m=\rho v$, we have

$$[m]=0 \quad , \qquad (6.4.4)$$

$$[p+mv]=0 \ , \tag{6.4.5}$$

$$m\left[\frac{2}{\Gamma(\Gamma-1)}C^2(1-Z)(\Gamma-Z)+v^2\right]=0 . \tag{6.4.6}$$

The Lax entropy condition is given as

$$\lambda_{k-1}\left(U_1^*\right)<\sigma<\lambda_k\left(U_1^*\right) \ , \ \lambda_k\left(U_2^*\right)<\sigma<\lambda_{k+1}\left(U_2^*\right) \ , \ k=1,3 \ . \tag{6.4.7}$$

We now see what further implication comes from the shock inequalities, knowing that they hold all along the shock curve.

From equation (6.4.7) for 1-shock waves we have $\sigma<u_1-C_1$ and $u_2-C_2<\sigma<u_2$, which implies that $C_1<v_1$ and $0<v_2<C_2<v_2+\sigma$ respectively. Therefore for 1-shock wave we have $C_1<v_1$ and $0<v_2<C_2$ which implies that $u_1>\sigma$ and $u_2>\sigma$. Thus the gas speed on both sides of the shock is greater than the shock speed, so for 1-shock, particles cross the shock from left to right. Similarly in the case of 3-shock waves we have $u_1<\sigma<u_1+C_1$ and $u_2+C_2<\sigma$, therefore for 3-shock we have $\sigma>u_1$ and $\sigma>u_2$. Thus for 3-shock waves the shock speed is greater than the gas speed on both sides of the shock, therefore particles cross a 3-shock from right to left.

Note that for both the shock families (i.e. 1-shock and 3-shock) $v_1\neq 0$ and $v_2\neq 0$ so $m=\rho_1 v_1=\rho_2 v_2\neq 0$. Since for 1-shock, we have $v_1>C_1>0$ and $C_2>v_2>0$ which implies that $v_1^2>C_1^2$ and $C_2^2>v_2^2$. While for 3-shock, we have $v_1<-C_1<0$ and $-C_2<v_2<0$ which implies that $v_1^2>C_1^2$ and $C_2^2>v_2^2$ respectively. Thus in both the shock families we have $v_1^2>C_1^2$ and $C_2^2>v_2^2$. Since $m\neq 0$ so from equation (6.4.6) we obtain

$$\frac{2}{\Gamma(\Gamma-1)}C_1^2(1-Z_1)(\Gamma-Z_1)+v_1^2=\frac{2}{\Gamma(\Gamma-1)}C_2^2(1-Z_2)(\Gamma-Z_2)+v_2^2 . \tag{6.4.8}$$

Using $v_1^2 > C_1^2$ and $C_2^2 > v_2^2$, above equation yields

$$\frac{2}{\Gamma(\Gamma-1)} C_1^2 (1-Z_1)(\Gamma - Z_1) + C_1^2 < \frac{2}{\Gamma(\Gamma-1)} C_2^2 (1-Z_2)(\Gamma - Z_2) + C_2^2 . \qquad (6.4.9)$$

Hence from the above equation we have $C_2^2 > C_1^2$ and thus $v_1^2 > v_2^2$ which implies that $C_2 > C_1$ and $|v_1| > |v_2|$ respectively.

Since $m = \rho v$ is constant i.e. $\rho_1 v_1 = \rho_2 v_2$ implies that $\rho_2 > \rho_1$. Also from equation (6.4.5) we get $p_2 > p_1$.Thus for 1-shock wave we have $\rho_2 > \rho_1$ and $p_2 > p_1$. Similarly we can easily prove that for 3-shock wave $\rho_2 < \rho_1$ and $p_2 < p_1$.Therefore both the shock waves are compressive waves in nature.

We now explicitly calculate the one parameter family of shock waves. We begin with 1-shock and define the constants

$$\pi = p_2/p_1, \quad \alpha = \rho_2/\rho_1, \quad \beta = (\Gamma+1)/(\Gamma-1), \quad \tau = (\Gamma-1)/2\Gamma . \qquad (6.4.10)$$

Clearly here $\pi > 1$ and $\alpha > 1$.

Since $C^2 = \Gamma p / ((1-Z)\rho)$ where $Z = \theta\rho$ therefore we have

$$\left(\frac{C_2}{C_1}\right)^2 = \frac{\pi}{\alpha}\frac{(1-Z_1)}{(1-Z_2)}, \qquad \text{where } Z_1 = \theta\rho_1 \text{ and } Z_2 = \theta\rho_2 . \qquad (6.4.11)$$

Now for ideal gas equation we have $\rho_1 v_1 = \rho_2 v_2$ which gives

$$v_2/v_1 = 1/\alpha . \qquad (6.4.12)$$

Using equation (6.4.11) and (6.4.12) in equation (6.4.6) we get

$$\left(\frac{v_1}{C_1}\right)^2 = \frac{2}{\Gamma(\Gamma-1)}\alpha(1-Z_1)\frac{\left[(\Gamma-Z_1)\alpha - \pi(\Gamma-Z_2)\right]}{(1-\alpha^2)} . \qquad (6.4.13)$$

Also from equation (6.4.5) we have

$$p_1 + m_1 v_1 = p_2 + m_2 v_2 . \qquad (6.4.14)$$

Since $m=\rho v$ and $p=C^2\rho(1-Z)/\Gamma$ therefore equation (6.4.14) yields

$$\left(\frac{v_1}{C_1}\right)^2=\frac{\alpha(1-Z_1)(1-\pi)}{\Gamma(1-\alpha)}. \tag{6.4.15}$$

Thus comparing equation (6.4.13) and (6.4.15) we get

$$\alpha=\frac{1+\pi\beta-2\pi Z_2/(\Gamma-1)}{\pi+\beta-2Z_1/(\Gamma-1)}. \tag{6.4.16}$$

Note that this implies $\alpha<\beta$ and since $1<\alpha$, we find $\rho_1<\rho_2<\beta\rho_1$ .This gives a bound on the density ρ_2 in terms of ρ_1.

