Chapter 3

Resonantly interacting non-linear waves in a van der Waals gas

3.1 Introduction

A large number of physical phenomena such as nuclear explosions, chemical explosion, bomb blast, collision of two or more galaxies, supersonic flow etc.; happening in nature may be described in terms of mathematical model described by quasilinear hyperbolic partial differential equations (Whitham (1974), Courant and Hilbert (1962), Anile et al. (1993)). In non-linear science and engineering, the study of shock waves, acceleration waves, weakly non-linear waves, interaction of shock waves etc. has been of great interest for researchers since long time, due to its wide application in nuclear physics, plasma physics, astrophysical sciences and interstellar gas masses. Asymptotic method for the solution of weakly non-linear hyperbolic waves has received considerable attention of researchers in the last few decades. A significant contribution on small amplitudes non-linear waves has been made by Choquet-Bruhat (1969) in which they have considered a shockless solution of system of hyperbolic equations that depends only on single phase function. Earlier Germain (1971) has studied the single phase progressive wave solution for the weekly non-linear waves. Latter, some authors such as Fusco (1982), Fusco and Engelbrecht (1984), Sharma et al. (1987, 1989) and Radha et al. (1995) have used the progressive wave approach to analyze the wave propagation problem in various gasdynamic regimes. In this reference, the contribution of various

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authors like Becker (1972), Ockendon and Spence (1969), Chu (1970), Moodie et al. (1991) and He and Moodie (1995) are worth mentioning. Hunter and Keller (1983) proposed a general non-resonant multi wave mode theory based on the weakly non-linear geometrical optics, which has led to several important generalizations by Majda and Rosales (1984) and then Hunter et al. (1986) in which they have derived the uniformly valid asymptotic theory of resonantly interacting high frequency waves for non-linear hyperbolic system of equations.

If the temperature of the gas is very high and density is too low then the ideal gas model is no longer valid therefore in this situation the alternative of the ideal gas model is a modified van der Waals gas model. The study of shock related phenomena in van der Waals gas is more complex as compared to general ideal gas model. In recent years the study of shock related phenomena in van der Waals gas have received great attention of scientist and engineers due its application in space physical science such as chemical processes, nuclear reaction, aerospace engineering and sciences etc. see (Smirnov et al. (2014, 2017a, 2017b),). Zhao et al. (2011) has studied a complete classification of shock waves and shock splitting phenomena together with their admissibility in van der Waals fluid. Further, the theory of progressive wave is used to study the finite and moderately small amplitude disturbances in van der Waals gas see Ambika (2014). Singh et al. (2015) have used the progressive wave approach to analyze the main feature of weakly non-linear waves in non-ideal gas. Further Nath et al. (2017) used the same technique to analyze the feature of weakly non-linear waves in van der Waal gas. Sharma et al. (2005) have studied the wave interaction in a non-equilibrium gas flow by using the method of multiple time scale. Using the same technique Arora et al. (2008, 2009) have obtained the asymptotic solution of system of hyperbolic equations in various material media. Further the method of weakly non-linear geometrical optics is used to study the evolution of fast magnetosonic waves of high frequency propagating into an axisymmetric equilibrium plasma and the geometry of fast magnetosonic rays Manuel (2016a, 2016b).

The aim of the present chapter is to study the propagation of weekly non-linear small amplitude high frequency asymptotic waves for one dimensional unsteady, compressible flow in a non-ideal gas followed by van der Waals equation of state with generalized geometry. The resonantly interacting multi wave theory is used to derive the evolution equation governing the growth and decay behaviour of wave amplitude for the nonplanar waves. Also the interaction coefficients are determined which measure the strength of coupling between different waves mode. The small amplitude high frequency asymptotic solution of one dimensional hyperbolic system of equations for the planar and nonplanar flow in van der Waals gas is obtained. Further, the existence of shock and its path (parametrically) in non-ideal gas followed by van der Waals equation of state is examined.

