Chapter1

Introduction

1.1 Integral Equations

The philosophy of integral equations or the inversion of integrals has come unexpectedly into prominence among mathematicians. Du Bois-Reymond prescribed the principal meaning of integral equation in 1888, as an equation which contains one or more integral operator defined on the unknown function [1]. An integral equation in $u(x)$ can be defined as [2],

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt, \tag{1.1}$$

where λ is some parameter, $u(x)$ and $h(x)$ are integration limits, and $k(x,t)$ is called the kernel of an integral equation. The function $u(x)$ is unknown which has to be find out. The functions $f(x)$ and $k(x,t)$ are known.

Integral equations have several forms. An integral equation has two types on the basis of the character of limits of integration, namely:

i. If integration limits are constants, then Eq. (1.1) referees to Fredholm integral equation (FIE),

$$u(x) = f(x) + \int_a^b K(x,t)u(t)dt. \tag{1.2}$$

ii. If either lower or upper limit is not fixed, i.e., only one limit is fixed, then Eq. (1.1) is known as Volterra integral equation (VIE) and written as,

$$u(x) = f(x) + \int_a^x k(x,t)u(t)dt. \tag{1.3}$$

Moreover, integral equations can be characterized on the presence of $u(x)$:

iii. If $u(x)$ is present inside the integral sign only, the integral equation is referred to integral equation of first kind.

iv. If an integral equation contains an integral operator defined on the unknown functions and also the unknown function is present outside the integral, then it is known as the integral equation of second kind.

1.2 Singular Integral Equations

Singular integral equations (SIEs) are integral equations in which either,

I. integration limits are not finite, or

II. $k(x,t)$ tends to infinity inside the range of integration.

Kernel $k(x,t)$ is said to be weakly singular if it can be written as,

$$k(x,t) = a(x,t)(x-t)^{-\alpha}, \quad 0 < \alpha < 1, \tag{1.4}$$

where $a(x,t)$ is a smooth function. When $\alpha = 1$, the SIE is called hyper singular integral equation.

1.2.1 Abel's Integral Equations

The concept of integral equations was introduced by the great mathematician Niels Henrik Abel in the theory of mathematical physics in 1823 [3, 4]. Abel integral equation

(AIE) arises in mechanics as the derivation of the equation of motion of a particle which slides through a smooth curve, given by,

$$f(x) = \int_0^x \frac{u(t)}{(x-t)^{1/2}} dt, \qquad 0 < x < 1, \tag{1.5}$$

A physical particle sliding under the effect of gravity along a smooth curve is taking time $f(x)$ to travel to the initial point 0 from the vertical height x on the curve, and the equation of the curve is to find out.

Abel generalized this problem by presenting the SIEs [5],

$$f(x) = \int_0^x \frac{u(t)}{(x-t)^{\alpha}} dt, \qquad 0 < \alpha < 1, \tag{1.6}$$

referred to Generalized Abel's integral equation (GAIEs) where $0 < \alpha < 1$ is a given constant. The AIEs considered above is a particular case of the GAIEs Eq. (1.6) with $\alpha = 1/2$. The term $(x-t)^{-\alpha}$ is known as Abel's kernel.

The integral equation (1.5) is occasionally named Abel's integral equation of the first type. A trivial generalization of first type AIE is Eq. (1.6). Abel's integral equation in form of second kind Volterra integral equation has been studied in [6, 7] written as,

$$\int_0^t \frac{\psi(x)}{(t-x)^{\mu}} dx + f(t) = \psi(t). \tag{1.7}$$

The AIEs of the second type is,

$$f(x) = \int_x^1 \frac{u(t)}{(t-x)^{1/2}} dt, \qquad 0 < x < 1, \tag{1.8}$$

The general form of AIEs (1.6) is,

$$f(x) = \int_0^x \frac{u(t)}{(h(x)-h(t))^{\alpha}} dt, \qquad 0 < x < 1, \tag{1.9}$$

where $0 < \alpha < 1$ and $h(x)$ is a differentiable and strictly monotonically increasing function of x on [0,1] and $h'(x) \neq 0$ on [0,1] [2].

The more generalized form of AIEs [8, 9],

$$a(x)\int_0^x \frac{u(t)}{(x-t)^\alpha}dt + b(x)\int_x^1 \frac{u(t)}{(t-x)^\alpha}dt = f(x). \qquad 0 < \alpha < 1, \tag{1.10}$$

where $a(x)$ and $b(x)$ are not zero simultaneously. Eq. (1.10) was first studied by Chakrabarty [8] in 1993. Chakrabarty and George [8] derived a formula to obtain the analytical solution of GAIEs. This formula is derived by using the fractional operators namely right and left Riemann-Liouville operators and some standard results of fractional calculus. In 2008, Chakrabarty [9] again provided a theoretic method based on direct function to obtain the closed form of the exact solution. By a suitable variable transformation, Eq. (1.10) can be rewrite as,

$$a(x)\int_0^x \frac{t^{\gamma-1}u(t)}{(x^\gamma-t^\gamma)^\alpha}dt + b(x)\int_x^1 \frac{t^{\gamma-1}u(t)}{(t^\gamma-x^\gamma)^\alpha}dt = f(x). \tag{1.11}$$

Eqs. (1.10) and (1.11) belongs to the class of GAIEs. The Eq. (1.11) was studied in Gakhov's book with the expectations that the coefficients $a(x)$ and $b(x)$ satisfy Holder's condition in [0,1] and $f(x)$ and $u(x)$ can be write as,

$f(x) = \big(x(1-x)\big)^\delta f^*(x)$, and $u(x) = \frac{u^{*(x)}}{\big(x(1-x)\big)^{1-\alpha-\delta}x^{\gamma-1}}$, $\delta > 0$.