If we use equation (6.4.16) in equation (6.4.15) we get

$$\left(\frac{v_1}{C_1}\right)^2=\frac{(1-Z_1)(1-\pi)\alpha\left(\pi+\beta-2Z_1/(\Gamma-1)\right)}{\Gamma\left(\pi+\beta-\frac{2Z_1}{\Gamma-1}-1-\pi\beta+2\pi Z_2/(\Gamma-1)\right)}. \tag{6.4.17}$$

Let $\eta=2Z_1/(\Gamma-1)$, $\xi=2Z_2/(\Gamma-1)$ and using $v=u-\sigma$ the above equation yields

$$\frac{u_2-u_1}{C_1}=\pm\frac{\left[(2-\eta(\Gamma-1))(1-\pi)\left((\beta-1)(1-\pi)-\eta+\pi\xi\right)\right]^{1/2}}{\left[2\Gamma\left(1+\pi(\beta-\xi)\right)\right]^{1/2}}. \tag{6.4.18}$$

Equation (6.4.18) represents the change in velocity across a shock transition. Here positive sign represents for 1-shock and negative sign for 3-shocks.

To make these somewhat more explicit, we introduce a new parameter ψ (Smoller (1994)) where

$$\psi=-\log\pi\ , \tag{6.4.19}$$

Note that $e^{-\psi}=\pi=p_2/p_1>1$ so that $\psi<0$. In terms of this parameterization, we have following formulas for 1-shock (recall that $\beta=(\Gamma+1)/(\Gamma-1)$)

$$\frac{p_2}{p_1}=e^{-\psi}\ , \tag{6.4.20}$$

$$\frac{\rho_2}{\rho_1}=\frac{1+e^{-\psi}\left(\beta-\xi\right)}{\beta+e^{-\psi}-\eta} \quad , \tag{6.4.21}$$

$$\frac{u_2-u_1}{C_1}=\frac{\left[\left(2-\eta\left(\Gamma-1\right)\right)\left(1-e^{-\psi}\right)\left(\left(\beta-1\right)\left(1-e^{-\psi}\right)-\eta+e^{-\psi}\xi\right)\right]^{1/2}}{\left[2\Gamma\left(1+e^{-\psi}\left(\beta-\xi\right)\right)\right]^{1/2}} \quad . \tag{6.4.22}$$

Similarly we have the following formulas for 3-shock

$$\frac{p_1}{p_2}=e^{\psi} \quad , \tag{6.4.23}$$

$$\frac{\rho_1}{\rho_2}=\frac{1+e^{\psi}\left(\beta-\xi\right)}{\beta+e^{\psi}-\eta} \quad , \tag{6.4.24}$$

$$\frac{u_1-u_2}{C_2}=\frac{\left[\left(2-\eta\left(\Gamma-1\right)\right)\left(e^{\psi}-1\right)\left(\left(\beta-1\right)\left(e^{\psi}-1\right)-\eta+e^{\psi}\xi\right)\right]^{1/2}}{\left[2\Gamma\left(1+e^{\psi}\left(\beta-\xi\right)\right)\right]^{1/2}}. \tag{6.4.25}$$

6.5 Simple waves

For a system of hyperbolic partial differential equations in 1-dimensional space, a centered rarefaction wave is a simple wave in which one family of characteristics are straight lines and the dependent variables are constant along characteristics. For a rarefaction wave the two constant states U_1^* and U_2^* are connected through a smooth transition in a k^{th} genuinely non-linear characteristic field and satisfy following conditions (i) the Riemann invariants are constant across the wave and (ii) the characteristics on the left and the right of the wave diverge i.e. $\lambda_k\left(U_1^*\right)<\lambda_k\left(U_2^*\right), \quad k=1,3$. We now calculate the simple wave curves. Here we consider only 1-simple waves; the details for 3-simple waves are analogous.

Since the 1-Riemann invariant is constant on a 1-simple wave, we have

$$S_2=S_1, \tag{6.5.1}$$

and

$$u_2 + \frac{2C_2}{(\Gamma - 1)}(1 - Z_2) = u_1 + \frac{2C_1}{(\Gamma - 1)}(1 - Z_1) \quad . \tag{6.5.2}$$

Since the gas is polytropic ideal with dust particle so we have equation of state as

$$p = k e^{S/c_v} \left(\rho / (1 - \theta\rho)\right)^{\Gamma} . \tag{6.5.3}$$

So from equation (6.5.1) and (6.5.3) we get

$$\frac{p_2}{p_1} = \left(\frac{C_2}{C_1}\right)^{\frac{2\Gamma}{\Gamma-1}} \left(\frac{1 - Z_2}{1 - Z_1}\right)^{\frac{2\Gamma}{\Gamma-1}} = \left(\frac{\rho_2}{\rho_1}\right)^{\Gamma} \left(\frac{1 - Z_1}{1 - Z_2}\right)^{\Gamma} \quad . \tag{6.5.4}$$

Also, from equation (6.5.2) we have

$$\frac{u_2 - u_1}{C_1} = \frac{2}{\Gamma - 1}\left[(1 - Z_1) - \frac{C_2}{C_1}(1 - Z_2)\right]. \tag{6.5.5}$$

But $\lambda_1 = u - C$ must increase in a 1-rarefaction wave so $\lambda_1^{(2)} \geq \lambda_1^{(1)}$ gives $u_2 - u_1 \geq C_2 - C_1$. Thus from equation (6.5.5) we have

$$\frac{C_2 - C_1}{C_1} \leq \frac{2}{\Gamma - 1}\left[(1 - Z_1) - \frac{C_2}{C_1}(1 - Z_2)\right],$$

which gives

$$0 \leq \left(\frac{2}{\Gamma - 1} + 1\right)\left(1 - \frac{C_2}{C_1}\right) + \frac{2}{\Gamma - 1}\left(Z_2 \frac{C_2}{C_1} - Z_1\right) \quad . \tag{6.5.6}$$

