Chapter 2

Collocation Method for Generalized Abel's Integral Equations

2.1 Introduction

It is a well-known fact that many problems in engineering and applied sciences are converted to integral equations. Integral equation builds the base for modeling of various phenomena in basic and engineering sciences. Abel's equation is one of the integral equations which is directly derived from the problem of physics [81, 82, 83, 84]. This integral equation arises in various type of mathematical modeling. Abel's equation is one of the integral equations derived directly from a concrete problem of mechanics or physics (without using any differential equation).

In this chapter, we consider the generalized Abel's integral equation [8, 9],

$$a(x)\int_0^x \frac{\psi(t)}{(x-t)^\mu}\,dt + b(x)\int_x^1 \frac{\psi(t)}{(t-x)^\mu}\,dt = \zeta(x), \tag{2.1}$$

where $0 < \mu < 1$, the function $\psi \in L^2[0,1], \psi(t)$ is the unknown function.

Several methods have been purposed to solve integral equations in general and the Generalized Abel's integral equation Eq. (2.1) in particular. In [85], Pandey and Mandal

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used the Bernstein's polynomials for solving a system of Generalized Abel's integral equations using Bernstein's polynomials.

Collocation method has been established as one of the important methods to solve the problems of integral equations and differential equations. Some of the recent studies on Collocation methods are described as follows. Nemati [86] has applied Legendre collocation method for solving Volterra- Fredholm integral equations. Saadatmandi and Dehghan [87] applied collocation method to solve Abel's integral equations of first and second kinds using shifted Legendre's polynomials. Hashemizadeh et al. [88] have used the Sinc-collocation approach for solving Hammerstein integral equations. Assari et al. [89] discussed a numerical scheme based on the radial basis functions (RBFs) for solving weakly singular Fredholm integral equations by combining the product integration and collocation methods. Parand et al. [90, 91] presented the sinc-collocation methods and Gegenbauer collocation methods for solving the Thomas–Fermi equations and laminar boundary layer equations respectively. In [43], authors have used the collocation methods for solving fractional integro-differential equations and Sahu and Ray applied the Legendre-collocation method for solving a biological model in [92]. Some other methods of importance such as mesh less based methods [93, 94] and finite difference method [95] are also successfully applied to solve integral equations and Fredholm integro-differential equations. These methods may also be discussed for the Generalized Abel's integral equations in future.

In this chapter, we extend the collocation method to solve the general Abel's integral given by Eq. (1) using polynomials bases. The basic idea is to find an approximate solution of Eq. (1) from a finite-dimensional family of functions. The description of the proposed method is presented in Section 2. The error estimate of the collocation method for Generalized Abel's integral equation is established in Section 3. In Section 4, we use some standard polynomials to compute the numerical solutions of the illustrative examples.

2.2 Description of the Method

In this section, we extend the collocation method described in Chapter 1 to solve the integral equation given by Eq. (2.1). Consider Eq. (2.1) in an operator form as,

$$\chi(\psi) = \zeta \,, \tag{2.2}$$

where the operator χ is assumed on a Banach space X. We choose a sequence of finite dimensional subspaces $X_n \subset X, n \geq 1$, having dimension $n+1$. Let X_n has a basis $S = \{\theta_0, \theta_1, \ldots \theta_n\}$ in X. We seek a function $\psi_n \in X_n$, which is the best approximation ofψsuch that,

$$\psi_n(t) = \sum_{j=0}^{n} c_j \theta_j(t), \quad t \in [0,1]. \tag{2.3}$$

Substituting Eq. (2.3) into Eq. (2.1), coefficients $\{c_j \big| j = 0,1, \ldots, n\}$ are determined by forcing the equation to be exact in some sense. For later use, introduce,

$$\tau_n(t) = a(x) \int_0^x \frac{\sum_{j=0}^{n} c_j \theta_j(t)}{(x-t)^{\mu}} \mathrm{dt} + b(x) \int_x^1 \frac{\sum_{j=0}^{n} c_j \theta_j(t)}{(t-x)^{\mu}} \mathrm{dt} - \zeta(x). \tag{2.4}$$

This is called the residual in approximating Eq. (2.1) when ψ is replaced by ψ_n. Symbolically,

$$\tau_n = \chi(\psi_n) - \zeta, \tag{2.5}$$

Or,

$$\tau_n = \chi\left(\sum_{j=0}^{n} c_j \theta_j(t)\right) - \zeta. \tag{2.6}$$

The coefficients $\{c_j | j = 0,1,\ldots.,n\}$ are chosen by forcing $\tau_n(t)$ to be approximately zero. The hope and expectation are that the resulting $\psi_n(t)$ will be a good approximation of the true solution $\psi(t)$. We approximate the solution of Eq. (2.1) using collocation method. We pick distinct node points $t_0,\ldots\ldots,t_n \in [0,1]$, such that $\tau_n(t_i) = 0, (i = 0,1,2,\ldots..n)$.

$$\chi(\psi_n(t_i)) - \zeta(t_i) = 0. \tag{2.7}$$

$$\chi\left(\sum_{j=0}^{n} c_j\theta_j(t_i)\right) - \zeta(t_i) = 0. \tag{2.8}$$

This leads to determining $\{c_j | j = 0,1,\ldots.,n\}$ as the solution of the linear system,

$$a(t_i)\int_0^{t_i}\frac{\sum_{j=0}^{n} c_j\theta_j(t)}{(t_i-t)^{\mu}}dt + b(t_i)\int_{t_i}^{1}\frac{\sum_{j=0}^{n} c_j\theta_j(t)}{(t-t_i)^{\mu}}dt = \zeta(t_i), t_i \in (0,1), i = 0,1,\ldots\ldots.n \tag{2.9}$$

Now, Eq. (2.1) is converted into a system of linear equations as given by Eq. (2.9) in unknowns $\{c_j\}$.