3.2 Governing equations

Let us consider a general class of real gases whose equation of state is given as

$$\left(p+\frac{a}{V^2}\right)\left(V-b\right)=RT\,, \qquad (3.2.1)$$

where p is the pressure, V is the volume, R is the universal gas constant, T is the absolute temperature. Here a represents the amount of intermolecular force of attraction between the gas particles and b represents the excluded volume of the gas. It is well known that the gases behave like real gas at low temperature and high pressure. For the given equation of state (3.2.1), the expression of internal energy, with the help of $R=\left(\gamma-1\right)C_v$ where C_v is the specific heat at constant volume, may be written as

$$e = \frac{\left(p + a\rho^2\right)\left(1 - b\rho\right) - a\rho^2\left(\gamma - 1\right)}{\left(\gamma - 1\right)\rho} \quad , \tag{3.2.2}$$

where ρ is the density of the gas, γ is the adiabatic exponent and is defined as $\gamma = C_p/C_v$ where C_p is the specific heat at constant pressure. Here, if we take $a = 0$ and $b = 0$ in equation (3.2.1) then the equation of state of van der Waals gas turns to the equation of state of ideal gas.

Thus the governing equations for the one-dimensional compressible, inviscid, unsteady, planar and non-planar flow in van der Waals gas may be written as (Whitham (1974), Courant and Hilbert (1962), Thompson (1849))

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \frac{n\rho u}{x} = 0, \tag{3.2.3a}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \rho^{-1} \frac{\partial p}{\partial x} = 0, \tag{3.2.3b}$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \rho A^2 \left(\frac{\partial u}{\partial x} + \frac{nu}{x}\right) = 0. \tag{3.2.3c}$$

where u is the gas velocity, t is the time, x is the spatial coordinate. Here $(n = 0)$ represents the planar flow, $(n = 1)$ represents the cylindrically symmetric flow and $(n = 2)$ represents the spherically symmetric flow. Also A represents the sound velocity and is defined as

$$A = \left(\frac{\gamma p + a\rho^2\left(\gamma - 2 + 2b\rho\right)}{\rho\left(1 - b\rho\right)}\right)^{1/2} \quad . \tag{3.2.4}$$

Now equation (3.2.3) can be written in matrix form as

$$\frac{\partial U}{\partial t} + M \frac{\partial U}{\partial x} + N = 0, \tag{3.2.5}$$

where $U=\left(\rho,\, u,\, p\right)^{tr}$, $N=\left(n\rho u/x,\, 0,\, \rho A^2 nu/x\right)^{tr}$ and M is the coefficient matrix of order 3×3 having components M^{ij} and the non zero components of M are as

$$M^{11}=M^{22}=M^{33}=u,$$

$$M^{12}=\rho,\ \ M^{23}=\rho^{-1} \text{ and } M^{32}=\rho A^2. \qquad (3.2.6)$$

The eigen values of the system (3.2.5) are given as $\lambda_1=u+A$, $\lambda_2=u$, $\lambda_3=u-A$, therefore the system of partial differential equation (3.2.5) is strictly hyperbolic in nature and admits 3 characteristics curves, among them two represent the waves propagating in the $+x$ direction and $-x$ with the shock speed $u+A$ and $u-A$ respectively. While other represents the particle path propagating with the shock speed u. Here we are considering only those waves which are propagating through the constant state U_0 and is defined as $U_0=\left(\rho_0,\, 0,\, p_0\right)$. The characteristics speed at the constant state U_0 are given by $\lambda_1'=A_0$, $\lambda_2'=0$ and $\lambda_3'=-A_0$. Here the subscript 0 denotes the evaluation at $U=U_0$.

3.3 Interaction of high frequency waves

In this section the method of multiple time scale will be used to derive the asymptotic solution in the form of small amplitude high frequency waves for the system of equations (3.2.5), when the attenuation time scale (τ_{at}) is large as compared to the characteristic time scale (τ_{ch}), that is $\varepsilon=\tau_{ch}/\tau_{at}<<1$.