From equation (6.5.6) we observe that $0 < C_2 / C_1 \leq 1$. Using this in equation (6.5.4) we get

$$0 < \frac{p_2}{p_1} \leq 1 \quad . \tag{6.5.7}$$

From equation (6.4.19), we have

$$\psi = -\log \pi \quad . \tag{6.5.8}$$

Note that, $e^{-\psi} = \pi = p_2 / p_1 < 1$. So that $\psi > 0$. Therefore we can write the formula for 1-simple waves using equation (6.5.4) and (6.5.5) as

$$\frac{p_2}{p_1} = e^{-\psi} \quad , \tag{6.5.9}$$

$$\frac{\rho_2}{\rho_1} = e^{-\frac{\psi}{\Gamma}} \left(\frac{1-Z_2}{1-Z_1} \right) , \tag{6.5.10}$$

$$\frac{u_2 - u_1}{C_1} = \frac{2}{\Gamma - 1} (1 - Z_1) \left[1 - e^{-\psi\tau} \right] . \tag{6.5.11}$$

Similarly, for 3-simple waves we have the following formula

$$\frac{p_1}{p_2} = e^{\psi} , \tag{6.5.12}$$

$$\frac{\rho_1}{\rho_2} = e^{\frac{\psi}{\Gamma}} \left(\frac{1-Z_1}{1-Z_2} \right) , \tag{6.5.13}$$

$$\frac{u_1 - u_2}{C_2} = \frac{2}{\Gamma - 1} (1 - Z_1) \left[e^{\psi\tau} - 1 \right] . \tag{6.5.14}$$

6.6 Contact discontinuities

Contact discontinuities are the surfaces which separate two zones of different density and temperature. This type of wave comes from the linear degeneracy of the second characteristics family. There are no shocks or rarefaction waves in this family, but instead a one parameter family of contact discontinuities. In other words we can say that for a contact discontinuity the two constant states U_1^* and U_2^* are connected through a single jump discontinuity with speed σ_2 in the second characteristic field, which is linearly degenerate and the following condition holds (i) the R-H jump condition $F\left(U_2^*\right) - F\left(U_1^*\right) = \sigma\left(U_2^* - U_1^*\right)$ and (ii) the parallel characteristic conditions

$\lambda_2\left(U_2^*\right)=\lambda_2\left(U_1^*\right)=\sigma_2$. Thus we have the following formulas for the one parameter family of contact discontinuities

$$\frac{p_2}{p_1}=1\,, \tag{6.6.1}$$

$$\frac{\rho_2}{\rho_1}=e^{\psi}\quad,\qquad -\infty<\psi<\infty\quad, \tag{6.6.2}$$

$$u_2-u_1=0\ . \tag{6.6.3}$$

Now we can put together above formulas (i.e. of shocks, contact discontinuities and simple waves) for the one parameter families of curves as follows

For the 1- family (either 1-shock or 1-rarefaction waves depending on the sign of ψ), $\psi\in R$

$$\frac{p_2}{p_1}=e^{-\psi}\,, \tag{6.6.4}$$

$$\frac{\rho_2}{\rho_1}=g_1(\psi)=\begin{cases} e^{-\frac{\psi}{\Gamma}}\left(\dfrac{1-Z_2}{1-Z_1}\right), & \psi>0\\ \dfrac{1+e^{-\psi}\left(\beta-\xi\right)}{\beta+e^{-\psi}-\eta}, & \psi<0\end{cases}\quad, \tag{6.6.5}$$

$$\frac{u_2-u_1}{C_1}=h_1(\psi)=\begin{cases}\dfrac{2}{\Gamma-1}\left(1-Z_1\right)\left(1-e^{-\psi\tau}\right), & \psi>0\\ \dfrac{\left[\left(2-\eta\left(\Gamma-1\right)\right)\left(1-e^{-\psi}\right)\left(\left(\beta-1\right)\left(1-e^{-\psi}\right)-\eta+e^{-\psi}\xi\right)\right]^{1/2}}{\left[2\Gamma\left(1+e^{-\psi}\left(\beta-\xi\right)\right)\right]^{1/2}}, & \psi<0\end{cases}. \tag{6.6.6}$$

For the 2-family, $\psi\in R$

$$\frac{p_2}{p_1}=1,\quad \frac{\rho_2}{\rho_1}=e^{\psi},\quad u_2-u_1=0. \tag{6.6.7}$$

For the 3-family (either 3-shock or 3-rarefaction waves depending on the sign of ψ),

$\psi \in R$

$$\frac{p_1}{p_2} = e^{\psi}, \tag{6.6.8}$$

$$\frac{\rho_1}{\rho_2} = g_3(\psi) = \begin{cases} e^{\frac{\psi}{\Gamma}}\left(\dfrac{1-Z_1}{1-Z_2}\right), & \psi > 0 \\ \dfrac{1+e^{\psi}(\beta-\xi)}{\beta+e^{\psi}-\eta}, & \psi < 0 \end{cases}, \tag{6.6.9}$$

$$\frac{u_1-u_2}{C_2} = h_3(\psi) = \begin{cases} \dfrac{2}{\Gamma-1}(1-Z_1)\left(e^{\tau\psi}-1\right), & \psi > 0 \\ \dfrac{\left[\left(2-\eta(\Gamma-1)\right)\left(e^{\psi}-1\right)\left((\beta-1)\left(e^{\psi}-1\right)-\eta+e^{\psi}\xi\right)\right]^{1/2}}{\left[2\Gamma\left(1+e^{\psi}(\beta-\xi)\right)\right]^{1/2}}, & \psi < 0 \end{cases}. \tag{6.6.10}$$

Lemma (i): $h_1^{'}(\psi) > 0$ and $h_1(R) = \left(-\infty, \dfrac{2}{\Gamma-1}(1-Z_1)\right]$.