2.3 Convergence Analysis

Here, we discuss the convergence analysis and error bound of the collocation method presented in the Section 2.2 to compute the approximate solution of the generalized Abel's integral equation Eq. (2.1). We use some notations as, $I = [0,1]$, $X = L^2(I)$, and $e_n(x) = \psi(x) - \psi_n(x)$.

To establish the error estimate, we use the following results.

Lemma 2.1 [84]. Let $X = L^2(I)$ and K be a Volterra integral operator on X with square summable kernel $\kappa(x,t)$i.e.,$\int_0^1\int_0^1 |\kappa(x,t)| dxdt = M^2$, where M is a constant. Let the operator K is defined by,

$$K(\phi(x)) = \int_0^x \kappa(x,t)\,\phi(t)dt, \tag{2.13}$$

then K is bounded. That is,

$$||K(\phi(x))|| \le M||\,\phi||. \tag{2.14}$$

Lemma 2.2 [86]. If $y(x)$ is a sufficiently smooth function on $[0,1]$ and $p_n(x)$ is the interpolating polynomial to $y(x)$ at pointsx_i, where $x_i, i = 0,1,\ldots\ldots,n$ are the roots of $(n+1)$ degree interpolating polynomial in $[0,1]$, then we have

$$y(x) - p_n(x) = \frac{y^{n+1}(x)}{(n+1)!}\prod_{i=0}^{n}(x - x_i)\ , x \in [0,1], \tag{2.15}$$

$$|y(x) - p_n(x)| \le \frac{M_n}{2^{2n+1}(n+1)!}\ , \tag{2.16}$$

where, $M_n = \max\{\,|\,y^{n+1}(x)|\ x \in (0,1)\}$.

Lemma 2.3 Let us define the linear operators A and B on X by

$$A(\psi\,(x)) = \int_0^x \frac{\psi\,(t)}{(x-t)^{\mu}}dt, \tag{2.17}$$

and,

$$B(\psi\,(x)) = \int_x^1 \frac{\psi\,(t)}{(t-x)^{\mu}}dt, \tag{2.18}$$

where $0 < \mu < 1$, $x \in [0,1]$ and for all $\psi \in L^2(I)$.

Thus, by Lemma 1, both operator A and B defined above are bounded. If A is one-one and onto, then A^{-1} is bounded.

Theorem 2.1 Suppose that the known functions in Eq. (2.1) are real $(n+1)$-times continuously differentiable function on the interval $(0,1)$ and $\psi_n{}'(t) = \sum_{j=0}^{n} c_j\theta_j(t)$ be the expansion of exact solution $\psi\ (x)$ with respect to the basis function in S. Let $\psi_n(x) = \sum_{j=0}^{n} a_j\theta_j(x)$ be the approximate solution obtained by the purposed method discussed in Section 2.2 and $M_n = \max\{\,|\,\psi^{n+1}(x)|\,;\ x \in (0,1)\}$ .Then there exists constants Λ_1 , Λ_2 and ε_n such that,

$$\parallel e_n(x) \parallel_2 \leq \Lambda_1\Lambda_2\left(\frac{M_n}{2^{2n+1}(n+1)!} + \varepsilon_n \parallel c - a \parallel_2\right), \tag{2.19}$$

where $c = (c_0, c_1, \dots, c_n)^T$ and $a = (a_0, a_1, \dots, a_n)^T$.

Proof: From the notation mentioned in Eq. (2.17) and Eq. (2.18), Eq. (2.1) takes the form,

$$A\big(\psi\ (x)\big) + B\big(\psi\ (x)\big) = \zeta(x).$$

If A is bijective then,

$$\psi\ (x) + A^{-1}\Big(B\big(\psi\ (x)\big)\Big) = A^{-1}(\zeta(x)). \tag{2.20}$$

$\psi_n(x)$ is the approximate solution of Eq.(2.1), then Eq. (2.20) can be written as

$$\psi_n(x) + A^{-1}\Big(B\big(\psi_n(x)\big)\Big) = A^{-1}(\zeta(x)). \tag{2.21}$$

Subtracting Eq. (2.20) from Eq. (2.21), we get,

$$e_n(x) = A^{-1}\Big(B\big(\psi_n(x) - \psi\ (x)\big)\Big), \tag{2.22}$$

Using Lemma 2.3, and we have B and A^{-1} are bounded therefore, there exist two constants Λ_1 and Λ_2 such that,

$$e_n(x) = \Lambda_1\Lambda_2 \parallel \psi(x) - \psi_n(x) \parallel_2. \tag{2.23}$$

Let $\mathbb{R}_n(x)$ be the space of all real-valued polynomials of degree $\leq n$. Using the definition, $\psi\,(x)$ and $\psi_n(x)$ are in $\mathbb{R}_n(x)$ .Therefore, we have,

$$\parallel \psi(x) - \psi_n(x) \parallel_2 \leq \parallel \psi(x) - \psi_n{}'(t) \parallel_2 + \parallel \psi_n{}'(t) - \psi_n(x) \parallel_2. \tag{2.24}$$

Now using Lemma 2.2, we get,

$$\parallel \psi(x) - \psi_n{}'(t) \parallel_2 \leq \frac{M_n}{2^{2n+1}(n+1)!}\,, \tag{2.25}$$

where, $\qquad M_n = \max\{\,|\,\psi^{n+1}(x)|:\ x \in (0,1)\},$

and,

$$\parallel \psi_n{}'(t) - \psi_n(x) \parallel_2 = \left[\int_0^1 \{\sum_{i=0}^{n}(c_i - a_i)\,\theta_i(x)\}^2\right]^{1/2}$$

$$\leq \varepsilon_n[\{\sum_{i=0}^{n}(c_i - a_i)\}^2]^{1/2} \tag{2.26}$$

where,

$$\varepsilon_n = \left[\sum_{i=0}^{n}\int_0^1 |\theta_i(x)|^2 dx\right]^{1/2}. \tag{2.27}$$

Thus, $\parallel \psi(x) - \psi_n(x) \parallel_2 \leq \frac{M_n}{2^{2n+1}(n+1)!} + \varepsilon_n \parallel c - a \parallel_2.$ (2.28)

Now from Eq. (2.23) and Eq. (2.28), we have,

$$\parallel e_n(x) \parallel_2 \leq \Lambda_1\Lambda_2\left(\frac{M_n}{2^{2n+1}(n+1)!} + \varepsilon_n \parallel c - a \parallel_2\right), \tag{2.29}$$

This completes the proof.