The left and right eigenvectors of M_0 corresponding to the eigenvalues $\lambda_1'=A_0, \lambda_2'=0$, $\lambda_3'=-A_0$ are denoted by $L^{(i)}$ and $R^{(i)}$ ($i=1,2,3$) respectively. These eigenvectors satisfy the normalization conditions $L^{(i)}R^{(j)}=\delta_{ij}$ $(1\le i\le 3, 1\le j\le 3)$, where δ_{ij} is

Krönecker delta. Therefore in view of above conditions the left and right eigenvectors may be written as

$$L^{(1)}=\left(0,\ \rho_0/2A_0\,,\ 1/2A_0^{\,2}\right),\ \ R^{(1)}=\left(1,\ A_0/\rho_0\,,\ A_0^{\,2}\right)^{tr}\ ,$$

$$L^{(2)}=\left(-A_0^{\,2},\ 0,\ 1\right),\ R^{(2)}=\left(-1/A_0^{\,2},\ 0,\ 0\right)^{tr}, \tag{3.3.1}$$

$$L^{(3)}=\left(0,\ -\rho_0/2A_0\,,\ 1/2A_0^{\,2}\right),\ R^{(3)}=\left(1,\ -A_0/\rho_0\,,\ A_0^{\,2}\ \right).$$

We now seek the asymptotic solution of (3.2.5) in the following form with $\varepsilon \to 0$

$$U\left(x,t\right)=U_0+\varepsilon U_1\left(x,t,\tilde{\xi}\right)+\varepsilon^2 U_2\left(x,t,\tilde{\xi}\right)+O\left(\varepsilon^3\right), \tag{3.3.2}$$

where U_0 is the constant solution of system of equations (3.2.5), U_1 is a smooth bounded function of its arguments while U_2 is a bounded function in $\left(x,t\right)$ coordinate within some bounded region of interest having at most sub-linear growth in $\tilde{\xi}$ as $\tilde{\xi}\to\pm\infty$ (Sharma et al. (2005)). Here $\tilde{\xi}=\left(\xi_1,\ \xi_2,\ \xi_3\right)$ denotes the "fast variable" and is characterized as $\tilde{\xi}=\tilde{\psi}/\varepsilon$, where $\tilde{\psi}=\left(\psi_1,\psi_2,\psi_3\right)$ indeed $\psi_i\left(1\le i\le 3\right)$ is the phase function of i^{th} wave associated with the characteristic speed λ_i'. Using the Taylor series expansion of M and N, in powers of ε about the constant state U_0 and equation (3.3.2) in equation (3.2.5) and replace the partial derivatives $\partial/\partial X$ (X will be either x or t) by $\partial/\partial X+\varepsilon^{-1}\sum_{i=1}^{3}\left(\partial\psi_i/\partial X\right)\partial/\partial\xi_i$ and equating the coefficients of ε^0 and ε^1 to zero, we have

$$\sum_{i=1}^{3}\left(I\,\frac{\partial\psi_i}{\partial t}+M_0\,\frac{\partial\psi_i}{\partial x}\right)\frac{\partial U_1}{\xi_i}=0\,, \tag{3.3.3}$$

$$\sum_{i=1}^{3}\left(I\frac{\partial \psi_i}{\partial t}+M_0\frac{\partial \psi_i}{\partial x}\right)\frac{\partial U_2}{\partial \xi_i}= \\ -\frac{\partial U_1}{\partial t}-M_0\frac{\partial U_1}{\partial t}-\left(U_1.\nabla N\right)_0-\sum_{i=1}^{3}\frac{\partial \psi_i}{\partial x}\left(U_1.\nabla M\right)_0\frac{\partial U_1}{\partial \xi_i}, \tag{3.3.4}$$

where I is an identity matrix of order 3×3, ∇ is the gradient operator with respect to the flux vector U. Since all the phase functions $\psi_i\left(i=1,\,2,\,3\right)$ satisfy the eikonal equation

$$\det\left(I\frac{\partial \psi_i}{\partial t}+M_0\frac{\partial \psi_i}{\partial x}\right)=0. \tag{3.3.5}$$

