Chapter 4

Galerkin and Collocation Methods for Weakly Singular Fractional Integro-Differential Equations

4.1 Introduction

Theory of fractional integration and differentiation is applicable in various fields. Specially, FIDEs has been found applicable in fluid dynamics, rheology, biology, porous media, physics [44, 114-117]. As the exact solution of most of the FIDEs is not easy to find, so it is necessary to develop numerical methods for the approximation of FIDEs.

The main motivation of this chapter is to develop some efficient numerical methods to approximate the solution of FIDEs [43] of the form,

$$\mathcal{D}^{\upsilon} w(x) = g(x) + f(x)w(x) + \int_0^x \kappa(x,u)\mathcal{G}\big(w(u)\big)du \ \ ,0 < \upsilon < 1, \quad (4.1)$$

with given initial condition,

$$w(0) = w_0, \quad (4.2)$$

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where $w(x)$ is to find out and $g(x), f(x)$ are given functions. Here $\mathcal{D}^{\upsilon}$ is fractional derivative in Caputo sense (Definition 1.4). $\mathcal{G}$ is some operator which may be linear or nonlinear and $\kappa(x, u)$ is weakly singular kernel defined by,

$$\kappa(x, u) = (x - u)^{-\mu}, \qquad 0 < \mu < 1. \tag{4.3}$$

This type of equations has been solved using various approaches. The present chapter is based on the comparison of Jacobi collocation and Jacobi Galerkin methods for FIDEs. In last decade, collocation and Galerkin methods attracted mathematician as well engineers. Here, we cite some most recent works which investigate the collocation and Galerkin methods for integral equations. These recent works such as a spectral iterative method [117], collocation method [43], and meshless discrete Galerkin method [94] are devoted to solve the different type of integral equations.

In this chapter, Jacobi collocation and Jacobi Galerkin methods are presented to solve linear and nonlinear FIDEs. The main focus of this chapter is to consider FIDEs with the weakly singular kernel as these type of problem is not easy to solve analytically. First, we introduce a change of variables so that the exact solution becomes sufficiently smooth near the origin. After that exact solution is approximated in terms of Jacobi polynomials with unknown coefficients, and by using collocation and Galerkin methods the FIDEs are reduced to a system of equations. This makes easy solving such problem and also accelerate the computational efficiency. Such behaviour were also addressed by Mustapha and co-authors in their research works [118-121]. In these works, non-uniform meshes have been employed to reimburse the singular behaviour of the solution near origin by concentrating the mesh elements near zero. High order convergences were also obtained.

The chapter is arranged in the following manner. In section 2, some elementary definitions and lemmas are presented. In section 3, an introduction of Jacobi polynomials is added and it is shown that any continuous function can be approximated as a series of Jacobi polynomials and this series converges uniformly on the respective domain. Existence and uniqueness of the solution of Eq. (1) with the initial condition given by Eq. (2) is discussed in section 4. Section 5, describes the variable transformation which removes the singularity of the solution at the origin. Also, collocation and Galerkin methods are discussed in this section. Numerical experiments with different parameters for the linear and nonlinear case are performed in section 6. The last section concludes the chapter.

4.2 Approximation of Function

To calculate the solution of Eq. (4.1) in [0,1], we use shifted Jacobi polynomials $j_{a,b}^{Q}(x)$ and the shifted weight function which are defined as,

$$P_{a,b}^{Q}(x) = j_{a,b}^{Q}(2x-1), \tag{4.4}$$

and,

$$s(x) = S(2x-1). \tag{4.5}$$

A function w belonging to $L^2[0,1]$ can be approximated using Jacobi polynomials as

$$w(x) = \sum_{r=0}^{\infty} k_r P_{a,b}^{r}(x), \quad x \in [0,1], \tag{4.6}$$

where $k_r = \dfrac{\langle w(x) | P_{a,b}^{r}(x) \rangle_s}{\langle P_{a,b}^{r}(x) | P_{a,b}^{r}(x) \rangle_s}$. (4.7)