2.4 Approximations Using Polynomials Bases and Numerical Results

Here, we analyze the collocation method presented in the previous section for Eq. (2.1) using the polynomials bases. Different polynomials such as Jacobi, Legendre, Chebyshev and Gegenbauer polynomials are used to get the approximate solution of Eq. (2.1). We named these polynomial schemes as S1, S2, S3 and S4 respectively and corresponding bases comparison is also presented in Table 2.1. These polynomials are described as follows:

1) Jacobi Polynomials (S1)

Jacobi polynomial of degree n, denoted as $P_n^{\alpha,\beta}$, form a basis for the vector space of polynomials of degree at most n. Jacobi polynomials [81] are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formulae:

$$P_n^{\alpha,\beta} = \sum_s \binom{n+\alpha}{s}\binom{n+\beta}{n-s}\left(\frac{x-1}{2}\right)^{n-s}\left(\frac{x+1}{2}\right)^{s}, \quad n \geq s \geq 0, \tag{2.30}$$

where, $\binom{n}{i}$ is binomial coefficient. Jacobi polynomials also satisfy the recurrence formula

$$2n(n+\alpha+\beta)(2n+\alpha+\beta-2)P_n^{\alpha,\beta}(x) = (2n+\alpha+\beta-1)\{(2n+\alpha+\beta)(2n+\alpha+\beta-2)x+\alpha^2-\beta^2\}P_{n-1}^{\alpha,\beta}(x)$$

$$-2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)P_{n-2}^{\alpha,\beta}(x). \tag{2.31}$$

Jacobi polynomials $P_n^{\alpha,\beta}(x)$ satisfy the orthogonality condition with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ on the interval $[-1,1]$ and defined as,

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\, P_n^{\alpha,\beta}(x) P_m^{\alpha,\beta}(x)dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{(n+\alpha)!(n+\beta)!}{n!(n+\alpha+\beta)!}\delta_{nm},$$

where, $\alpha, \beta, \alpha+\beta > -1$, and δ_{nm} is Kronecker Delta function. We choose Jacobi polynomials as basis functions on [0,1]. In order to use these polynomials on the interval [0,1], we shift Jacobi polynomials from $[0,1]$ to $[-1,1]$ by introducing a linear transformation $x \rightarrow 2x-1$ and obtain,

$$\psi_n(t) = \sum_{j=0}^{n} c_j P_n^{\alpha,\beta}(2t-1), \quad t \in [0,1]. \tag{2.32}$$

Substituting Eq. (2.32) in Eq. (2.4) and collocating at points $t_i \in [0,1], i = 0,1, \dots \dots . n$, we obtain,

$$a(t_i)\int_0^{t_i} \frac{\sum_{j=0}^{n} c_j P_j^{\alpha,\beta}(2t-1)}{(t_i-t)^{\mu}}dt + b(t_i)\int_{t_i}^{1} \frac{\sum_{j=0}^{n} c_j P_j^{\alpha,\beta}(2t-1)}{(t-t_i)^{\mu}}dt = \zeta(t_i),$$

$t_i \in (0,1), i = 0,1, \dots \dots . n,$

$$\sum_{j=0}^{n} c_j \left(a(t_i)\int_0^{t_i} \frac{P_j^{\alpha,\beta}(2t-1)}{(t_i-t)^{\mu}}dt + b(t_i)\int_{t_i}^{1} \frac{P_j^{\alpha,\beta}(2t-1)}{(t-t_i)^{\mu}}dt \right) = \zeta(t_i), \tag{2.33}$$

This leads to a linear system of equations in unknowns $\{c_j\}$ which has a unique solution by Eq. (2.12).

2) Legendre Polynomials (S2)

Jacobi polynomials are defined as Legendre polynomials $P_n(x)$ for $\alpha = \beta = 0$. Legendre polynomials [86] also form an orthogonal set with respect to the weight function 1 on the interval $[-1,1]$. We choose Legendre polynomials as basis functions of all polynomials of degree $\leq n$ defined over [0,1] by transforming intervel$[-1,1]$ to [0,1] (shifted by a linear transformation $t \rightarrow 2t-1$). And hence Eq. (2.4) takes the form,

$$\sum_{j=0}^{n} c_j \left(a(t_i) \int_0^{t_i} \frac{P_j(2t-1)}{(t_i-t)^{\mu}} dt + b(t_i) \int_{t_i}^{1} \frac{P_j(2t-1)}{(t-t_i)^{\mu}} dt \right) = \zeta(t_i). \tag{2.34}$$

The remaining steps are performed similar to the S1 and numerical solutions are obtained.

3) Chebyshev Polynomials (S3)

Jacobi polynomials with $\alpha = -1/2, \beta = -1/2$, are defined as Chebyshev polynomials and denoted by $T_n(x)$. Chebyshev polynomials [67] also form an orthogonal set with a weight $(1-x)^{-1/2}$ on the interval $[-1,1]$. Using the transformation $t \to 2t-1$, Chebyshev polynomials on $[0,1]$ together with Eq. (2.4) converts into a system of linear equations,

$$\sum_{j=0}^{n} c_j \left(a(t_i) \int_0^{t_i} \frac{T_j(2t-1)}{(t_i-t)^{\mu}} dt + b(t_i) \int_{t_i}^{1} \frac{T_j(2t-1)}{(t-t_i)^{\mu}} dt \right) = \zeta(t_i), \tag{2.35}$$

By solving the system Eq. (2.35), the approximate solution can be obtained.