Here "det" represents the determinant. Now let us take a simplest phase function of the above equation as

$$\psi_i\left(x,t\right)=x-\lambda_i' t,\quad 1\le i\le 3. \tag{3.3.6}$$

From equation (3.3.3) we conclude that for each phase function ψ_i the terms $\partial U_1/\partial \xi_i$ are parallel to the right eigenvectors R^i of the matrix M_0, therefore we have

$$U_1=\sum_{i=1}^{3}\mu_i\left(x,t,\xi_i\right)R^{(i)}, \tag{3.3.7}$$

where $\mu_i=\left(L^{(i)}.U_1\right)$ is a scalar function known as wave amplitude that depends on the i^{th} fast variable ξ_i. The dependency of μ_i on ξ_i describes the wave form whether it is an oscillatory wave or a pulse. Here we consider that $\mu_i\left(x,t,\xi_i\right)$ has zero mean value with respect to the fast variables ξ_i, that is

$$\lim_{P\to\infty}\frac{1}{2P}\int_{-P}^{P}\mu_i\left(x,t,\xi_i\right)d\xi_i=0. \tag{3.3.8}$$

Now we use equation (3.3.7) and solve the equation (3.3.4) for U_2. To begin with we write

$$U_2 = \sum_{j=1}^{3} n_j R^{(j)} . \tag{3.3.9}$$

Using equation (3.3.9) in equation (3.3.4) and then pre multiplying the resulting equation by $L^{(i)}$ yields the following decoupled inhomogeneous system of first order partial differential equations

$$\begin{aligned} &\sum_{j=1}^{3}\left(\lambda_i' - \lambda_j'\right)\frac{\partial n_i}{\partial \xi_j} = -\frac{\partial \mu_i}{\partial t} - \lambda_i'\frac{\partial \mu_i}{\partial x} - L^{(i)}\left(U_1.\nabla N\right)_0 \\ &-\sum_{j=1}^{3} L^{(i)}\left(U_1.\nabla M\right)_0 \frac{\partial U_1}{\partial \xi_j}, \quad 1 \le i \le 3 \end{aligned} . \tag{3.3.10}$$

Here $\partial/\partial t + \lambda_i' \partial/\partial x$ represents the Ray derivative.

The i^{th} characteristics ordinary differential equation in equation (3.3.10) is given as

$$\dot{\xi}_j = \lambda_i' - \lambda_j' \quad \text{for } i \ne j , \quad \dot{\xi}_i = 0 , \ \dot{n}_i = K_i , \tag{3.3.11}$$

where $K_i\left(x, t, \xi_1, \xi_2, \xi_3\right) = -\dfrac{\partial \mu_i}{\partial t} - \lambda_i'\dfrac{\partial \mu_i}{\partial x} - L^{(i)}\left(U_1.\nabla N\right)_0 - \sum_{j=1}^{3} L^{(i)}\left(U_1.\nabla M\right)_0 \dfrac{\partial U_1}{\partial \xi_j}$.

Now we will determine the asymptotically average of equation (3.3.10) along the characteristics and then appeal to the sub linearity of U_2 in ξ, which makes sure that the expression (3.3.2) does not contain any secular terms. Since ξ_i 's are constant along the characteristics, thus along with this, the vanishing asymptotic mean value of $\dot{n}_i$ along the characteristics implies that the wave amplitudes $\mu_i\left(1 \le i \le 3\right)$, satisfy the following system of coupled integro-differential equations (Hunter et al. (1986))

$$\begin{aligned} &\frac{\partial \mu_i}{\partial t} + \lambda_i'\frac{\partial \mu_i}{\partial x} + \alpha_i \mu_i + \Pi_{ii}^{i} \mu_i \frac{\partial \mu_i}{\partial \xi_i} \\ &+ \sum_{i \ne j \ne k} \Pi_{jk}^{i} \lim_{P \to \infty} \frac{1}{2P} \int_{-P}^{P} \mu_j\left(\xi_i + \left(\lambda_i' - \lambda_j'\right)s\right) \tilde{\mu}_k\left(\xi_i + \left(\lambda_i' - \lambda_k'\right)s\right) ds = 0 \end{aligned} , \tag{3.3.12}$$

where $\tilde{\mu}_k = \partial \mu_k / \partial \xi_k$ and the coefficients α_i and Π_{jk}^{i} are given as

$$\alpha_i = L^{(i)}\left(R^{(i)}.\nabla N\right)_0, \qquad \Pi^i_{jk} = L^{(i)}\left(R^{(j)}.\nabla M\right)_0 R^{(k)}. \tag{3.3.13}$$