If we truncate the infinite series in Eq. (4.6) we have,

$$w(x) = \sum_{r=0}^{Q} k_j P_{a,b}^{r}(x). \tag{4.8}$$

Theorem 4.1 Let $w \in L^2[0,1] \cap C[0,1]$ and $sup|w| \leq \mathcal{F}$, then the Jacobi approximation of w given by Eq.(4.8) converges uniformly and also we have

$$\| k_r \| \leq \frac{\mathcal{F}}{2} \frac{(2r+a+b+1)\Gamma(b+1)}{\Gamma(r+b+1)}. \tag{4.9}$$

Proof A function $w \in L^2[0,1] \cap C[0,1]$ can be written by Eq. (4.8) and the coefficients are determined by,

$$k_r = \frac{1}{\|P_{a,b}^{r}(x)\|_2{}^2} \int_0^1 w(x)\, s(x) P_{a,b}^{r}(x) dx,$$

$$\leq \frac{1}{\|P_{a,b}^{r}(x)\|_2{}^2} sup|w(x)| \int_0^1 |s(x) P_{a,b}^{r}(x)| dx.$$

Substitute $x = (y+1)/2$, we obtain,

$$k_r \leq \frac{\mathcal{F}}{2\|P_{a,b}^{r}(x)\|_2{}^2} \int_{-1}^{1} |S(y) j_{a,b}^{r}(y)| dy,$$

$$k_r \leq \frac{\mathcal{F}}{2\|P_{a,b}^{r}(x)\|_2{}^2} \sum_{i=0}^{r} A(i,a,b,r) \int_{-1}^{1} |(1-y)^a (1+y)^b \left(\frac{y-1}{2}\right)^i | \, dy,$$

where $A(i,a,b,r) = \binom{r}{i} \frac{\Gamma(a+r+1)}{r!\Gamma(a+b+r+1)} \frac{\Gamma(a+b+r+i+1)}{\Gamma(a+i+1)}$. (4.10)

$$k_r \leq \frac{\mathcal{F}}{2\|P_{a,b}^{r}(x)\|_2{}^2} \sum_{i=0}^{r} A(i,a,b,r)\, 2^{a+b+i+1} \frac{\Gamma(a+i+1)\Gamma(b+1)}{\Gamma(a+b+i+2)}. \tag{4.11}$$

Substituting the values of $A(i,a,b,r)$ from Eq. (4.10) and $\| P_{a,b}^{r}(x) \|_2{}^2$ in above equation, we get,

$$\| k_r \| \leq \frac{\mathcal{F}}{2}\frac{(2r+a+b+1)\Gamma(b+1)}{\Gamma(r+b+1)}. \tag{4.12}$$

For particular values of $a = b = 0$ ($Legendre\ polynomials$),

$$\| k_r \| \leq \frac{\mathcal{F}}{2}\frac{2r+1}{\Gamma(r+1)},$$

and for $a = b = -1/2$ ($Chebyshev\ polynomials$),

$$\| k_r \| \leq \frac{\sqrt{\pi}\mathcal{F}}{2}\frac{2r}{\Gamma(r+1/2)}.$$

It is clear from Eq. (4.12), $\sum_{r=0}^{Q} k_r$ converges absolutely and hence $\sum_{r=0}^{Q} k_r P_{a,b}^{r}(x)$ converges uniformly to $w(x)$.

4.3 Uniqueness of the Solution

Eq. (4.1) can be written as,

$$\mathcal{D}^{v} w(x) = g(x) + f(x)w(x) + \chi(w(x)), \tag{4.13}$$

where $\chi(w(x)) = \int_0^x \kappa(x,u)\mathcal{G}(w(u))du.$ (4.14)

Operating $\mathcal{J}^{v}$ on Eq. (4.13) gives,

$$w(x) = O(x) + \mathcal{J}^{v}[g(x) + f(x)w(x) + \chi(w(x))], \tag{4.15}$$

where, $O(x) = w(0^{+})$.