1.2.2 Literature Review of Abel's Integral Equations

AIEs, solved and treated as a first integral equation, have a wide background in literature. An extensive range of problems of physics are described by AIEs, for example, transfer of heat [10], the transmission of nonlinear waves [11], diffusion [12], and appliance in neutron transport and traffic theory. There are numerous applications and methods in the

literature involving numerical analysis related to the study of AIEs [13–22]. In 1930, Tamarkin [23] applied numerous integral operators to deliberate the integral solution of AIEs with definite conditions. In 1956, convolution transform was defined in terms of AIEs by Summer [24]. Minerbo and Levy [25] examined the inversion of AIEs by means of orthogonal polynomials. In this paper, four methods for inversion of AIEs were investigated for smooth testing functions including orthogonal polynomials. Hatcher [26] solved a nonlinear equation in closed form by converting the main problem to a GAIEs. In [27], a stable solution of AIEs is approximated in form of Abel's inversion using Bernstein polynomials where a series of orthonormal Bernstein polynomials was constructed to form a basis. Using this basis, the integration was reduced to a matrix and then converted in the system of algebraic equations. In [28], a positive illustration of the solution to the integral equation of Abel's type was provided using fractional ordered Mikusinski operator and discussed a new explanation to the solution of AIEs. AIEs of the first kind was numerically solved by Jahanshahi et al. [29] by approximating the fractional integrals and Caputo derivatives. Kumar et al. [30] suggested a novel and easy approach (HPLTM) for AIEs based on the homotopy perturbation method coupled with Laplace transform. The main benefit of this approach is that it is easy to apply and computationally efficient. Li and Clarkson [31] applied a simple approach for solving AIEs using fractional integrals and Babenko's approach. In this paper, the AIEs of second kind is extended to a distributional space using some definitions and results of the fractional operator.

1.3 Integro-Differential Equations

Integro differential equations (IDEs) [2] can be defined as an integral equation which contains a derivative term of the unknown function $u\,(x)$, written as,

$$u^n(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt, \tag{1.12}$$

where $u^n(x) = \frac{d^n u(x)}{dx^n}$ and $f(x)$ is a known function .

IDEs perform a major role in many scientific and engineering areas, especially when IVPs or BVPs are converted to integral equations. IDEs are classified in the same category as we used before to classify the integral equations.

1.3.1 Volterra Integro-Differential Equations

Volterra integro-differential equations (VIDEs) [2] arise in the inversion of initial value problems into integral equations. In VIDEs, one of the integration limits is a variable like in VIEs. It is very important to notice here that for a specified solution the initial conditions must be given for VIDEs. The VIDEs is defined as,

$$u^n(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt, \tag{1.13}$$

where $f(x)$ is known function, $k(x,t)$ is the kernel and $u\,(x)$ is an unknown function.

1.3.2 Fredholm Integro-Differential Equations

Fredholm integro-differential equations (FdIDEs) arise in the inversion of boundary value problem to an integral equation. In FdIDEs, the integration limits are fixed as in FIEs. To obtain a specified solution, the boundary conditions must be given for FdIDEs. The FdIDE has the form,

$$u^n(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt. \tag{1.14}$$

where $f(x)$ and $k(x,t)$ are known function and $u\,(x)$ is unknown function. Other derivatives of $u(x)$ may appear in the above IDEs.

1.3.3 Volterra-Fredholm Integro-Differential Equations

The Volterra-Fredholm integro-differential equations (VFIDEs) occurs in literature in two forms [2], specifically

$$u^n(x) = f(x) + \lambda_1 \int_a^x K(x,t,u(t))dt + \lambda_2 \int_a^b L(x,t,u(t))dt, \tag{1.15}$$

$$u^n(x,t) = f(x,t) + \lambda \int_a^x \int_D G(x,t,w,z,u(w,z))\, dwdz, \tag{1.16}$$

where D is some closed subset of R^n, $(x,t) \in D \times [0,T]$and $G(x,t,w,z,u(w,z))$ and $f(x,t)$ are analytic on $D \times [0,T\,]$. Initial and boundary conditions must be specified to conclude the solution.

The equations of the form (1.15)-(1.16) play an important role in abstract formulation of many initials, boundary value problems of perturbed differential equations, partial differential equations and partial integro-differential equations which arise in various applications like chemical reaction kinetics, population dynamics, heat-flow in material with memory, viscoelastic and reaction-diffusion problems [32-35].