Here it may be observed that for planar flow $(n=0)$, the coefficients α_i becomes zero as the source terms in the system of equations becomes zero. Further, in absence of van der Waals parameters a,b the equation (3.3.12) reduces to the similar equation as derived in [17]. Thus the coefficients α_i in equation (3.3.12) describe the growth or decay behaviour of the wave amplitude μ_i due to non-planar wave form and the non-idealness of the medium. Further the interaction coefficients Π^i_{jk}, which are asymmetric in j and k, measure the strength of coupling between the j^{th} and k^{th} wave modes ($j \neq k$) that can generate a i^{th} wave$\left(i \neq j \neq k\right)$. Also the coefficients Π^i_{ii} refer to the non-linear self-interaction which are non-zero for genuinely non-linear waves and are zero for linearly degenerate waves. Further it is observed that, if all the coupling coefficients Π^i_{jk} $\left(i \neq j \neq k\right)$ are zero or the integral in (3.3.12) vanishes, then the wave will not resonate and therefore the equation (3.3.12) reduces to the system of coupled Burger's equation.

The coefficients α_i, Π^i_{jk} and Π^i_{ii}, as given in equation (3.3.13), gives the qualitative picture of the non-linear interaction process present in the system under consideration and can be determined with the formulae as given in (3.3.13). Thus the coefficients are given by

$$\alpha_1 = \frac{nA_0}{2x}, \quad \alpha_2 = 0, \quad \alpha_3 = -\frac{nA_0}{2x},$$

$$\Pi^1_{23} = -\Pi^3_{21} = \frac{2a\rho_0\left(\gamma - 2 + 3b\rho_0\right) + A_0^2}{2A_0^3\rho_0\left(1 - b\rho_0\right)} = \omega \text{ (Say)},$$

$$\Pi_{13}^{2}=-\Pi_{31}^{2}=-\frac{2a\rho_0 A_0\left(\gamma-2+3b\rho_0\right)+\left(\gamma-1+2b\rho_0\right)A_0^3}{\rho_0\left(1-b\rho_0\right)}, \tag{3.3.14}$$

$$\Pi_{32}^{1}=\Pi_{12}^{3}=\Pi_{22}^{2}=0,$$

and

$$\Pi_{11}^{1}=-\Pi_{33}^{3}=\frac{2a\rho_0\left(\gamma-2+3b\rho_0\right)+\left(\gamma+1\right)A_0^2}{2A_0\rho_0\left(1-b\rho_0\right)}=\Pi \text{ (Say).}$$

Further the resonant asymptotic equation (3.3.12) can now be written as, after some simplification

$$\frac{\partial\mu_1}{\partial t}+A_0\frac{\partial\mu_1}{\partial x}+\frac{nA_0}{2x}\mu_1+\Pi\mu_1\frac{\partial\mu_1}{\partial\xi_1}+\lim_{P\to\infty}\frac{1}{2P}\int_{-P}^{P}\zeta\left(x,t,\frac{\xi_1+\psi}{2}\right)\mu_3\left(x,t,\psi\right)d\psi=0, \tag{3.3.15}$$

$$\frac{\partial\mu_2}{\partial t}=0, \tag{3.3.16}$$

$$\frac{\partial\mu_3}{\partial t}-A_0\frac{\partial\mu_3}{\partial x}-\frac{nA_0}{2x}\mu_3-\Pi\mu_3\frac{\partial\mu_3}{\partial\xi_1}-\lim_{P\to\infty}\frac{1}{2P}\int_{-P}^{P}\zeta\left(x,t,\frac{\xi_3+\psi}{2}\right)\mu_1\left(x,t,\psi\right)d\psi=0, \tag{3.3.17}$$

where ζ is kernel and is defined as

$$\zeta\left(x,t,\frac{\xi+\psi}{2}\right)=\frac{\omega}{2}\frac{\partial\mu_2}{\partial\xi_2}\left(x,t,\frac{\xi+\psi}{2}\right). \tag{3.3.18}$$

The integral average term in equation (3.3.15) exhibits contribution to the wave amplitude μ_1 due to the non-linear interactions of the wave field μ_2 with the wave field μ_3. Similarly the integral average term in equation (3.3.17) exhibits contribution to the wave amplitude μ_3 due to the non-linear interaction of the wave field μ_2 with the wave field μ_1.

