Chapter 5

Evolution and decay of acceleration waves in dusty gas flow

5.1 Introduction

A large number of physical phenomena such as nuclear explosion, chemical explosion, supersonic flow etc.; taking place in nature may be described by mathematical models in terms of quasilinear hyperbolic system of partial differential equations (Whitham (1974), Anile et al (1993), Sharma and Venkatraman (2012)). In the unsteady flow of a compressible fluid, a surface may exist, on which the acceleration of fluid particles has a discontinuity, known as acceleration wave. The study of acceleration waves has received considerable attention in last few decades because of the fact that they are a special class of non-linear wave processes which may be treated rigorously by analytical or numerical methods.

It is well known that the weak discontinuities of a non-linear hyperbolic system of equations propagate along characteristic curves. If the magnitude of the jumps in the derivatives of the flow variable across the wave front becomes unbounded, the weak discontinuity turns into a shock wave and propagation then ceases to be along the leading characteristics. The presence of such a shock wave was identified with the physical process of steepening or breaking of a wave (Whitham (1974), Courant and Friedrichs (1948)). The analysis of possible steepening of compressive wave leading to shock waves has received great attention in past. Some authors such as Thomas (1957), Coleman and Gurtin (1967), Menon et al. (1983), Anile and Russo (1986), Jordan

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(2005) have studied the problems of propagation with some consideration of the non-linear effect. Shyam et al. (1981) have used the singular surface theory to study the growth and decay of weak shock waves in a radiating gas. For moderately weak shock strength, the complete history of shock may be determined approximately by means of a method due to Whitham (1974), Courant and Friedrichs (1948), Jeffrey and Taniuti (1964). Shankar (1989) has successfully applied the well-known ray theory given by Keller (1954) to study the propagation of shock wave in radiative magnetogasdynamics. Singh et al. (1987, 2015) have used the progressive wave approach to analyze the growth and decay of weak shock waves in various gasdynamic regimes. Further, Nath et al. (2017) used the same technique to analyze the main features of weakly non-linear waves and determined the transport equation governing the growth and decay of shock waves in non-ideal magnetogasdynamics. Ram (1978) have studied the propagation of acceleration waves and the effect of radiative heat transfer on the process of steepening of acceleration waves and determined the shock formation time and distance using the method of characteristics. Further Singh et al. (2012, 2014) used the same method to study the evolution and decay of acceleration waves in different material media.

In general the mixture of gas and small solid particles are known as dusty gas. The study of shock related phenomena in a mixture of small solid particles and gas is of great importance due to its wide application in lunar ash flow, bomb blast, coal-mine blast, cosmic explosions, propellant rocket, supersonic flight in polluted air, particle acceleration in shock and in many other engineering problems (Miura and Glass (1983), Pai et al. (1980), and Pai (1977)).When a shock wave is propagated through a gas with small solid particles the pressure, temperature and the entropy changes across the shock and the other features differ greatly from those which arise when the shock passes through the dust-free gas. The method of group theoretic is used to solve the class of self similar solution to the problem of propagation of shock waves in dusty gas (Jena and Sharma (1999)). Further the method of Lie group of transformation is used to obtain the self similar solution to the problem of propagation of shock waves in non-ideal gas with dust particles (Chadha and Jena (2014)). Anand (2014) have derived the shock jump relation for dusty gas medium.

In the present chapter we consider the problem of propagation of acceleration waves in a one dimensional unsteady, inviscid dusty gas with generalized geometry. Propagation of acceleration waves along the characteristics path is investigated by using the characteristics of the governing system as a reference coordinate. Further it has been shown that a linear solution in the characteristic plane exhibits non-linear behaviour of acceleration wave in the physical plane. Also the transport equation, governing the growth and decay behaviour of acceleration waves in an ideal gas with dust particles, is derived. It may be noted here that all compressive waves, irrespective of their initial strength, terminates into shock wave for planar and non-planar flows. Also the variation of mass fraction of the dust particles and the ratio of specific density of the solid particles and specific density of the gas at constant pressure on the evolution and decay behaviour of acceleration waves is presented. We also compare contrast the nature of solution in dusty gas medium with the solution in dust free gas.