4) Gegenbauer Polynomials (S4)

Here, we consider the well-known Gegenbauer polynomials [96] denoted as $C_n^{\alpha}(x)$ on interval $[-1,1]$ and determined using the following recurrence formulae,

$$C_0^{\alpha} = 1 \; and \; C_1^{\alpha} = 2\alpha x. \tag{2.36}$$

$$C_n^{\alpha}(x) = \frac{1}{n}[2x(n+\alpha-1)C_{n-1}^{\alpha}(x) - (n+2\alpha-2)C_{n-2}^{\alpha}(x)]. \tag{2.37}$$

These polynomials are also the special case of the Jacobi polynomials for $\alpha = \beta = \alpha - 1/2$ and in terms of Jacobi polynomials it is defined as,

$$C_n^{\alpha}(x) = \frac{(2\alpha)_n}{(\alpha+1/2)_n} P_n^{\alpha-1/2,\alpha-1/2}(x). \tag{2.38}$$

Gegenbauer polynomials $C_n^{\alpha}(x)$ satisfy the ortghogonality condition on $[-1,1]$ with respect to the weight function $(1-x)^{\alpha-1/2}$ and given by,

$$\int_{-1}^{1}(1-x^2)^{\alpha-1/2}C_n^{\alpha}(x)C_m^{\alpha}(x)\,dx = \frac{\pi 2^{(1-2\alpha)}(n+2\alpha-1)!}{n!(n+\alpha-1)![(\alpha-1)!]^2}\delta_{nm}, \tag{2.39}$$

where δ_{nm} is Kronecker Delta function. Now, we shift the properties of Gegenbauer polynomials from $[-1,1]$ to $[0,1]$ by introducing a linear transformation $x \to 2x-1$ and define,

$$\psi_n(t) = \sum_{j=0}^{n} c_j C_j^{\alpha}(2x-1), \quad t \in [0,1]. \tag{2.40}$$

Substituting Eq. (2.40) in Eq. (2.4) and then collocating at points $t_i, i = 0,1, \dots \dots . n$, one obtains,

$$a(t_i)\int_0^{t_i}\frac{\sum_{j=0}^{n} c_j C_j^{\alpha}(2t-1)}{(t_i-t)^{\mu}}dt + b(t_i)\int_{t_i}^{1}\frac{\sum_{j=0}^{n} c_j C_j^{\alpha}(2t-1)}{(t-t_i)^{\mu}}dt = \zeta(t_i),\ t_i \in (0,1), i = 0,1, \dots \dots . n,$$

$$\sum_{j=0}^{n} c_j\left(a(t_i)\int_0^{t_i}\frac{C_j^{\alpha}(2t-1)}{(t_i-t)^{\mu}}dt + b(t_i)\int_{t_i}^{1}\frac{C_j^{\alpha}(2t-1)}{(t-t_i)^{\mu}}dt\right) = \zeta(t_i), \tag{2.41}$$

Eq. (2.41) leads the system of linear equations and that can be solved using any standard method. The unknowns $\{c_j\}$ are obtained uniquely under the condition of Eq. (2.12).

Test Example 2.1 Consider the Generalized Abel integral equation [97] given by Eq. (2.1) with $a(x) = b(x) = 1, \nu = 1/2$ and $\zeta(x) = \frac{4}{105}x^{3/2}(35-24x^2) + \frac{8}{105}(1-x)^{1/2}(5+13x-6x^2+12x^3)$. It has exact solution $\psi(x) = x - x^3$.

This example is solved for the values of $n = 3$ and 5 and the numerical results are obtained. It is observed that the presented method discussed in Section 2 works well and

Table 2.1 Comparison of the different polynomial bases for $n = 0, 1, 2,$ and 3.

n	Jacobi	Legendre	Chebyshev	Gegenbauer
0	1	1	1	1
1	$\left(\frac{1}{2}\right)(\alpha-\beta+(2+\alpha+\beta)x)$	x	x	$2\alpha x$
2	$\left(\frac{1}{2}\right)(1+\alpha)(2+\alpha)+\left(\frac{1}{2}\right)(2+\alpha)(3+\alpha+\beta)(-1+x)+\left(\frac{1}{8}\right)(3+\alpha+\beta)(-1+x^2)$	$\frac{1}{2}(3x^2-1)$	$2x^2-1$	$-\alpha+2\alpha(1+\alpha)x^2$
3	$\left(\frac{1}{6}\right)(1+\alpha)(2+\alpha)(3+\alpha)+\left(\frac{1}{4}\right)(2+\alpha)(3+\alpha)(3+\alpha+\beta)(-1+x)+\left(\frac{1}{8}\right)(3+\alpha)(4+\alpha+\beta)(5+\alpha+\beta)(-1+x)^2+\left(\frac{1}{48}\right)(4+\alpha+\beta)(5+\alpha+\beta)(6+\alpha+\beta)(-1+x)^3$	$\frac{1}{2}(5x^3-3x)$	$4x^3-3x$	$-2\alpha(1+\alpha)x+\frac{4}{3}\alpha(1+\alpha)(2+\alpha)x^3$

achieve high accuracy in the numerical results. Since the exact solution in the present test example is a polynomial of degree 3 so it is sufficient to choose only the basis polynomials up to degree 3 to compute the approximate solution which turns out to the exact solution. All four polynomials based schemes works well and the obtained errors for each schemes are shown in Table 2.1. The respective errors for each scheme are also plotted and shown through Figs. 2.1-Figs. 2.4. In each Figures, $n = 3$ represents the thik line and $n = 5$ indicates the dashed line.

Test Example 2.2 In this case, Generalized Abel integral Eq. (2.1) is considered from [97] with $a(x) = b(x) = 1$, $\nu = 1/5$ and $\zeta(x) = e^x \left[(1-x)^{4/5}(x-1)^{-4/5}\left\{\Gamma\left(\frac{4}{5}\right) - \Gamma\left(\frac{4}{5}, x-1\right)\right\} + \left\{\Gamma\left(\frac{4}{5}\right) - \Gamma\left(\frac{4}{5}, x\right)\right\}\right]$, where Γ denotes the Gamma function. In this case, the exact solution is given by $\psi(x) = e^x$.