Let us suppose the initial value of μ_i at time $t=0$ is $\mu_i^0\left(x,\xi_i\right)$. Therefore from equation (3.3.16) we have $\mu_2\left(x,t,\xi_1\right)=\mu_1^0\left(x,\xi_1\right)$ and consequently the system of equations (3.3.15)-(3.3.17) transforms to a pair of equations for the wave fields μ_1 and μ_3 coupled through the linear integral operator involving the kernel

$$\zeta(x,t,\xi)=\frac{\omega}{2}\frac{\partial\mu_2^0}{\partial\xi_2}(x,\xi). \tag{3.3.19}$$

If the initial data $\mu_i^0(x,\xi)$ are periodic function of 2π of the phase variable ξ, then the pair of resonant asymptotic equations takes the following form

$$\frac{\partial\mu_1}{\partial t}+A_0\frac{\partial\mu_1}{\partial x}+\frac{nA_0}{2x}\mu_1+\Pi\mu_1\frac{\partial\mu_1}{\partial\xi_1}+\frac{1}{2\pi}\int_{-\pi}^{\pi}\zeta\left(x,t,\frac{\xi+\psi}{2}\right)\mu_3(x,t,\psi)\,d\psi=0, \tag{3.3.20}$$

$$\frac{\partial\mu_3}{\partial t}-A_0\frac{\partial\mu_3}{\partial x}-\frac{nA_0}{2x}\mu_3-\Pi\mu_3\frac{\partial\mu_3}{\partial\xi_3}-\frac{1}{2\pi}\int_{-\pi}^{\pi}\zeta\left(x,t,\frac{\xi+\psi}{2}\right)\mu_1(x,t,\psi)\,d\psi=0, \tag{3.3.21}$$

where ζ is defined as in equation (3.3.19).

3.4 Non-linear geometrical acoustics solution

The approximate asymptotic solution (3.3.2) of the system of hyperbolic equations (3.2.5) satisfying small amplitude oscillating initial data

$$U(x,0)=U_0+\varepsilon U_1^0(x,x/\varepsilon)+O(\varepsilon^2), \tag{3.4.1}$$

will be non-resonant if the functions $U_1^0(x,x/\varepsilon)$ are smooth with a compact support (Majda et al. (1984)). Indeed the expansion (3.3.2), with U_1 as given by equation (3.3.7) is uniformly valid to the leading order until shock waves have formed in the solution.

The characteristic equations are

$$\frac{d\xi_i}{dx}=\frac{\Pi\mu_r}{A_0}\ ,\quad \frac{dt}{dx}=\frac{\tau_i}{A_0}, \tag{3.4.2}$$

where τ_i takes the value $+1$ and -1 according as $i=1$ or $i=3$ respectively.

Now in view of (3.4.2), the decoupled equations (3.3.20) and (3.3.21) may be written as

$$\frac{d\mu_i}{dx} = -\frac{n\mu_i}{2x}. \tag{3.4.3}$$

Equation (3.4.3) yields on integration

$$\mu_i = \mu_i^0\left(s_i, \chi_i\right)\left(x/s_i\right)^{-n/2}, \tag{3.4.4}$$

along the rays $s_i = x - \tau_i A_0 t =$ constant, here the function μ_i^0 is obtained from the initial condition (3.4.1), and the "fast' variable χ_i parameterizes the set of characteristics curve $(3.4.2)_1$.

Therefore we obtain from equation (3.4.2)

$$\chi_i = \xi_i - \tau_i \Pi \mu_i^0\left(s_i, \chi_i\right) I_i^{(n)}\left(t\right), \tag{3.4.5}$$

where $I_i^{(n)}(t) = \int_0^t \left(1 + \frac{\tau_i A_0 t}{s_i}\right)^{-n/2} dt$.