Rewriting the Eq. (4.15) as fixed point problem $H(w(x)) = w(x)$, where H can be defined as,

$$H(w(x)) = O(x) + \mathcal{J}^{v}[g(x) + f(x)w(x) + \chi(w(x))]. \tag{4.16}$$

Let $C[0,1]$ be the space of continuous functions equipped with sup norm $\|.\|_{\infty}$. Then $(C[0,1], \|.\|_{\infty})$ forms a Hilbert space. We suppose $\mathcal{G}$ to be satisfied the Lipchitz condition on [0,1] given by,

$$|\mathcal{G}(w_1(x) - w_2(x))| \leq \mathcal{L}|w_1(x) - w_2(x)|, \tag{4.17}$$

where $\mathcal{L}$ defines the Lipschitz constant. Then uniqueness of the solution can be described by the following condition:

Theorem 4.2 The condition of uniqueness of solution of the IVP (4.1) is given by

$$\| f(x) \|_{\infty} + \mathcal{L} \| \chi \|_{\infty} \leq \Gamma(\upsilon + 1). \tag{4.18}$$

Proof: Pl. see theorem 2 in [122] (In Press).

4.4 Proposed Method

The classical approximation theory states that if we consider the Eq. (4.1) with integer order and the forcing terms of Eq.(4.1) are smooth on the respective domains, then the solution of Eq.(4.1) is also smooth with the equal degree of smoothness. But, for $0 < \mu < 1$ in Eq. (4.1), all the smooth forcing functions $g(x)$, $f(x)$, $\kappa(x,u)$ show that the behaviour of the solution will be like $O(x^{\mu})$ and the derivative will be like $O(x^{\mu-1})$ near the left boundary of $I = [0,1]$ and thus the solution becomes unbounded near left boundary of the interval. Now it can be observed that the solution of Eq. (4.1) may not be continuously differentiable near the left boundary of the interval [0,1]. Due to the unboundedness of the derivative of the solution, numerical methods for solving such FIDEs don't allow a fast convergence. To obtain the rapid convergence, it is essential to consider the smooth behaviour of the exact solution. In particular, we can build methods for solving these type of FIDEs by using Jacobi polynomials.

The above discussion implies that derivative of $w(x)$ is unbounded near $x = 0$ which depends on both parameters υ, μ. In order to remove this singularity, a variable transformation is being applied,

$$x = t^{\gamma}, u = y^{\gamma}, \tag{4.19}$$

where γ is the least common multiple of denominators of υ, μ. Now, Eq. (4.1)-(4.2) transform into

$$\mathfrak{D}^{\upsilon} A(t) = W(t) + P(t)A(t) + \int_0^t K(t,y)\mathcal{G}\big(A(y)\big)\gamma y^{\gamma-1} dy \ , \tag{4.20}$$

$$A(0) = w_0, \tag{4.21}$$

where,

$$\mathfrak{D}^{\upsilon} A(t) = \frac{1}{\Gamma(1-\upsilon)} \int_0^t (t^{\gamma} - y^{\gamma})^{-\upsilon} A'(y) dy.$$

$$W(t) = g(t^{\gamma}), P(t) = f(t^{\gamma}), K(t,y) = \kappa(t^{\gamma}, y^{\gamma}),$$

and the exact solution $A(t)$ can be written as,

$$A(t) = w(t^{\gamma}).$$

In above definition we use, $A'(y) = \gamma t^{\gamma-1} w'(t)$ and $du = \gamma t^{\gamma-1} dy \Longrightarrow dy = \frac{1}{\gamma t^{\gamma-1}} du$.

Now, the Jacobi collocation and Jacobi Galerkin methods will be stated to solve FIDE given by Eq. (4.20) with (4.21).

4.4.1 Collocation Method

A function $w_Q(t)$ is to find out such that,

$$w_Q(t) = \sum_{r=0}^{Q-1} k_r P_{a,b}^r(t), \quad t \in [0,1], \tag{4.22}$$

and $w_Q(t)$ must satisfy the following condition,

$$\mathfrak{D}^{\upsilon} \sum_{r=0}^{Q-1} k_r P_{a,b}^r(t_l) = W(t_l) + P(t_l) \sum_{r=0}^{Q-1} k_r P_{a,b}^r(t_l) +$$
$$\int_0^{t_l} K(t_l, y)\mathcal{G}\left(\sum_{r=0}^{Q-1} k_r P_{a,b}^r(y)\right)\gamma y^{\gamma-1} dy. \tag{4.23}$$