(ii) $h_3(\psi) = e^{\psi/2} h_1(\psi)\sqrt{g_1(\psi)\dfrac{1-Z_1}{1-Z_2}}$. Where $\psi \in R$, while h_1, h_3 and g_1 are defined by equations (6.6.6), (6.6.10) and (6.6.5) respectively.

Proof: The proof of Lemma (i) and (ii) are completely straightforward and can be omitted.

It is useful to write the above one parameter families of curves in a general notation and for this let us consider transformations $T_{\psi}^{(i)}$, $i = 1,2,3$ as

$$T_{\psi}^{(1)}(\rho, p, u) = \left(g_1(\psi)\rho,\ e^{-\psi}p,\ u + C\,h_1(\psi)\right),$$

$$T_{\psi}^{(2)}(\rho, p, u) = \left(e^{\psi}\rho,\ \ p,\ u\right),$$

$$T_{\psi}^{(3)}(\rho, p, u) = \left(g_3(\psi)\rho,\ \ e^{\psi}p,\ \ u + C\,h_3(\psi)\right),$$

where the $g_i^{'s}$ and $h_i^{'s}$ are defined as above. Note that in this notation, we really mean that the state (ρ, p, u) can be connected to the state $T_\psi^{(i)}$, by an i-shock or i-rarefaction wave if $i=1$ or 3 and $\psi<0$ or $\psi>0$, respectively, and if $i=2$, it can be connected to the state $T_\psi^{(2)}$ by a contact discontinuity .

Theorem 1: Consider the system of equations (6.2.10) for an ideal polytropic dusty gas whose equation of state is given by (6.2.5). Let $U_1(\rho, p, u)$ and $U_2(\rho, p, u)$ be any two states which is not necessarily closed, then there is a unique solution (in the case of shocks, centered rarefaction waves and contact discontinuities separating constant states) to the Riemann problem with these initial states, if and only if

$$u_2 - u_1 < \frac{2}{\Gamma - 1}(1 - Z_1)(C_2 + C_1). \qquad (6.6.11)$$

And if equation (6.6.11) is violated, then a vacuum is present in the solution.

Proof: let us consider two states U_1 and U_2 which is defined as $U_1 = (\rho_1, p_1, u_1)$, $U_2 = (\rho_2, p_2, u_2)$. To solve the Riemann problem with this data means we have to find the real numbers ψ_1, ψ_2 and ψ_3 for which

$$U_2 = T_{\psi_3}^{(3)} T_{\psi_2}^{(2)} T_{\psi_1}^{(1)}(U_1). \qquad (6.6.12)$$

So from equation (6.6.12) clearly we get

$$\begin{pmatrix} \rho_2 \\ p_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} g_1(\psi_1) e^{\psi_2} g_3(\psi_3) \rho_1 \\ e^{\psi_3 - \psi_1} p_1 \\ u_1 + C_1 \left[h_1(\psi_1) + \left(\dfrac{e^{-(\psi_1+\psi_2)}}{g_1(\psi_1)} \right)^{1/2} h_3(\psi_3) \right] \end{pmatrix}. \qquad (6.6.13)$$

Let us define

$$M = \frac{\rho_2}{\rho_1}, \quad N = \frac{p_2}{p_1}, \quad \text{and} \quad Q = \frac{u_2 - u_1}{C_1}. \tag{6.6.14}$$

From the second component of equation (6.6.13) we have

$$\psi_3 - \psi_1 = \log N. \tag{6.6.15}$$

The first component of equation (6.6.13) gives

$$g_1(\psi_1) e^{\psi_2} g_3(\psi_3) = M, \tag{6.6.16}$$

and from the third component of equation (6.6.13), we have

$$Q = h_1(\psi_1) + \left(\frac{e^{-(\psi_1+\psi_2)}}{g_1(\psi_1)} \right)^{1/2} h_3(\psi_3). \tag{6.6.17}$$

Using the fact $g_1 g_3 = 1$ and equation (6.6.16) in equation (6.6.17), we have

$$Q = h_1(\psi_1) + \left(\frac{e^{-\psi_1}}{M\, g_1(\psi_3)} \right)^{1/2} h_3(\psi_3).$$

Now using Lemma (ii) in above equation we get

$$Q = h_1(\psi_1) + \frac{e^{(\psi_3-\psi_1)/2}}{\sqrt{M}} \left(\frac{1-Z_1}{1-Z_2} \right)^{1/2} h_1(\psi_3). \tag{6.6.18}$$

In view of equation (6.6.15) equation (6.6.18) can be written as

$$Q = h_1(\psi_1) + \sqrt{\frac{N}{M}\left(\frac{1-Z_1}{1-Z_2} \right)}\, h_1(\psi_1 + \log N). \tag{6.6.19}$$

Now Lemma (i) shows that this equation (6.6.19) has a unique solution if and only if

$$\frac{2}{\Gamma-1}(1-Z_1)\left[1 + \left(\frac{N}{M}\left(\frac{1-Z_1}{1-Z_2} \right) \right)^{1/2} \right] > Q. \tag{6.6.20}$$

This on simplification gives us

$$u_2 - u_1 < \frac{2}{\Gamma-1}(1-Z_1)(C_2 + C_1). \tag{6.6.21}$$

Now we can find ψ_1 from equation (6.6.19), then ψ_3 from equation (6.6.15), and finally equation (6.6.16) gives us ψ_2. Further if equation (6.6.11) is violated, then the relative velocities on both sides of states are so large that a vacuum is formed. Thus we place a vacuum $(\rho = 0)$ between the two parts of the gas, and the other variables are left undefined. This completes the proof of the theorem.

Now we will prove how these methods provide simple criteria for knowing which of the two possibilities, shock or simple wave, occur in 1-family and 3-family of curves.