5.2 Governing equations

The governing equations for the one-dimensional unsteady planar, cylindrically symmetric and spherically symmetric motion of an ideal gas in presence of dust particles obeying the following equation of state of Mie Grüneisen type

$$p=\left(1-k_p\right)\left(1-Z\right)^{-1}\rho RT\,, \tag{5.2.1}$$

may be written as (Whitham (1974), Miura and Glass (1983), Pai et al. (1980), Pai (1977))

$$\rho_t + u\rho_x + \rho u_x + n\rho u/x = 0, \tag{5.2.2}$$

$$u_t + uu_x + p_x/\rho = 0, \tag{5.2.3}$$

$$E_t + uE_x - \left(p/\rho^2\right)\left(\rho_t + u\rho_x\right) = 0, \tag{5.2.4}$$

where ρ is the density, p is the pressure, u is the velocity, x is the spatial coordinate, R is the gas constant, T is the absolute temperature and E is the internal energy. Here $(n = 0)$ represents the planar flow, $(n = 1)$ represents the cylindrically symmetric flow and $(n = 2)$ represents the spherically symmetric flow. The subscripts denote partial differentiation unless stated otherwise. The internal energy E per unit mass of the mixture of solid particles and gas is given as

$$E = \frac{(1-Z)\,p}{(\Gamma-1)\rho}, \tag{5.2.5}$$

where Z is the volume fraction of the dust particles in the mixture and is defined as $Z = V_{sp}/V_g$, $k_p = m_{sp}/m_g$ is the mass fraction of the small solid particles in the mixture, while V_{sp} and m_{sp} are the volumetric extension and total mass of the solid particles in the mixture respectively, V_g and m_g are the total volume and total mass of the gas respectively. Γ is the Grüneisen coefficient and is defined as $\Gamma = \gamma\left(1+\lambda\beta\right)/\left(1+\lambda\beta\gamma\right)$, with $\lambda = k_p/\left(1-k_p\right)$, $\beta = c_{sp}/c_p$ and $\gamma = c_p/c_v$, where c_{sp} is the specific heat of the solid particles, c_p is the specific heat of the gas at constant pressure and c_v is the specific heat of the gas at constant volume. The entities Z and k_p are related via the expression $Z = \theta\rho$, where $\theta = k_p / \rho_{sp}$ with ρ_{sp} is the species density of the solid particles. Here, it may be noted that $\theta = 0$ corresponds to an ideal gas case.

In view of equation (5.2.5) equation (5.2.4) may be written as

$$p_t + up_x + \Gamma p/(1-\theta\rho)(u_x + nu/x) = 0. \tag{5.2.6}$$

Equations (5.2.2), (5.2.3) and (5.2.6) may be written in matrix form as

$$F_t + MF_x + N = 0, \tag{5.2.7}$$

where $F = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}$, $M = \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \Gamma p/(1-\theta\rho) & u \end{bmatrix}$ and $N = \begin{bmatrix} n\rho u/x \\ 0 \\ \Gamma pnu/(1-\theta\rho)x \end{bmatrix}$.

Equation (5.2.7) is a system of quasilinear hyperbolic partial differential equations having three real characteristic curves among two are genuinely non-linear while other is linearly degenerate and along these characteristics, acceleration waves propagate. Thus the function $F(x,t)$ satisfies (5.2.7) everywhere except at the characteristic curve $\Psi(t)$, where $F(x,t)$ is continuous but its first order derivative F_t, F_x and their higher order derivatives undergo finite jump across $\Psi(t)$, such type of discontinuities are known as “acceleration waves”. In other words this type of discontinuity is known as weak discontinuity.

If the jump of the quantity F across $\Psi(t)$ is denoted by $[F]$, therefore along $\Psi(t)$ we have

$$\frac{\partial}{\partial t}[F] = [F_t] + \frac{d\Psi}{dt}[F_x], \tag{5.2.8}$$

where $\partial/\partial t$ represents the time derivative as observed from the wave front. In view of equation (5.2.8), taking jump in (5.2.7) and the condition of continuity $[F] = 0$ yields

$$\left(M-\frac{d\Psi}{dt}I\right)[F_x]=0, \tag{5.2.9}$$

where I is an identity matrix of order 3×3.

From equation (5.2.9), it is observed that, if there exist a finite discontinuity of acceleration waves along the characteristic curve, the characteristics speed of propagation $d\Psi/dt$ is the characteristic root of the matrix M. Therefore, it follows immediately that there exists three families of characteristics curves and two of them are given as

$$\frac{dx}{dt}=u\pm A, \tag{5.2.10}$$

where A is the sound speed and is given as $A=\left(\Gamma p/(1-\theta\rho)\rho\right)^{1/2}$. Equation (5.2.10) represents the outgoing and incoming wavelets along the x-axis with A as a speed of sound in dusty gas medium and the remaining one

$$\frac{dx}{dt}=u\ , \tag{5.2.11}$$

represents the trajectory of the fluid particles.