Here, we study the performance of the scheme discussed in Section 2. We consider the schemes S1, S2, S3 and S4 with the values of $n = 3$ and 5 and obtain the numerical results. The obtained errors corresponding to these schemes are given in Table 2.3 and the respective errors are described through Figs. 2.5-2.8. Here, we observe that as we increase the number of basis functions in the presented method the errors decreases respectively.

Test Example 2.3 Here, we consider a case similar to one as discussed in [97] where the exact solution of Generalized Abel integral Eq. (2.1) is a fractional polynomial. For $a(x) = b(x) = 1$, $\nu = 1/2$ and $\zeta(x) = \frac{\sqrt{\pi}x^{\frac{5}{6}}\Gamma(-\frac{5}{6})}{\Gamma(-\frac{1}{3})} + \frac{\sqrt{\pi}x^{\frac{5}{6}}\Gamma(\frac{4}{3})}{\Gamma(\frac{11}{6})} + \frac{6}{5}\ {}_2F_1[-\frac{5}{6}, \frac{1}{2}, \frac{1}{6}, x]$, the exact solution of Eq.(2.1) is given by $\psi(x) = x^{1/3}$.

Here, the presented method in testing on the Generalized Abel's integral equation which has an exact solution as the fractional power of x. The numerical investigations are performed for $n = 3$ and 5 and respective errors are obtained. The comparison of the errors obtained through the different polynomials are given in Table 2.4. We observe that all the schemes work well. The errors in each case are shown through Figs. 2.9-2.12.

Table 2.2 Maximum absolute errors with various schemes for Test example 2.1.

n	S1	S2	S3	S4
3	1.666E-16	2.376E-16	1.301E-16	1.804E-16
5	4.037E-17	1.034E-15	3.789E-17	1.998E-16

Table 2.3 Maximum absolute errors with various schemes for Test example 2.2.

n	S1	S2	S3	S4
3	1.667E-3	1.667E-3	1.667E-3	1.667E-3
5	4.123E-5	2.182E-7	1.123E-5	1.123E-5

Table 2.4 Maximum absolute errors with various schemes for Test example 2.3.

n	S1	S2	S3	S4
3	7.499E-2	7.499E-2	7.499E-2	7.499E-2
5	5.781E-2	5.781E-2	5.781E-2	5.781E-2

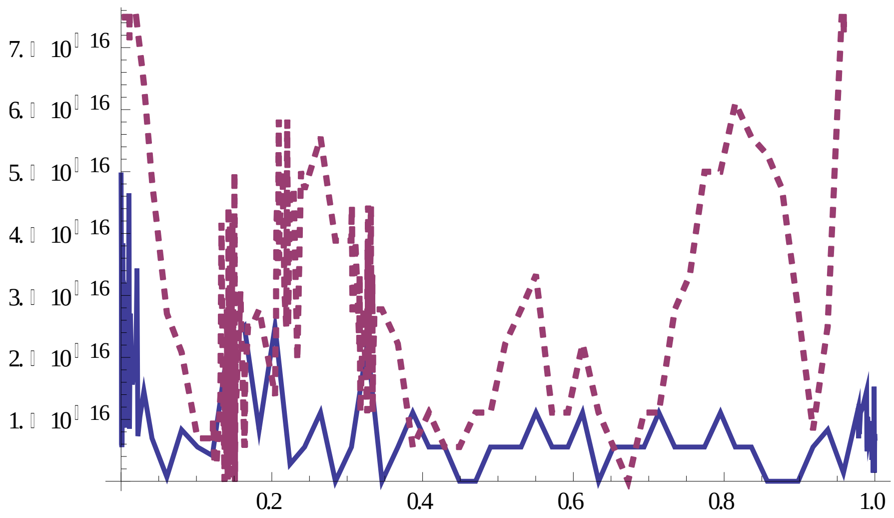

Fig. 2.1 Absolute error plot for Test example 2.1 using scheme S1 for $n = 3$ (thik line) and $n = 5$ (dashed line).

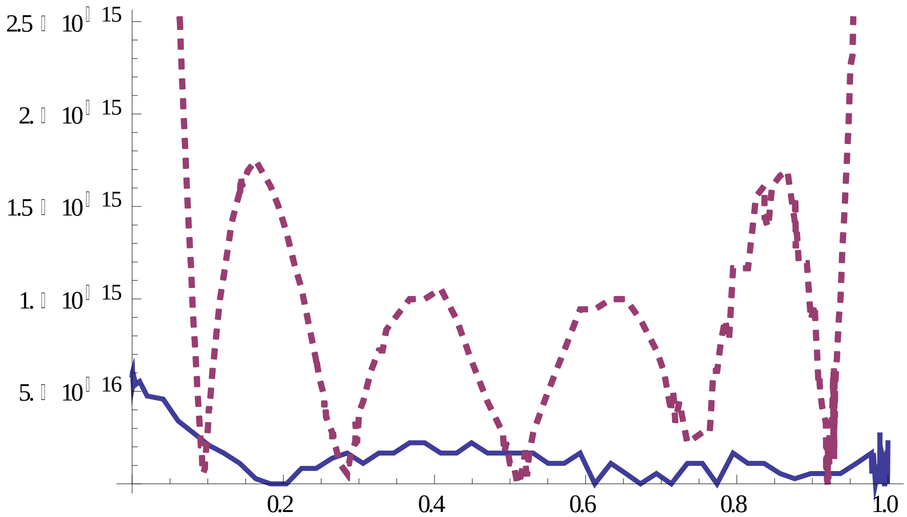

Fig. 2.2 Absolute error plot for Test example 2.1 using scheme S2 for $n = 3$ (thik line) and $n = 5$ (dashed line).