Corollary: Consider the solution of the Riemann problem obtained in theorem1. Then the following condition holds:

(a) The first component of the solution will be a simple wave if and only if

$$\sqrt{\frac{N}{M}\left(\frac{1-Z_1}{1-Z_2}\right)}\, h_1(\log N) < Q < \frac{2}{\Gamma-1}(1-Z_1)\left(1+\sqrt{\frac{N}{M}\left(\frac{1-Z_1}{1-Z_2}\right)}\right),$$

and will be a shock otherwise.

(b) The third component of the solution will be a simple wave if and only if

$$h_1(-\log N) < Q < \frac{2}{\Gamma-1}(1-Z_1)\left(1+\sqrt{\frac{N}{M}\left(\frac{1-Z_1}{1-Z_2}\right)}\right).$$

and will be a shock otherwise. Where M, N and Q are defined as in equation (6.6.14).

Proof: The right hand side inequalities in both (a) and (b) have already been proved in theorem1 so there is no need to prove once again. Now our main aim is to prove the left hand inequalities of (a) and (b). We consider first the 1-family.

Let $\zeta(\psi) = h_1(\psi) + \sqrt{\frac{N}{M}\left(\frac{1-Z_1}{1-Z_2}\right)}\, h_1(\psi + \log N)$, then $\zeta(0) = \sqrt{\frac{N}{M}\left(\frac{1-Z_1}{1-Z_2}\right)}\, h_1(\log N)$

and $\partial\zeta/\partial\psi > 0$. Thus equation (6.6.19) shows that the 1-wave is a simple wave if and only if $Q > \zeta(0)$ i.e. $Q > \sqrt{\frac{N}{M}\left(\frac{1-Z_1}{1-Z_2}\right)}\, h_1(\log N)$, Thus this proves (a). Now for (b) we use equation (6.6.15) in equation (6.6.19) to get

$$h_1(\psi_3 - \log N) + \sqrt{\frac{N}{M}\left(\frac{1-Z_1}{1-Z_2}\right)}\, h_3(\psi_3) = Q. \tag{6.6.22}$$

Equation (6.6.22) shows that 3-wave is a simple wave if and only if $Q > h_1(-\log N)$. This completes the proof of the corollary.

6.7 Results and discussion

The analytical solution of the Riemann problem for the quasilinear hyperbolic system of equations of an ideal polytropic dusty gas, that is, shock waves, rarefaction waves and contact discontinuities are obtained. It is observed that in the absence of dust particles (i.e., $\theta = 0$) the results obtained are identical with the earlier results derived for an ideal gas and non-magnetic case (Singh and Singh (2014a), Smoller (1994)). The density and velocity profiles for compressive waves (1-shock and 3-shock) and rarefaction waves (1-shock and 3-shock) versus ψ (psi) (where ψ is defined earlier as $\psi = -\log \pi$) are plotted in Figs. (2)-(9) using MATLAB for different values of mass fraction of dust particles k_p in the gas.

The values of the constants appearing in the computations are taken as

$\gamma = 1.4$, $\omega = 0.8$, $Z_1 = 0.03$, $Z_2 = 0.04$, k_p = 0.0, 0.2, 0.4, 0.6.

It may be noted here that the velocity profiles for 1-shock of compressive wave is concave upward while for 3-shock wave it is convex upward. Also the velocity profile for 1-shock of rarefaction wave is convex upward and for 3-shock it is concave upward. Further the density profiles for both the compressive waves as well as rarefaction waves are concave upward in nature. From the figures it is clear that the lines corresponding to increasing values of k_p move away from the line of exact solution of the Riemann problem for ideal case ($k_p = 0$). Further the effect of increasing values of k_p is to increase the density and velocity for compressive 1-shock wave whereas the same effect of k_p produces opposite trend in density and velocity profiles in case of compressive 3-shock wave. It is also observed that the effect of increasing values of k_p on the density and velocity profiles for rarefaction wave (1-shock and 3-shock) produces opposite trend. The reason for the effects obtained in the solution profiles is due to that the increasing values of k_p causes the Grüneisen coefficient Γ to decrease which results in the obtained behaviour of density and velocity profiles of 1-shock wave , 3-shock wave and 1-rarefaction wave and 3-rarefaction wave. Therefore, from the figs. (2)-(9) it may be concluded that the solution of the Riemann problem depends significantly on the presence of dust particles in the gas.