5.3 Characteristic transformation and boundary conditions

However for solving the problems involving quasilinear hyperbolic partial differential equations one may choose the characteristic coordinate as reference frame. Therefore we introduce the following two characteristic variables ϕ and Ω defined as

(i). ϕ is a wave tag so that ϕ is constant along an outgoing characteristics $dx/dt=u+A$ in the (x,t) plane and will be labeled as $\phi=t^*$, if an outgoing wave is generated at a time t^*.

(ii). Ω is a "particle tag" so that Ω is constant along the trajectory of the fluid particle $dx/dt = u$ in the (x,t) plane. If the characteristic wave front traverses a particle at time t', therefore the particle and its path will be labeled by $\Omega = t'$.

Now it is clear that for each pair of coordinate (ϕ,Ω) there is a corresponding pair of coordinate (x,t) so that $x = x(\phi,\Omega)$ and $t = t(\phi,\Omega)$. With the properties of ϕ and Ω the following partial differential equations will be obtained

$$x_\phi = ut_\phi, \qquad x_\Omega = (u+A)t_\Omega. \tag{5.3.1}$$

In view of the above transformations we have

$$F_t = \frac{F_\Omega x_\phi - F_\phi x_\Omega}{J}, \tag{5.3.2}$$

$$F_x = \frac{F_\phi t_\Omega - F_\Omega t_\phi}{J}, \tag{5.3.3}$$

where J is the Jacobian of transformation which plays an important role for solving the problem related to shock formation and is given as

$$J = \frac{\partial(x,t)}{\partial(\phi,\Omega)} = -At_\phi t_\Omega.$$

The above relations provide a clear picture that $J = 0$ if and only if $t_\phi = 0$, when two adjoining characteristics unify into a shock wave. Since doubling up or overlapping of fluid particles is restricted from physical consideration, therefore $t_\Omega \neq 0$. Hence $J = 0$ will provide us a condition for the steepening of the wave front or the formation of shock wave.

Now using equation (5.3.2) and (5.3.3) in equations (5.2.2), (5.2.3) and (5.2.6) we get

$$A\rho_\phi t_\Omega - \rho u_\phi t_\Omega + \rho u_\Omega t_\phi + \frac{n\rho u A t_\phi t_\Omega}{x} = 0, \tag{5.3.4}$$

$$\rho A u_\phi t_\Omega - p_\phi t_\Omega + p_\Omega t_\phi = 0\,, \tag{5.3.5}$$

$$A p_\phi t_\Omega - \frac{\Gamma p}{(1-\theta\rho)} u_\phi t_\Omega + \frac{\Gamma p}{(1-\theta\rho)} u_\Omega t_\phi + \frac{\Gamma p n u A t_\phi t_\Omega}{(1-\theta\rho)\, x} = 0\,. \tag{5.3.6}$$

Using equations (5.3.5) and (5.3.6) in equation (5.3.4) we get

$$p_\Omega + \rho A u_\Omega = -\frac{n\rho u A^2 t_\Omega}{x}\,. \tag{5.3.7}$$

The boundary conditions at the shock front are given by

$$[\rho] = 0\,,\quad [u] = 0\,,\quad [p] = 0\,,\quad [T] = 0\,,\qquad t = \Omega \text{ at } \phi = 0\,. \tag{5.3.8}$$

Since we assume that the flow of the fluid ahead of the shock front is homogeneous and at rest therefore from equation (5.3.8) we have

$$\rho_\Omega = 0\,,\quad u_\Omega = 0\,,\quad p_\Omega = 0\,,\quad T_\Omega = 0\,,\quad t_\Omega = 1 \text{ at } \phi = 0\,. \tag{5.3.9}$$

5.4 Solution of the problem

Using equations (5.3.8) and (5.3.9) in equations (5.3.4) and (5.3.1) we get

$$p_\phi = \rho_0 A_0 u_\phi\,,\qquad \text{at } \phi = 0\,, \tag{5.4.1}$$

$$x_\phi = 0\,,\quad x_\Omega = A_0\,,\quad \text{at } \phi = 0\,. \tag{5.4.2}$$

Here the subscript "0" denotes the fluid flow variables just ahead of the shock front (i.e. undisturbed region). In order to find the amplitude of the acceleration wave at the shock front we have to use equation (5.3.9) in equation (5.3.3), thus we have

$$\left[\frac{\partial u}{\partial x}\right] = \alpha = -\frac{u_\phi}{A_0 t_\phi}\,,\quad \text{at } \phi = 0\,, \tag{5.4.3}$$

where α is the amplitude of the acceleration wave.