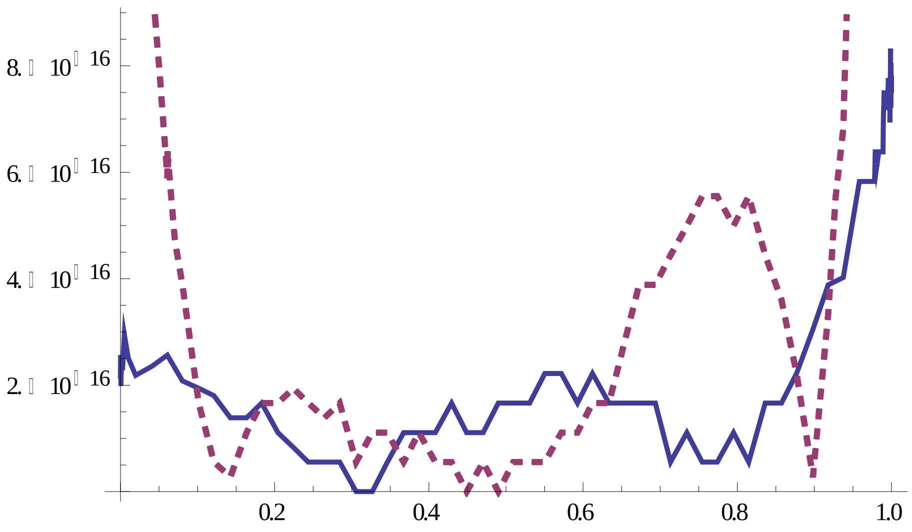

Fig. 2.3 Absolute error plot for Test example 2.1 using scheme S3 for $n = 3$ (thik line) and $n = 5$ (dashed line).

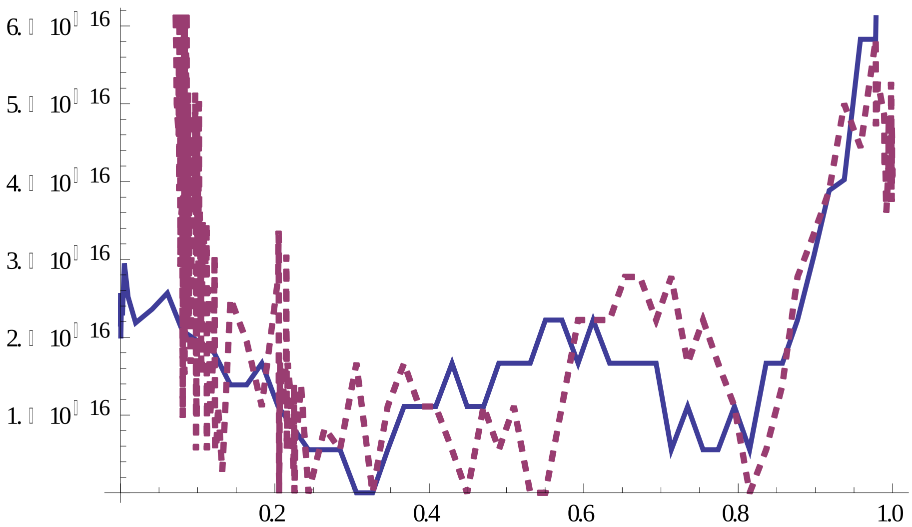

Fig. 2.4 Absolute error plot for Test example 2.1 using scheme S4 for $n = 3$ (thik line) and $n = 5$ (dashed line).

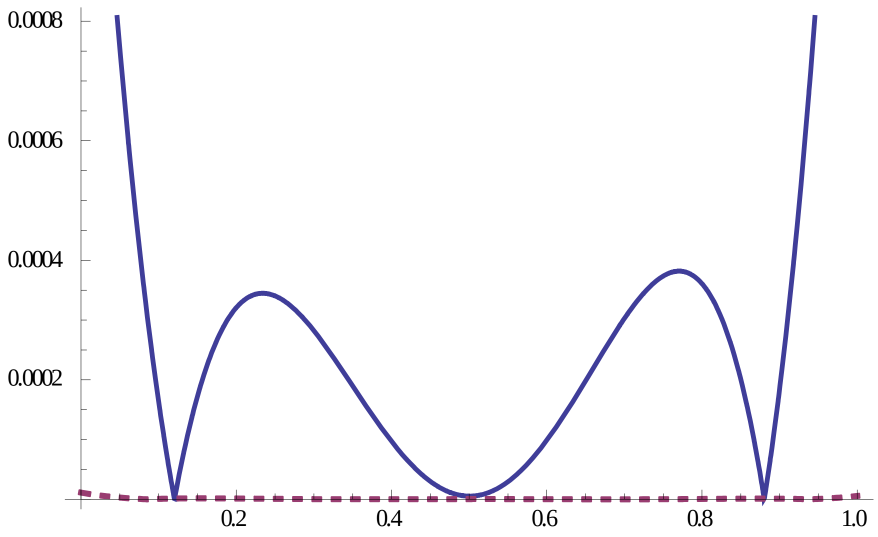

Fig. 2.5 Absolute error plot for Test example 2.2 using scheme S1 for $n = 3$ (thik line) and $n = 5$ (dashed line).