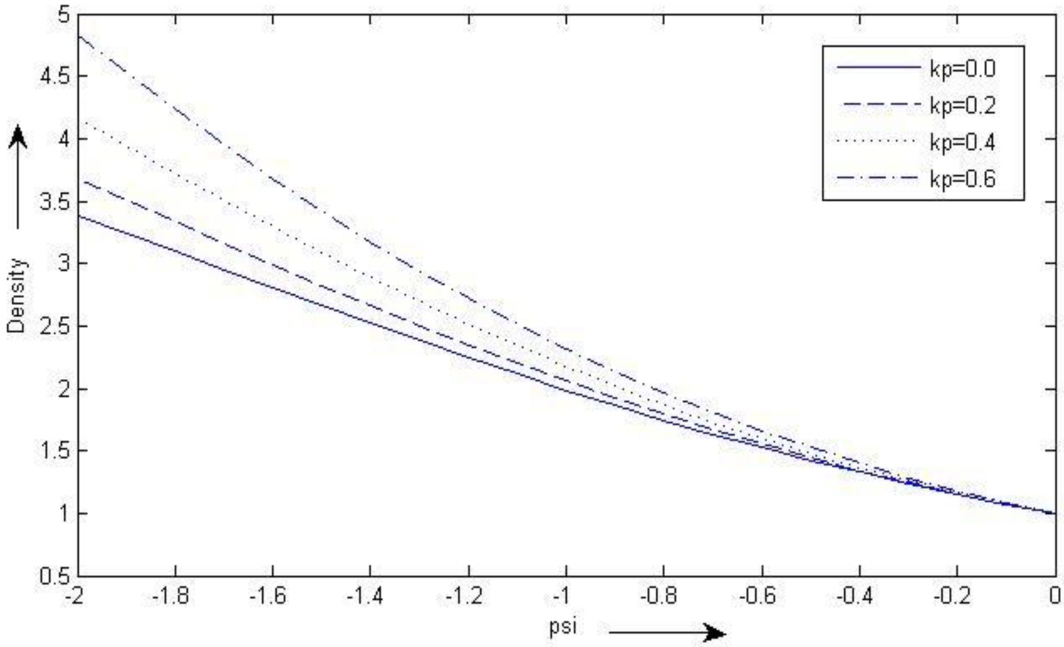

Fig.6.2 Density profile for compressive waves: 1-shock

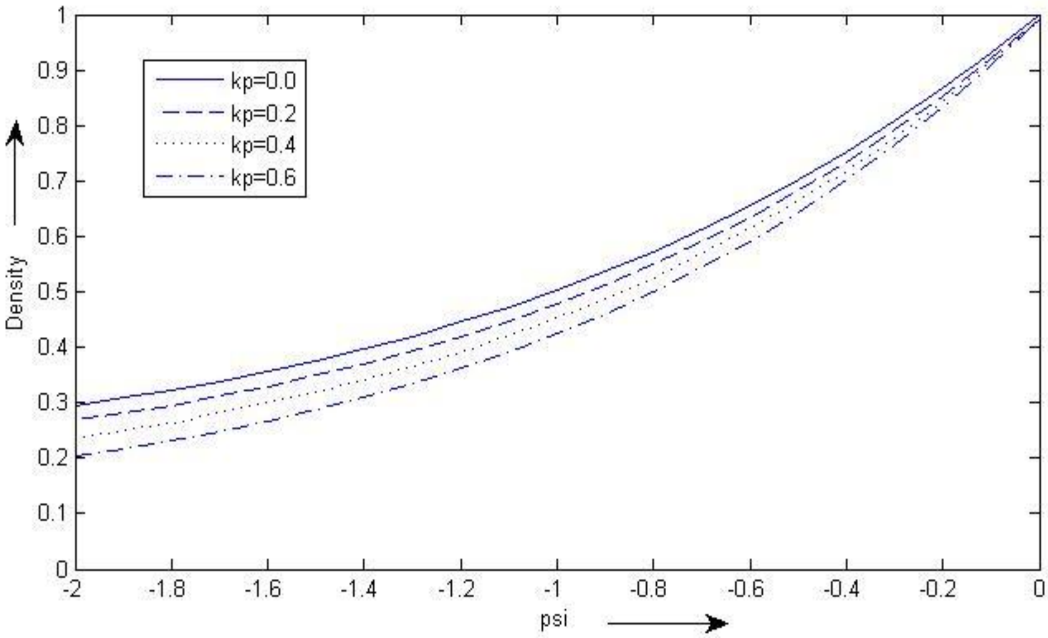

Fig.6.3 Density profile for compressive waves: 3-shock

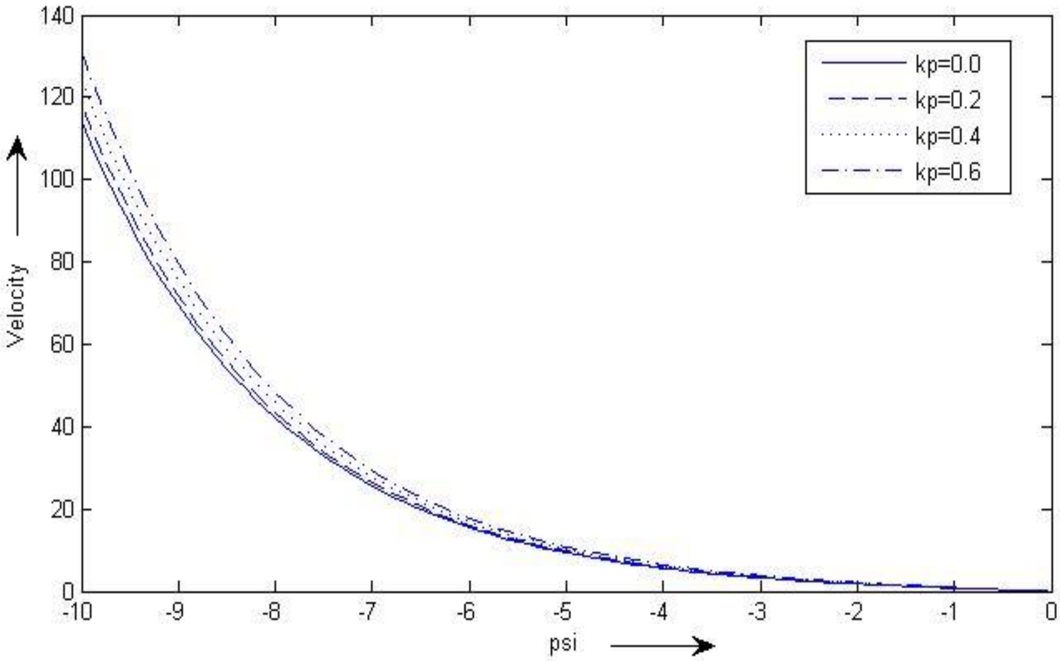

Fig.6.4 Velocity profile for compressive waves: 1-shock

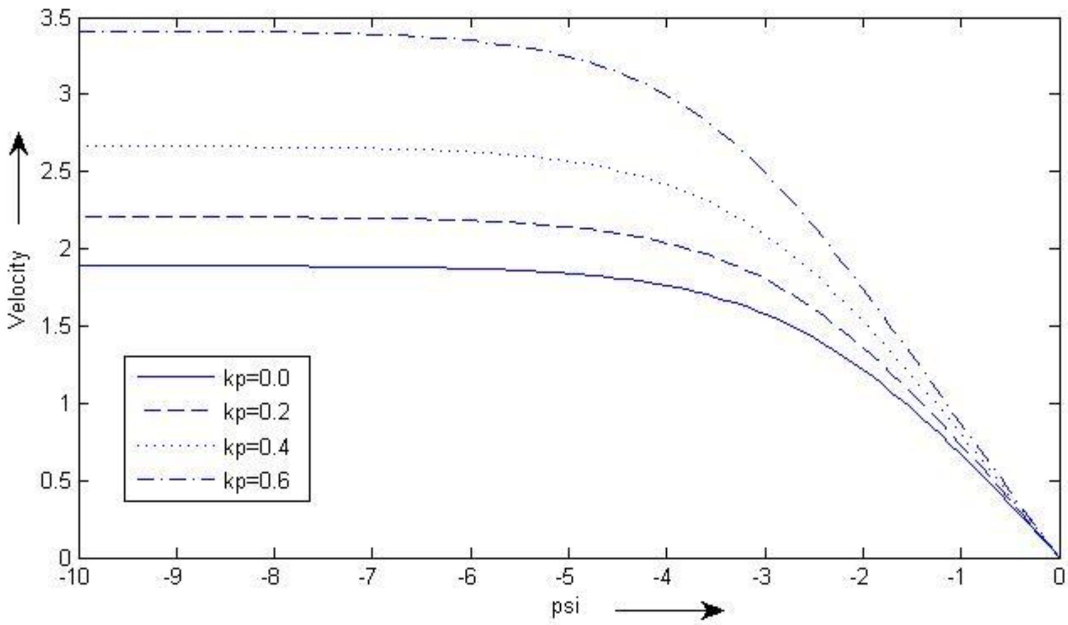

Fig.6.5 Velocity profile for compressive waves: 3-shock

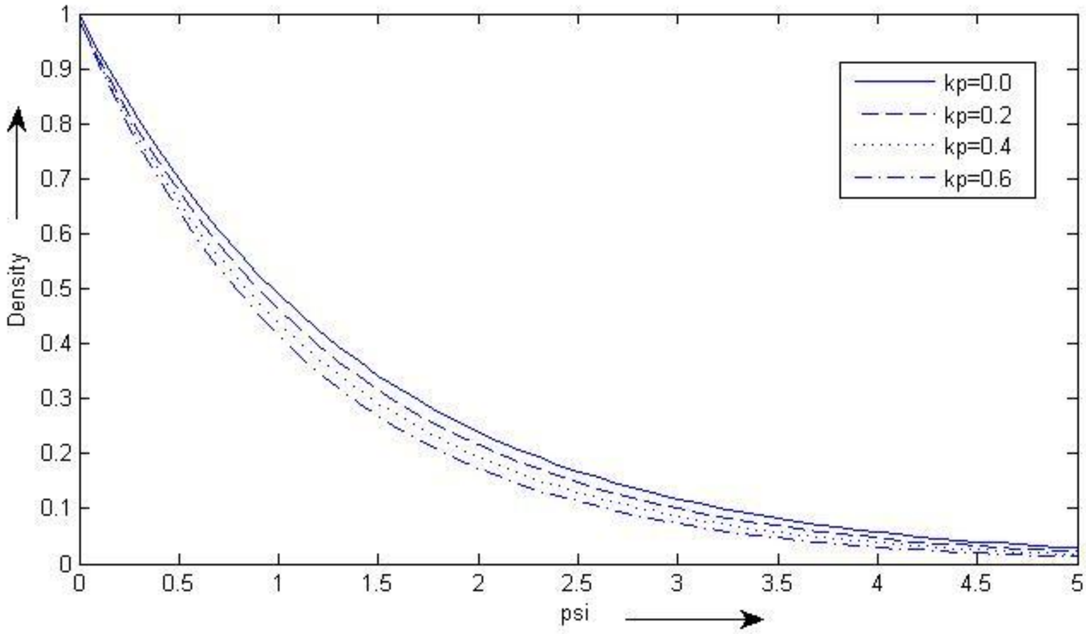

Fig.6.6 Density profile for rarefaction waves: 1-shock

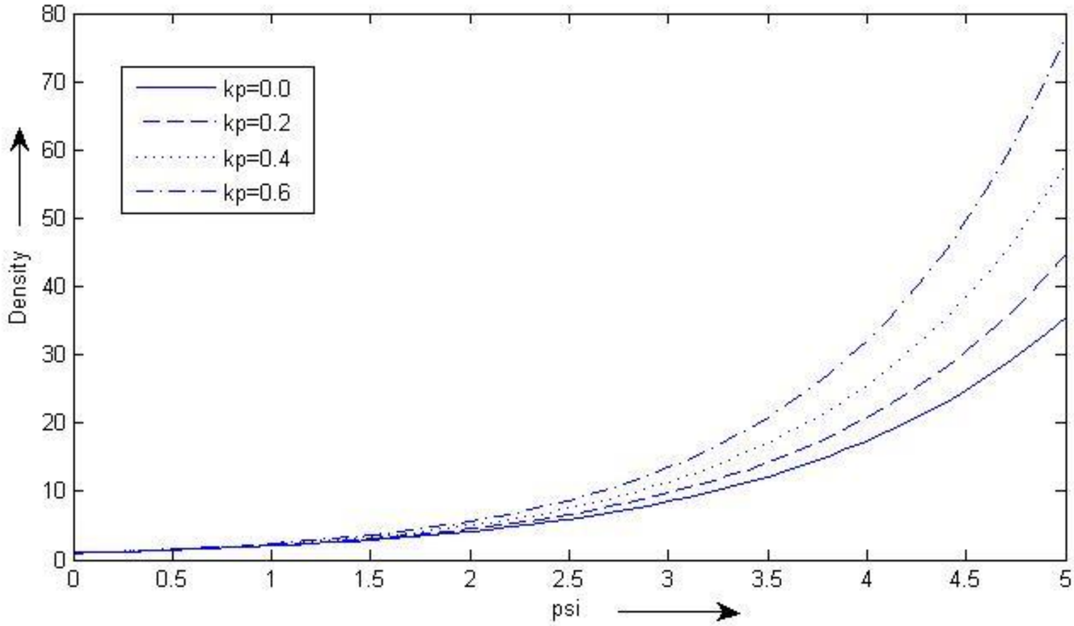

Fig.6.7 Density profile for rarefaction waves: 3-shock