For $t_l \in [0,1], l = 0,1, \dots Q-2$, Eq. (4.23) contains $Q-1$ equations with Q unknowns and initial condition,

$$\sum_{r=0}^{Q-1} k_r P_{a,b}^r(0) = w_0, \tag{4.24}$$

Eq. (4.23)-(4.24) provide us a set of algebraic equations and by applying Newton's iterative method [123], this nonlinear system can be solved and hence approximate solution w_Q is obtained.

4.4.2 Galerkin Method

Our aim is to find a function $w_Q(t)$ which can be approximated as

$$w_Q(t) = \sum_{r=0}^{Q-1} k_r P_{a,b}^r(t), \quad t \in [0,1], \tag{4.25}$$

and satisfying,

$$\left\langle \mathfrak{D}^{\upsilon} w_Q(t) \middle| \phi(t) \right\rangle = \langle W(t) | \phi(t) \rangle + \left\langle P(t) w_Q(t) \middle| \phi(t) \right\rangle$$

$$+ \left\langle \int_0^t K(t,y)\mathcal{G}\left(w_Q(y)\right)\gamma y^{\gamma-1} dy \,\middle|\, \phi(t) \right\rangle, \forall\, \phi \in L^2(I). \tag{4.26}$$

Take, $\phi = P_{a,b}^{j}(t), j = 0,1, \dots Q-2,$

$$\left\langle \mathfrak{D}^{v} w_Q(t) \middle| P_{\alpha,\beta}^{j}(t) \right\rangle = \left\langle W(t) \middle| P_{\alpha,\beta}^{j}(t) \right\rangle + \left\langle P(t) w_Q(t) \middle| P_{\alpha,\beta}^{j}(t) \right\rangle$$

$$+ \left\langle \int_0^t K(t,y) \mathcal{G}\left(w_Q(y)\right) \gamma y^{\gamma-1} dy \,\middle| P_{\alpha,\beta}^{j}(t) \right\rangle. \tag{4.27}$$

From Eq. (4.27), we obtain $Q-1$ equations with Q unknowns and remaining equation is obtained by Eq. (4.24). This system of equations is solved by Newton's method.

4.5 Convergence of the Proposed Method

Theorem 4.3 Suppose $w \in L^2(I)$ be smooth sufficiently and $\left(\frac{dw_Q(t)}{dt}\right)$ denote the Jacobi approximation of $\frac{dw(t)}{dt}$ with respect to the shifted Jacobi polynomials. If $\left|\frac{dw(t)}{dt}\right| \le \mathcal{M}$, then we have,

$$\| \frac{dw(t)}{dt} - \left(\frac{dw_Q(t)}{dt}\right) \|_2^{\,2} \le \sum_{r=Q}^{\infty} \frac{\mathcal{M}^2}{2^{1-a-b}} \frac{(2r+a+b+1)\left(\Gamma(b+1)\right)^2 \Gamma(r+a+1)}{r!\,\Gamma(r+b+1)\Gamma(r+a+b+1)}. \tag{4.28}$$

Proof: We assume,

$$\frac{dw(t)}{dt} = \sum_{r=0}^{\infty} k_r P_{a,b}^{r}(t). \tag{4.29}$$

After truncation up to $Q-1$ level, we get

$$\frac{dw_Q(t)}{dt} = \sum_{r=0}^{Q-1} k_r P_{a,b}^{r}(t). \tag{4.30}$$

Subtracting Eq. (4.29) from Eq. (4.30), we obtain,

$$\frac{dw(t)}{dt} - \left(\frac{dw_Q(t)}{dt}\right) = \sum_{r=Q}^{\infty} k_r P_{a,b}^{r}(t),$$

$$\| \frac{dw(t)}{dt} - \left(\frac{dw_Q(t)}{dt}\right) \|_2{}^2 = \int_0^1 \left(\frac{dw(t)}{dt} - \left(\frac{dw_Q(t)}{dt}\right)\right)^2 s(t)dt,$$

$$= \int_0^1 \left(\sum_{r=Q}^{\infty} k_r P_{a,b}^r(t)\right)^2 s(t)dt,$$

$$= \int_0^1 \left(\sum_{r=Q}^{\infty} k_r{}^2 \left(P_{a,b}^r(t)\right)^2 + 2\sum_{i=Q}^{\infty} \sum_{\substack{j=Q \\ j\neq i}}^{\infty} k_i k_j P_{a,b}^i(t) P_{a,b}^j(t)\right) s(t)dt.$$