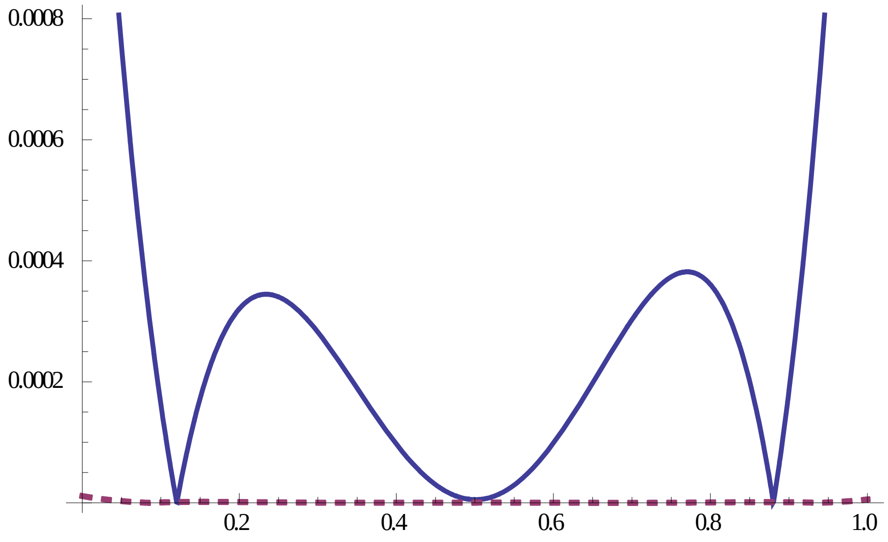

Fig. 2.6 Absolute error plot for Test example 2.2 using scheme S2 for $n = 3$ (thik line) and $n = 5$ (dashed line).

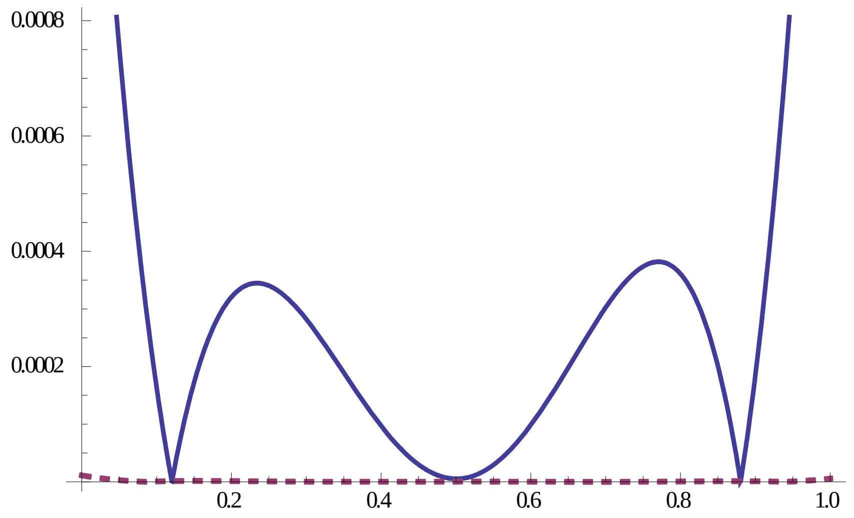

Fig. 2.7 Absolute error plot for Test example 2.7 using scheme S3 for $n = 3$ (thik line) and $n = 5$ (dashed line).

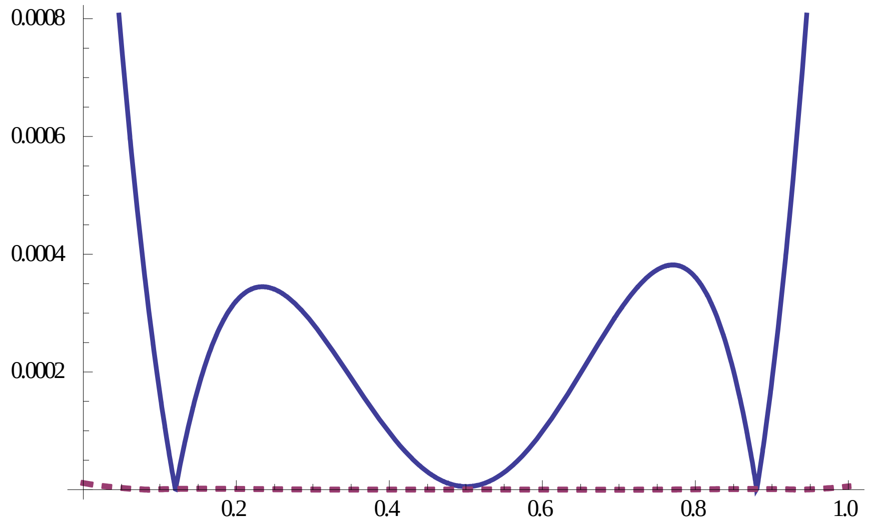

Fig. 2.8 Absolute error plot for Test example 2.2 using scheme S4 for $n = 3$ (thik line) and $n = 5$ (dashed line).

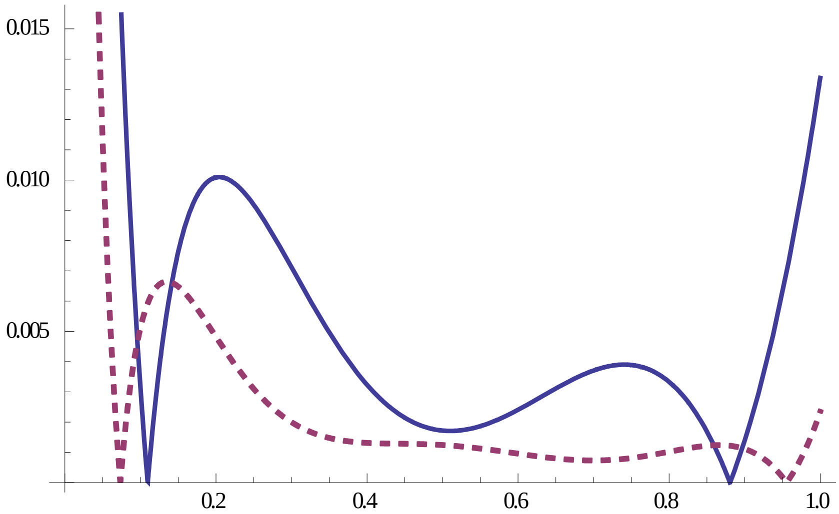

Fig. 2.9 Absolute error plot for Test example 2.3 using scheme S1 for $n = 3$ (thik line) and $n = 5$ (dashed line).