In order to discuss the dependency of u_ϕ and t_ϕ on time, we differentiate equation (5.3.1), (5.3.7) and (5.4.1) with respect to ϕ and Ω and then solving them we will get

$$\frac{u_{\phi\Omega}}{t_\phi} = \frac{nA_0}{2\Omega}\alpha\,, \quad \text{at } \phi = 0\,, \tag{5.4.4}$$

$$\frac{t_{\phi\Omega}}{t_\phi} = \frac{(\Gamma+1)}{2(1-\theta\rho_0)}\alpha\,, \quad \text{at } \phi = 0\,. \tag{5.4.5}$$

Now in order to find the transport equation governing the growth and decay behaviour of acceleration wave, we differentiate equation (5.4.3) with respect to Ω and then using equation (5.4.4) and (5.4.5) we get

$$\frac{d\alpha}{d\Omega} + \frac{n}{2\Omega}\alpha + \frac{1}{2}\left(\frac{\Gamma+1}{1-\theta\rho_0}\right)\alpha^2 = 0\,, \quad \text{at } \phi = 0\,. \tag{5.4.6}$$

Let us consider the following dimensionless parameters (Ram (1978))

$$\xi = \frac{\alpha}{\alpha^*}\,, \quad \chi = \frac{\Omega - \Omega^*}{2\Omega^*}\,, \quad \mu = \alpha^*\Omega^*\,, \tag{5.4.7}$$

where ξ is the dimensionless parameter of wave amplitude, χ is the dimensionless parameter of time, μ is the dimensionless parameter of the initial acceleration and the superscript "*" represents the initial wave level.

In view of equation (5.4.7) equation (5.4.6) may be written as

$$\frac{d\xi}{d\chi} + \frac{n}{(2\chi+1)}\xi + \left(\frac{\Gamma+1}{1-\theta\rho_0}\right)\mu\xi^2 = 0\,. \tag{5.4.8}$$

Equation (5.4.8) cannot be solved in present form therefore reducing equation (5.4.8) in linear differential equation form so that its analytical solution may be found as

$$\xi = \left\{\left(1 + \mu(\Gamma+1)(1-\theta\rho_0)^{-1}\,\Theta(\chi)\right)(1+2\chi)^{n/2}\right\}^{-1}, \tag{5.4.9}$$

where $\Theta(\chi) = \int_0^{\chi} \frac{1}{(1+2\chi)^{n/2}}\,d\chi$.

The function $\Theta(\chi)$ plays an important role in the breakdown of the characteristics solution. From equation (5.4.9) and (5.4.3) we observe that the shock wave will form when t_ϕ vanishes, i.e.

$$1+\mu\Theta(\chi)\left(\frac{\Gamma+1}{1-\theta\rho_0}\right)=0. \qquad (5.4.10)$$

Since the value of $\Theta(\chi)\geq 0$, therefore from equation (5.4.10) we observe that the only compressive wave fronts $(\mu<0)$ may terminate into shock waves.

5.5 Results and discussion

In the present chapter the transport equation governing the growth and decay behaviour of acceleration wave in one-dimensional unsteady planar and non-planar flows in dusty gas medium is derived. If we put $(\theta=0)$ in equation (5.4.9) the results which we have obtained here, are similar with the result as obtained previously (Ram (1978), Singh et al. (2012, 2014)) in absence of magnetic fields and radiative effects.

The values of the constants appearing in the computations are taken as

$\gamma=1.67$, $Z_0=0.04$, $\beta=0.2, 1.0$, $k_p=0.0, 0.3, 0.6$, $\mu=0.5, -0.5$.