Thus the solution of system of equations (3.2.5) satisfying equation (3.4.1) in view of $U_1^0\left(x, x/t\right)$ having compact support, is obtained as

$$\begin{aligned}\rho\left(x,t\right) &= \rho_0 + \varepsilon x^{-n/2}\left(\mu_1^0\left(s_1,\chi_1\right)\left(x - A_0 t\right)^{n/2} + \mu_3^0\left(s_3,\chi_3\right)\left(x + A_0 t\right)^{n/2}\right) \\ &- \frac{\varepsilon}{A_0^2}\mu_2^0\left(x, x/\varepsilon\right) + O\left(\varepsilon^2\right)\end{aligned}, \tag{3.4.6a}$$

$$u\left(x,t\right) = \varepsilon x^{-n/2}\frac{A_0}{\rho_0}\left(\mu_1^0\left(s_1,\chi_1\right)\left(x - A_0 t\right)^{n/2} - \mu_3^0\left(s_3,\chi_3\right)\left(x + A_0 t\right)^{n/2}\right) + O\left(\varepsilon^2\right), \tag{3.4.6b}$$

$$p\left(x,t\right) = p_0 + \varepsilon x^{-n/2} A_0^2\left(\mu_1^0\left(s_1,\chi_1\right)\left(x - A_0 t\right)^{n/2} + \mu_3^0\left(s_3,\chi_3\right)\left(x + A_0 t\right)^{n/2}\right) + O\left(\varepsilon^2\right), \tag{3.4.6c}$$

where the "fast" variable χ_i , as given in equation (3.4.5) are taken in such a way that at $t = 0$, $\chi_i = x/\varepsilon$, and the initial values for μ_i $\left(1 \le i \le 3\right)$ are obtained from the above solution (3.4.6) specified at $t = 0$ as

$$\mu_1^0\left(x,\chi_1\right) = \frac{\rho_0}{2A_0}u_1^0\left(x,\chi_1\right) - \frac{1}{2A_0^2}p_1^0\left(x,\chi_1\right), \tag{3.4.7a}$$

$$\mu_2^0\left(x, x/\varepsilon\right) = -A_0^2\rho_1^0\left(x, x/\varepsilon\right) + p_1^0\left(x, x/\varepsilon\right), \tag{3.4.7b}$$

$$\mu_3^0\left(x, \chi_3\right) = -\frac{\rho_0}{2A_0}u_1^0\left(x, \chi_3\right) + \frac{1}{2A_0^2}p_1^0\left(x, \chi_3\right). \tag{3.4.7c}$$

Thus we have found the complete solution of system of equations (3.2.5) and (3.4.1) and any multi valued overlap in this solution has to be resolved by introducing shock waves into the solution.

As we know that a shock wave may take place in the flow region and once it is formed it will propagate by separating the portion of the continuous regions. At shocks, the solution, which we have found above, satisfies the R-H jump conditions for the shock location $\xi_i^s\left(t\right)$ in the $t-\xi_i$ plane. Thus it can be shown that the shock location $\xi_i^s\left(t\right)$ satisfies the following relation (Hunter and Keller (1983))

$$\frac{d\xi_i^s}{dt} = \frac{1}{2}\Pi_{ii}^i\left(\mu_i^{(-)} + \mu_i^{(+)}\right), \quad i = 1, 3, \tag{3.4.8}$$

which is the shock speed in the $t-\xi_i$ plane. Here $\mu_i^{(-)}$ and $\mu_i^{(+)}$ denotes the value of the μ_i just ahead and behind the shock respectively. In the undisturbed region the value of $\mu_i^{(-)}$ will be zero for the shock front. In view of equation (3.4.4) and omitting the superscripts on ξ_i^s and $\mu_i^{(+)}$ we get

$$\frac{d\xi_i}{dt} = \frac{\Pi}{2}\tau_i\mu_i^0\left(s_i, \chi_i\right)\left(\frac{x}{s_i}\right)^{-n/2}. \tag{3.4.9}$$