Using orthogonal condition of Jacobi polynomials, we get

$$\| \frac{dw(t)}{dt} - \left(\frac{dw_Q(t)}{dt}\right) \|_2{}^2 = \sum_{r=Q}^{\infty} k_r{}^2 \| P_{a,b}^r(t) \|_2{}^2. \tag{4.31}$$

Now repeating the same procedure as in Theorem 1, we obtain,

$$\| k_r \| \leq \frac{\mathcal{M}}{2} \frac{(2r+a+b+1)\Gamma(b+1)}{\Gamma(r+b+1)}. \tag{4.32}$$

Substitute the value of k_r from Eq.(42) and $\| P_{a,b}^r(t) \|_2{}^2$in Eq.(41), we have,

$$\| \frac{dw(t)}{dt} - \left(\frac{dw_Q(t)}{dt}\right) \|_2{}^2 \leq B(Q,a,b). \tag{4.33}$$

where, $B(Q,a,b) = \sum_{r=Q}^{\infty} \frac{\mathcal{M}^2}{2^{1-a-b}} \frac{(2r+a+b+1)\left(\Gamma(b+1)\right)^2 \Gamma(r+a+1)}{r!\Gamma(r+b+1)\Gamma(r+a+b+1)}$. (4.34)

By ratio test, the series denoted by $B(Q,\alpha,\beta)$ converges to 0 as $Q \to \infty$.

Theorem 4.4 Let $A(t)$ and $w_Q(t)$ denote the exact and the approximated solution of Eq. (4.20) and operator $\mathcal{G}$ satisfies the Lipchitz condition given by Eq. (4.17). Then approximation error $\varepsilon_Q = A - w_Q \to 0$ and

$$\| A(t) - w_Q(t) \|_2 \leq \frac{1}{\|P(t)\|_2 - q\mathcal{L}} \frac{B(Q,a,b)}{\Gamma(1-v)}. \tag{4.35}$$

Proof: Since $w_Q(t)$ is the approximate solution of Eq. (4.20). So we have,

$$\mathfrak{D}^{v} w_Q(t) = W(t) + P(t) w_Q(t) + \int_0^t K(t,y) \mathcal{G}\left(w_Q(y)\right) \gamma y^{\gamma-1} dy. \qquad (4.36)$$

Subtracting above equation from Eq. (4.20), we have,

$$P(t)\left(A(t) - w_Q(t)\right) =$$

$$\mathfrak{D}^{v}\left(A(t) - w_Q(t)\right) - \int_0^t K(t,y) \mathcal{G}\big((A(y) - w_Q(y))\big) \gamma y^{\gamma-1} dy, \qquad (4.37)$$

$$\| P(t)\left(A(t) - w_Q(t)\right) \|_2 \leq \| \mathfrak{D}^{v}\left(A(t) - w_Q(t)\right) \|_2 + \| \int_0^t K(t,y) \mathcal{G}\left(\left(A(y) - w_Q(y)\right)\right) \gamma y^{\gamma-1} dy \|_2,$$

$$\leq \| \frac{1}{\Gamma(1-v)} \int_0^t (t^{\gamma} - y^{\gamma})^{-v} \left(A'(y) - w_Q{}'(y)\right) dy \|_2 + \mathcal{L} \| A(t) - w_Q(t) \|_2 \int_0^t K(t,y) \gamma y^{\gamma-1} dy \|_2,$$

$$\| P(t)\left(A(t) - w_Q(t)\right) \|_2 - \mathcal{L} \| A(t) - w_Q(t) \|_2 \| \int_0^t K(t,y) \gamma y^{\gamma-1} dy \|_2$$

$$\leq \frac{B(Q,a,b)}{\Gamma(1-v)} \int_0^t |(t^{\gamma} - y^{\gamma})^{-v}| dy,$$

$$\| A(t) - w_Q(t) \|_2 \leq \frac{1}{\|P(t)\|_2 - q\mathcal{L}} \frac{c}{\Gamma(1-v)}, \qquad (4.38)$$

where, $q = \| \int_0^t K(t,y) \gamma y^{\gamma-1} dy \|_2$.