Figs. (5.1)-(5.3) represent the variation of amplitude of the compressive wave 'ξ' versus χ for different values of mass fraction of the dust particles (k_p) and the ratio of specific heat of the solid particles and specific heat of the gas at constant pressure (β) for planar flow $(n=0)$, cylindrically symmetric flow $(n=1)$ and spherically symmetric flow $(n=2)$ respectively. From these curves we observe that all compressive waves, irrespective of their initial strength, terminate into shock wave as in absence of magnetic field as presented in (Schmitt (1972)). This is in contrast with the corresponding case of ideal radiating gas where one can always find the critical amplitude such that any compressive waves with initial amplitude greater than the critical amplitude always terminates into shock waves, while the initial amplitude less than the critical amplitude always results in decay of disturbance (Ram (1978)). In fig. (5.1), the dotted, dashed, dot-dashed and thin line represents the solution profile for different values of mass fraction of the dust particles and ratio of specific density of the solid particles and the specific density of the gas at constant pressure at fixed volume fraction. Here the vertical line represents the position of shock formation in dusty gas medium. From fig. (5.1), we observe that the increasing value of mass fraction (k_p) of the dust particles causes to delay the shock formation and also the increasing value of β causes to delay the shock formation time. Further as we move from planar symmetry to cylindrical symmetry and then from cylindrical symmetric flow to spherically symmetric flow the shock formation time is increased i.e. formation of shock will be delayed. From figs. (5.1)-(5.3) we observe that the process of steepening or flattening of compressive waves in case of cylindrically symmetric flow $(n=1)$ and spherically symmetric flow $(n=2)$ is more rapid as compare to the planar flow $(n=0)$. Figs. (5.4)-(5.6) represent the decay of expansive wave front $(\mu > 0)$ in planar, cylindrically symmetric and spherically symmetric flows in dusty gas respectively. In all the three cases the amplitude of the expansive wave decays and ultimately damped out. Further the effect of variation of mass fraction (k_p) of dust particles and β on the decay of expansive wave fronts are presented. It is observed that the increasing value of mass fraction of the dust particles causes to decrease the process of decay. Also similar effect of β on the decay of expansive wave fronts is obtained.

The solution curves of compressive waves and expansive waves corresponding to the cylindrically symmetric and spherically symmetric flows are presented in figs. (5.2), (5.3), (5.5) and (5.6). From these curves we observe that the solution profile is similar as in the case of planar flow however there is a slight variation in the sense that the process of steepening (or flattening) of compressive waves is less as compared to the planar case. These results are also in close agreement with result as obtained previously by some author's such as Ram (1978), Schmitt (1972) and Singh et al. (2012, 2014).

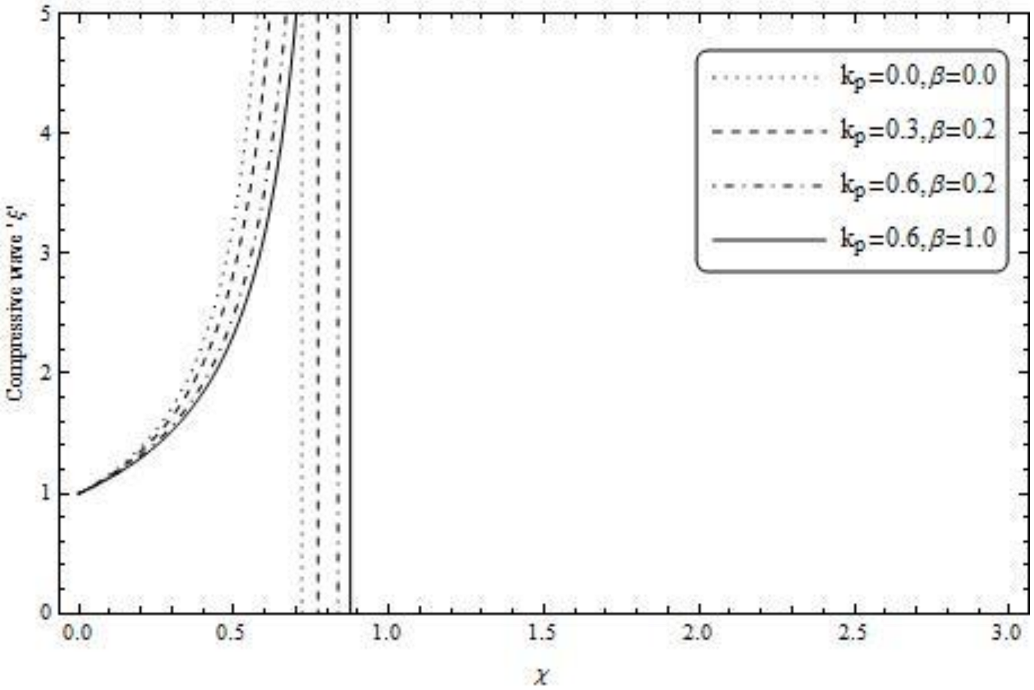

Fig.5.1 Effect of variation of mass fraction (k_p) and β on the growth of compressive waves for planar flow with $\mu = -0.5$.