With the help of equation (3.4.9) and equation (3.4.5) we have the following relation between χ_i and t on the shock

$$I_i^{(n)}\left(t\right) = -\left(\frac{2\tau_i}{\Pi\left(\mu_i^0\right)^2}\right)\int_0^{\chi_i}\mu_i^0\left(t\right)dt. \tag{3.4.10}$$

Further in view of equation (3.4.5), equation (3.4.10) yields the following relation which determines the path of shock wave parametrically.

$$\xi_i = \chi_i - \frac{2}{\mu_i^0} \int_0^{\chi_i} \mu_i^0 (t) \, dt \, . \tag{3.4.11}$$

Note that if $\mu_i^0 \neq 0$, then shock starts quickly right from the origin (Majda and Rosales (1984), Arora (2008)).

3.5 Results and discussion

The method of multiple time scale is used to find the small amplitudes high frequency asymptotic solution to the system of hyperbolic partial differential equations describing one-dimensional unsteady, compressible planar, cylindrically symmetric and spherically symmetric flow of a van der Waals gas. By using the theory of weakly non-linear geometrical acoustics those conditions are discussed in which wave interaction occur resonantly. The transport equations which we have derived here constitute a system of inviscid Berger's equation with quadratic non-linearity coupled through a linear integral operator with kernel which is known. The adjacent coefficients, which appear in the transport equations, provide a measure of coupling between the various modes and give the qualitative information about the interaction process which is happening there. Note that for the planar flow $(n = 0)$, the transport equation which we have derived here, are identical with the transport equation derived previously by some author's (Majda and Rosales (1984), Hunter and Majda (1986)). The effect of van der Waals parameter enters into the solution trough the parameter A_0. It is evident from equations (3.4.6) that the increasing values of b (keeping a fixed) causes the density of the small amplitude high frequency waves to increase and also the increasing values b have the same effect on the velocity and pressure of the small amplitude high frequency waves. While increasing the value of a (keeping b fixed) causes to decrease the density of the small amplitude high frequency waves. Similar effect of a are seen on the velocity and pressure of the small amplitude high frequency waves. From the above results it is found that the van der Waals parameters play an important role on the solution of the hyperbolic system of partial differential equations. In nonmagnetic case and without relaxation effect in ideal gas the results which we have obtained here are in close agreement with the results obtained previously by some authors (Nath et al. (2017), Sharma et al. (2005), Arora (2008)). It has been observed here that the wave fields associated with the particle path do not interact with each other. However the wave fields interact with an acoustic wave field to produce resonant contribution toward the other acoustic field. The acoustic wave fields may or may not interact but in either case their total contribution towards entropy field must be zero. Thus the system of hyperbolic equations reduces to a pair of resonant asymptotic equation for the acoustic wave fields. For a non-resonant multi wave mode as presented in the paper of Hunter and Keller (1983), the reduced system of transport equations get decoupled with vanishing integral average term and the occurrence of shock in the acoustic wave fields are discussed. Also the existence of shock and their location in the van der Waals gas is discussed. It is observed that in a contracting piston motion having spherical symmetry, a shock is always formed before the formation of focus no matter how small be the initial wave amplitudes, this is in contrast with the corresponding cylindrical situation where a shock forms before the focus only if the initial amplitude exceeds a critical value which is evident from equation (3.4.5).

3.6 Conclusions

In the present article we use the method of multiple time scales to obtain the small amplitude high frequency asymptotic solution to the system of hyperbolic partial differential equations describing one-dimensional unsteady, compressible planar and non-planar flow in a van der Waals gas. Those conditions are discussed in which wave interaction occur resonantly. Transport equations for the wave amplitude along the rays of the system of hyperbolic equations are derived; which constitute a system of inviscid Berger's equation with quadratic non-linearity coupled through a linear integral operator with known kernel. The growth and decay behaviour of wave amplitudes for the nonplanar waves are studied. It is observed that the increasing values of van der Waals parameter b is to enhance the values of flow parameters while the increasing values of van der Waals parameter a have reverse effect on flow parameters. Also the existence of shock and their location in the van der Waals gas is discussed.