Eq. (4.38) concludes $\| A(t) - w_Q(t) \|_2 \to 0$ as $Q \to \infty$ since $B(Q,a,b) \to 0$ as $Q \to \infty$.

4.6 Numerical Simulation and Results

To examine the validity of the proposed methods two test examples are considered and numerical simulations are performed as follows:

Test example 1 Consider the linear case of FIDE,

$$\mathcal{D}^{\upsilon}w(x) = g(x) + f(x)w(x) + \int_0^x \kappa(x,u)w(u)du \ , w(0) = 0, \tag{4.39}$$

$\upsilon = \frac{1}{3}, \kappa(x,u) = (x-u)^{-1/2}, f(x) = -\frac{32}{35}x^{1/2},$

where, $g(x) = \frac{6x^{8/3}}{\Gamma(11/3)} + (32/35 - \frac{\Gamma(1/2)\,\Gamma(7/3)}{\Gamma(17/6)]})x^{11/6} + \Gamma(7/3) * x,$

The exact solution for this case is $x^3 + x^{4/3}$.

We approximate the solution of above problem Eq. (4.39) by methods described in section 5 and MAEs are shown in Table 1.1 for different values of Q. In addition, as we know that the exact solution of above problem is not smooth, so we apply variable transformation $x = t^6, u = y^6$, which reduces to,

$$\mathfrak{D}^{\upsilon}A(t) = W(t) + P(t)A(t) + \int_0^t (t^6 - y^6)^{-1/2}A(y)6y^5dy \ ,A(0) = 0, \tag{4.40}$$

where,

$\mathfrak{D}^{\upsilon}A(t) = \frac{1}{\Gamma(\frac{2}{3})}\int_0^t (t^6 - y^6)^{-\upsilon}A'(y)dy,$

$W(t) = \frac{6t^{16}}{\Gamma(11/3)} + (32/35 - \frac{\Gamma(1/2)\,\Gamma(7/3)}{\Gamma(17/6)]})t^{11} + \Gamma(7/3) * t^6,$

$P(x) = -\frac{32}{35}t^3,$

and exact solution in this case is $t^8 + t^{18}$.

By using collocation and Galerkin method, we approximate the solution of Eq. (4.40) with $a = b = 0$ (Legendre polynomials) for $Q = 12{,}15{,}18$ and obtained MAEs are shown in Table 4.1. The obtained MAEs by collocation and Galerkin methods are 7.786E-16 and 6.188E-15 respectively.

It is clear from Table 4.1 and Table 4.2 that after applying the variable transformation, we obtain the better numerical accuracy as a solution of Eq. (4.40) becomes smooth. This problem is also considered in [43, 124, 125]. In [43], the solution is approximated by collocation method coupled with Simpson and Newton's 3/8 rule whereas Chebyshev spectral method is applied in [124]. Mokhtary [125] solved Eq. (4.39) by operational Tau method using Jacobi polynomials and numerical solution closer to exact solution is obtained.