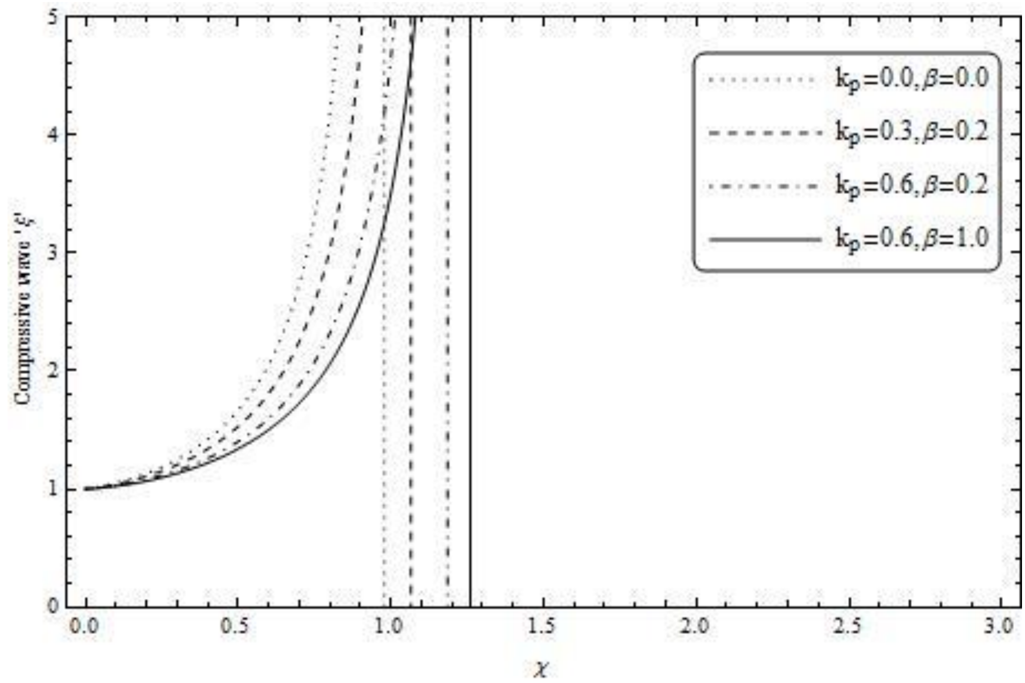

Fig.5.2 Effect of variation of mass fraction (k_p) and β on the growth of compressive waves in cylindrically symmetric flow with $\mu = -0.5$.

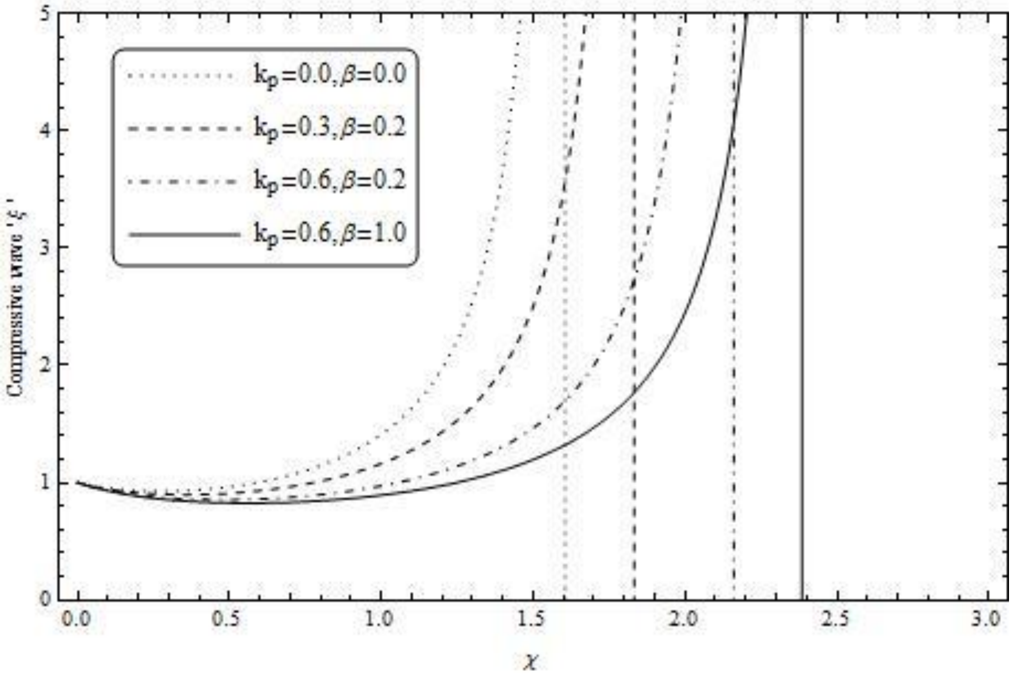

Fig.5.3 Effect of variation of mass fraction (k_p) and β on the growth of compressive waves in spherically symmetric flow with $\mu = -0.5$.