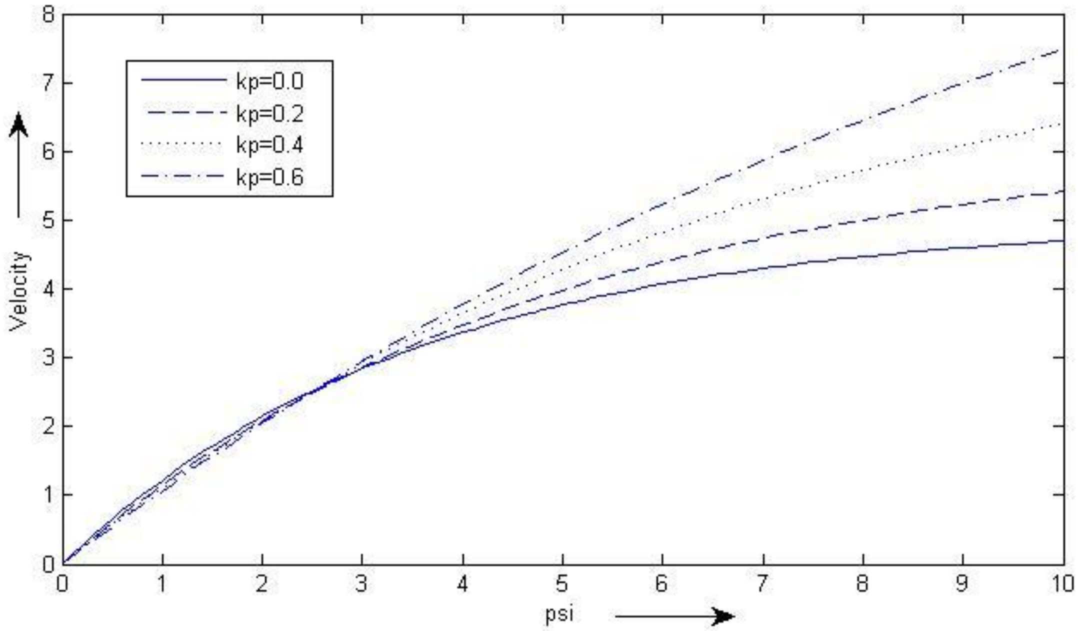

Fig.6.8 Velocity profile for rarefaction waves: 1-shock

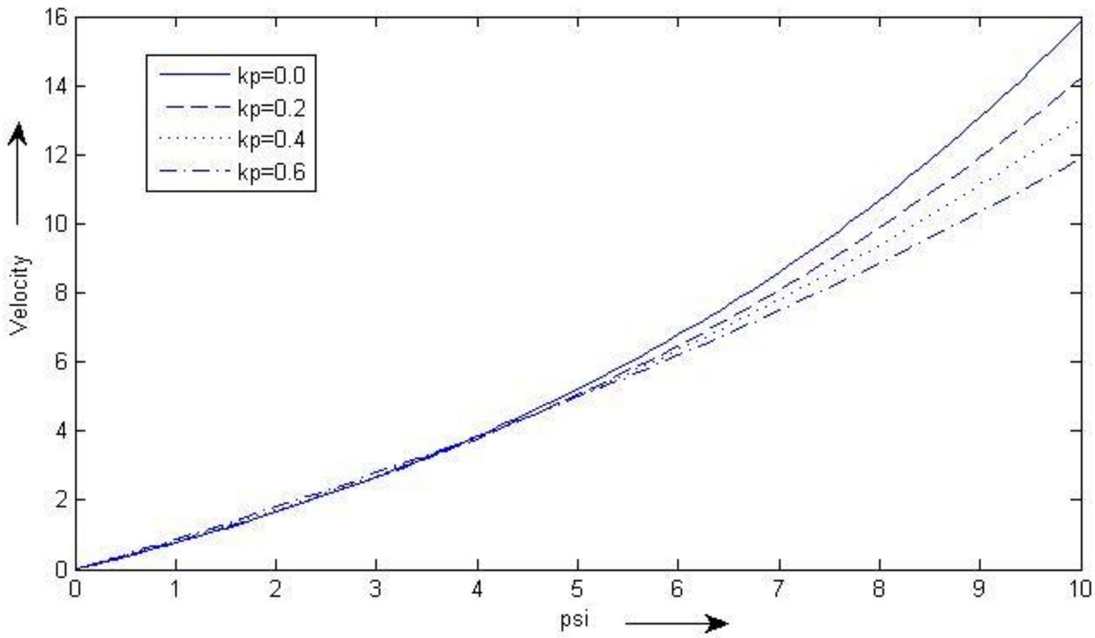

Fig.6.9 Velocity profile for rarefaction waves: 3-shock

6.8 Conclusions

In the present chapter the analytical solution of the Riemann problem for the quasilinear hyperbolic system of equations of an ideal polytropic dusty gas i.e., 1-shock waves (1-simple waves) and 3-shock waves (3-simple waves) is presented without any restriction on magnitude of the initial states. A necessary and sufficient condition for the existence of a unique solution to the Riemann problem in dusty gases is derived. Also those cases are discussed which gives information about the existence of shock waves or simple waves for a 1-family and for 3-family of curves. It is observed that the presence of dust particles in an ideal polytropic gas exhibits more complex expression as compared to the corresponding ideal case, however all the parallel results remain same. It is also shown that the results obtained are identical for the case of an ideal polytropic gas without dust particles.