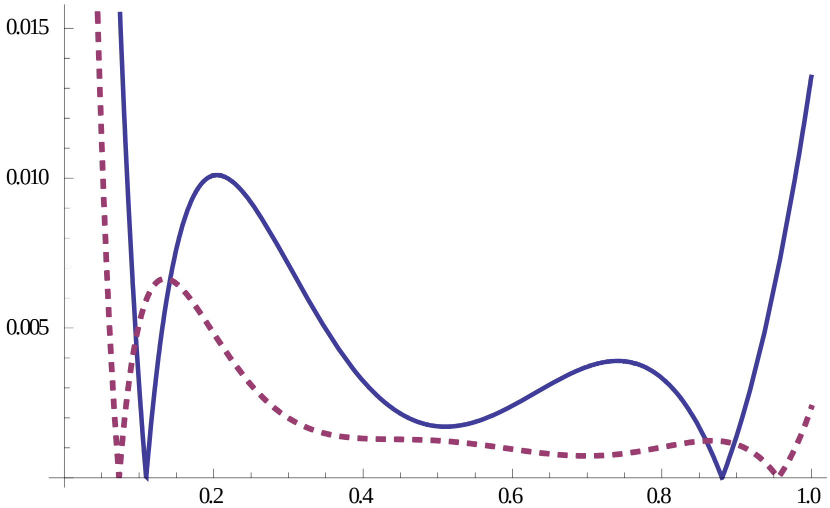

Fig. 2.10 Absolute error plot for Test example 2.3 using scheme S2 for $n = 3$ (thik line) and $n = 5$ (dashed line).

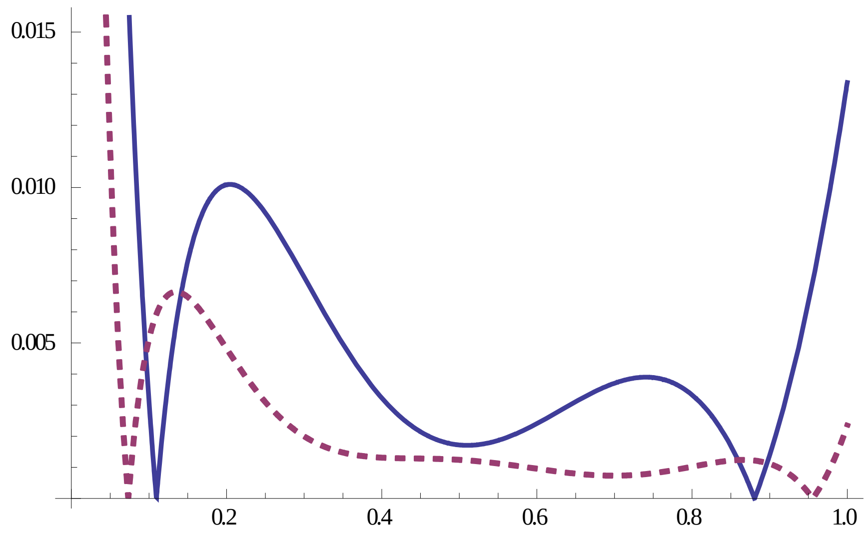

Fig. 2.11 Absolute error plot for Test example 2.3 using scheme S3 for $n = 3$ (thik line) and $n = 5$ (dashed line).

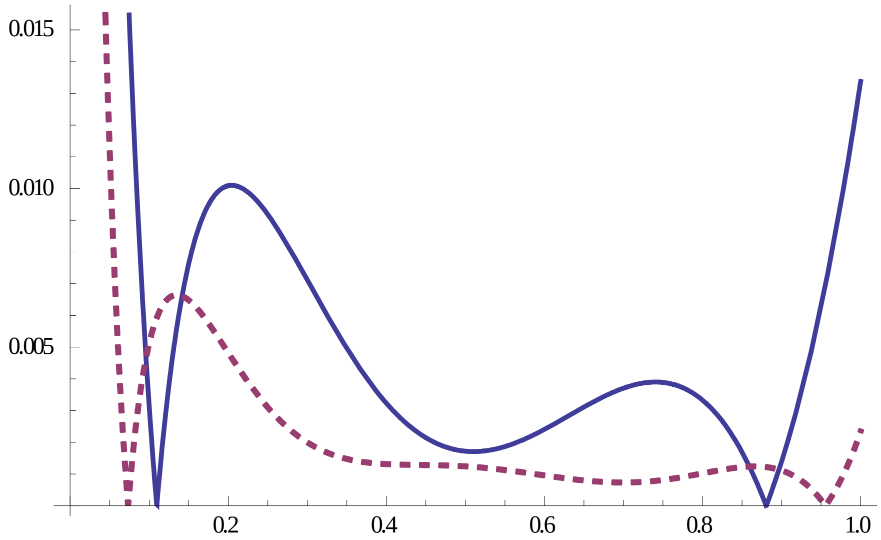

Fig. 2.12 Absolute error plot for Test example 2.3 using scheme S4 for $n = 3$ (thik line) and $n = 5$ (dashed line).

2.5 Conclusion

The presented method is a simple approach to get the approximate solution of generalized Abel's integral equation. The approximate method is based on the polynomial based Collocation methods for Volterra integral equations. Different polynomials are used to get the approximate solutions. Illustrative examples which involve different values of μ

and for different forcing function are considered to examine the accuracy of the proposed method. Approximate solutions are also compared with exact solutions and it is found that the approximate solutions are close to known results. We investigated the performance of the presented methods on the test examples having different solution behavior. From Table 2.2, it is observed that the scheme S3 achieves minimum error and converges rapidly to the exact solution when the exact solution is a polynomial. And in case, where the exact solution is exponential the scheme S2 converges to the exact solution with lesser number of basis elements (table 2.3). From Test example 2.3, we conclude that the each scheme produces similar results. The present study shows that, for any choice of $t_i, (i = 0,1, \dots, n)$, the approximate solution can be obtained uniquely. It is also noticed that present work is more straight forward and computationally efficient than the existing methods for the problem [97].