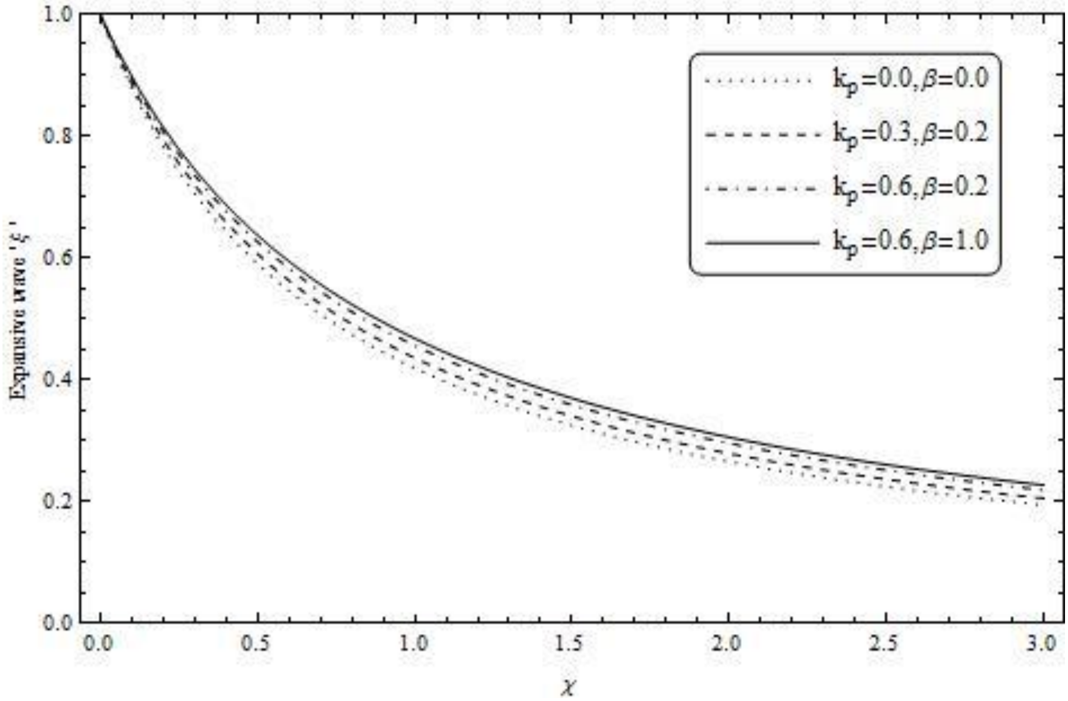

Fig.5.4 Effect of variation of mass fractions (k_p) and β on the decay of expansive waves for planar flow with $\mu = 0.5$.