Table 4.1 MAEs for Eq. (4.39) using collocation and Galerkin methods

N	**Collocation method**	**Galerkin method**
3	$4.021E-3$	$3.241E-3$
6	$6.798E-4$	$2.124E-4$
9	$3.033E-4$	$3.033E-4$
12	$1.435E-4$	$1.435E-4$
15	$5.654E-5$	$5.654E-5$
18	$3.138E-5$	$1.028E-5$

Table 4.2 MAEs for Eq. (4.40) using collocation and Galerkin methods

N	$Collocation\ method$	$Galerkin\ method$
12	$1.468E-6$	$1.468E-6$
15	$4.787E-9$	$5.495E-9$
18	$7.786E-16$	$6.188E-15$

Table 4.3 Numerical error for test example 1 by the method presented in [124]

N	**weighted L^2-error**
2	$1.34E-2$
4	$5.14E-4$
6	$1.56E-4$
8	$6.46E-5$
16	$7.38E-6$
18	$4.62E-6$
20	$2.85E-6$

Table 4.4 Numerical error for test example 1 in [43].

h	**Numerical Errors**	
	Simpson's collocation method	**Newton's 3/8 collocation method**
$\frac{1}{2}$	$5.32E-2$	$1.72E-2$
$\frac{1}{4}$	$7.04E-3$	$1.11E-3$
$\frac{1}{8}$	$6.91E-4$	$7.03E-5$
$\frac{1}{16}$	$5.61E-5$	$4.41E-6$
$\frac{1}{32}$	$4.15E-6$	$2.75E-7$
$\frac{1}{64}$	$2.93E-7$	$1.72E-8$
$\frac{1}{128}$	$2.02E-8$	$1.07E-9$

Test example 2 Consider the nonlinear case of FIDE,

$$\mathcal{D}^{\upsilon}w(x) = g(x) + f(x)w(x) + \int_0^x \kappa(x,u)\big(w(u)\big)^2 du \ , w(0) = 0, \quad (4.41)$$

$\upsilon = \frac{1}{2}, \kappa(x,u) = (x-u)^{-1/5}, f(x) = 1,$

where, $g(x) = 0.88622 - \sqrt{x} - \frac{25}{36}x^{\frac{9}{5}},$

For problem, we have $w(x) = \sqrt{x}$.

As the solution of Eq. (4.41) is not smooth, so we apply variable transformation $x = t^{10}, u = y^{10}$, which reduces the above problem in

$$\mathfrak{D}^{\upsilon} A(t) = W(t) + P(t)A(t) + \int_0^t (t^{10} - y^{10})^{-1/2} \big(A(y)\big)^2 10y^9 dy \ , A(0) = 0, \quad (4.42)$$

where,

$$\mathfrak{D}^{\upsilon} A(t) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t^{10} - y^{10})^{-\frac{1}{2}} A'(y) dy,$$

$$W(t) = 0.88622 - t^5 - \frac{25}{36} t^{18},$$

$$P(t) = 1,$$

and exact solution in this case is t^5.

By using collocation and Galerkin method, the solution of Eq. (4.41) is approximated using the methods described in section 5 but we don't get a better approximation as analytic solution is not smooth. So we apply the variable transformation and obtained Eq. (4.42). We solve Eq. (4.42) by Jacobi collocation and Galerkin methods and obtained solution which coincides with the exact solution t^5 for $Q \geq 5$. The obtained MAEs by collocation and Galerkin methods are shown in Table 4.5.

Table 4.5 MAEs for Eq. (4.42) using collocation and Galerkin methods

N	***Collocation method***	***Galerkin method***
3	$4.361E-3$	$1.025E-2$
4	$7.492E-4$	$6.822E-4$
5	$5.290E-17$	$1.925E-15$

4.7 Conclusion

Two numerical schemes such as collocation and Galerkin methods using Jacobi polynomials to approximate the solution of weakly singular FIDEs are presented and analyzed. In numerical experiments, we consider the FIDEs on [0,1] which have a nonsmooth solution. The obtained MAEs verify the theoretical convergence and have the good agreement with the known results. It can be observed by the obtained maximum absolute errors that the accuracy and convergence of collocation and Galerkin methods provide better results after applying the variable transformation (Table 4.1 and Table 4.2). The smoothness of the solution makes approximation better and help us to improve the convergence rate. Further, it is noticed from the numerical errors that collocation method is more computationally efficient than Galerkin method for solving FIDEs and this is due to the calculation of the double integration term in Galerkin method. Also, collocation method provides better results with a faster convergence rate. The presented methods are more computationally efficient and provide the comparatively better solution than the methods described in [43, 124] for FIDEs.