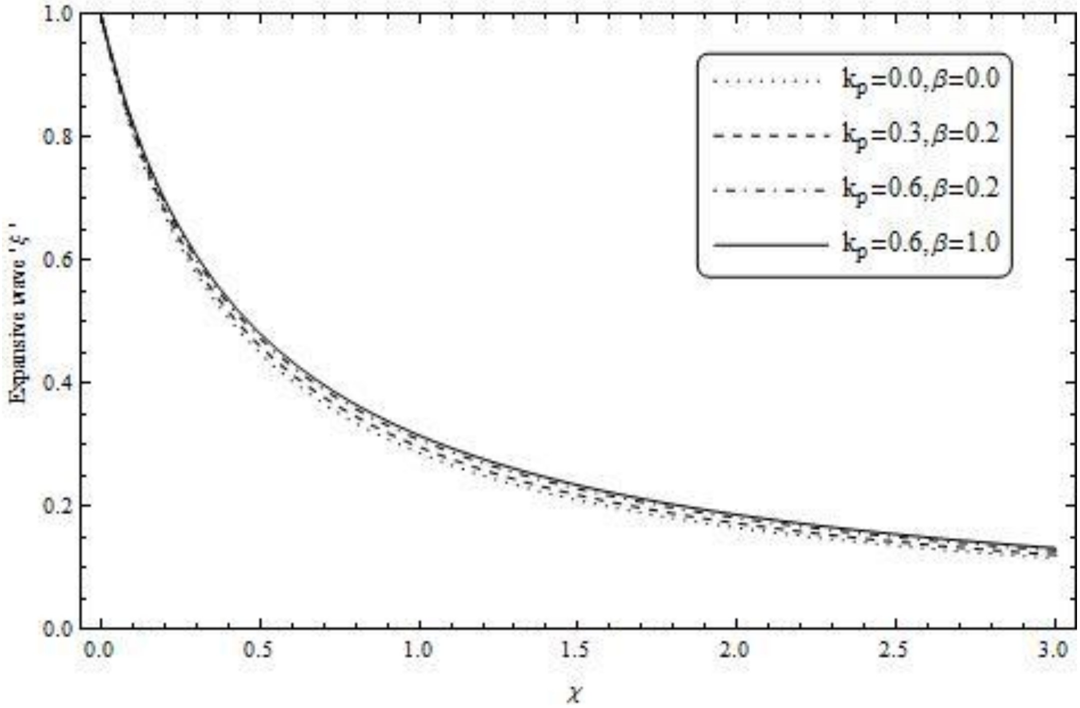

Fig.5.5 Effect of variation of mass fraction (k_p) and β on the decay of expansive waves in cylindrically symmetric flow with $\mu = 0.5$.

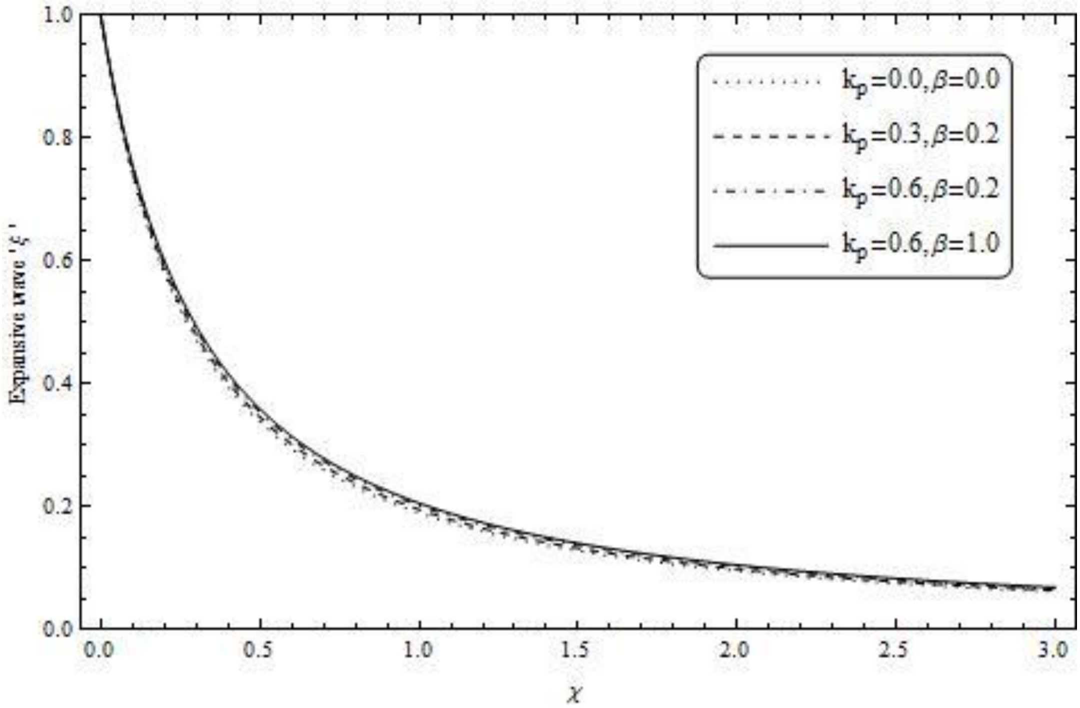

Fig.5.6 Effect of variation of mass fraction (k_p) and β on the decay of expansive waves in spherically symmetric flow with $\mu = 0.5$.

5.6 Conclusions

The evolution and decay behaviour of acceleration waves in one dimensional unsteady, inviscid, compressible planar and non-planar flow in dusty gas is examined. The solution of the problem of the propagation of acceleration waves along the characteristics path, by using the characteristic of the governing system of equations as a reference coordinate, is derived. It has been shown that a linear solution in the characteristic plane may exhibit non-linear behaviour in the physical plane. The transport equation governing the evolution and decay behaviour of acceleration waves in dusty gas is derived. It may be noted here that all compressive waves, irrespective of their initial strength, terminate into shock wave. It has been observed that the increasing value of mass fraction (k_p) of the dust particles causes to delay the shock formation. Also a similar effect of the increasing values of β was observed on the shock formation time. It is shown that in all the three cases i.e. in planar, cylindrically symmetric and spherically symmetric flows the amplitude of the expansive wave fronts decays and ultimately damped out. Also the presence of dust particles causes to delay the decay process of expansive waves as compared to the dusty free gas.