1.4 Linear Integral Equation

If we define an integral equation in operator form and this operator satisfies linearity in $u(x)$ then the integral equation is called linear otherwise it is nonlinear. In other words, an integral equation is nonlinear if the exponent of $u(x)$ is not equal to one, or if the equation encloses nonlinear functions of $u(x)$, the integral equation is called nonlinear.

1.5 Homogeneous condition for Integral Equation

This category is defined only for second kind integral equations. Second kind integral equations are categorized as homogeneous if the function $f(x)$ in Eq. (1.1) vanishes, otherwise it is known as inhomogeneous.

1.6 Fractional Calculus

In 1695, a letter was written by Leibniz to L`Hospital querying *"Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders*?" L`Hospital found this query interesting and replied to Leibniz with another query: "*What if the order will be 1/2*?" On September 30, 1695, —fractional calculus's exact birthday — Leibniz replied: "*It will lead to a paradox, from which one-day useful consequences will be drawn.*" The query boosted by Leibnitz for non-integer ordered derivative was an unfinished argument for more than 3 centuries. Many prominent mathematicians devoted themselves to this discussion, like J. Liouville, B. Riemann, H. Weyl, J. Fourier, N. H. Abel, S. F. Lacroix, G. Leibniz, A. K. Grunwald and A. V. Letnikov. Further, it was the start of a new branch of mathematics namely fractional calculus [36], which deals with arbitrary ordered differentiations and integrations.

In 1975, B. Ross [37] wrote a book and mentioned that Abel solved an integral equation named Tautochrone problem using fractional derivative in 1823. The first explanation of fractional derivative was given by J. Liouville explicitly in 1832. G. Boole (1844) established a symbolic approach for solving linear differential equations with constant coefficients with the help of fractional calculus. According to Ross's book (1975), B. Riemann (1847) [37], suggested this definition of fractional integration,

$$\mathcal{D}^{-q}f(x) = \frac{1}{\Gamma(q)}\int_a^x (x-t)^{q-1}f(t)dt + \varepsilon(x), \tag{1.17}$$

where $\varepsilon(x)$ is used for Riemann's complementary function.

Though the mathematics of fractional calculus occurred in mathematical history for more than 300 years, its applicability has been executed a bit recently. Nowadays, fractional calculus [38, 39, 40] has been found applicable in many areas of engineering and science. Few of them are cellular diffusion processes, atmospheric diffusion of pollution, dynamics of visco-elastic materials, network traffic, electronics etc.

1.6.1 Fractional Derivatives

Here, we will mention some renowned and broadly applied definitions of fractional derivative which help us to form fractional order modeling [39].

1.6.1.1 Riemann-Liouville (RL) Fractional Derivative

Definition 1.1: If $f \in C[a,b]$ and $q > 0$, then

$$J_{a+}^{q}f(x) = \frac{1}{\Gamma(q)}\int_a^x (x-u)^{q-1}f(u)du, \qquad x > a, \tag{1.18}$$

$$J_{b-}^{q}f(x) = \frac{1}{\Gamma(q)}\int_x^b (u-x)^{q-1}f(u)du, \qquad x < b, \tag{1.19}$$

are known as the left and the right RL fractional integral of order q respectively.

Definition 1.2: $${}^{RL}D_a^q f(x) = \frac{1}{\Gamma(1-q)}\frac{d}{dx}\int_a^x (x-t)^q f(t)dt, \tag{1.20}$$

for $0 < q < 1$, is known as left Riemann-Liouville fractional derivative (RL derivative) of order q whenever the RHS exists [39].

Now, we define RL derivative for an arbitrary value of q as follows.

Definition 1.3: Let $n-1 < q \leq n$, then the left and right RL derivative of order q are defined as [39],

$$ {}^{RL}D_{a+}^{q} f(x) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-q-1} f(t) dt = D^n J_{a+}^{n-q} f(x), x > a, \quad (1.21) $$

$$ {}^{RL}D_{b-}^{q} f(x) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_x^b (t-x)^{n-q-1} f(t) dt = D^n J_{b-}^{n-q} f(x), x < b, \quad (1.22) $$

respectively, whenever the RHSs exist.

In further discussion, unless mentioned otherwise, we denote ${}^{RL}D_{a+}^{q} f(x)$ by ${}^{RL}D_{a}^{q} f(x)$ and $J_{a+}^{q} f(x)$ by $J_{a}^{q} f(x)$, respectively. Also ${}^{RL}D^{q} f(x)$ and, $J^{q} f(x)$ refer to ${}^{RL}D_{0^+}^{q} f(x)$ and $J_{0^+}^{q} f(x)$,respectively.

Properties: (i) The RL derivative is non-zero for a constant.

$$ {}^{RL}D^{q} C = \frac{C t^{-q}}{\Gamma(1-q)} \neq 0. \quad (1.23) $$

(ii) Initial value problem (IVP) containing RL derivative requires initial conditions of the form ${}^{RL}D^{q} f(0)$ *i.e.*,

$$ J^{q}\left({}^{RL}D^{q} f(x)\right) = f(t) - \sum_{j=1}^{n} {}^{RL}D^{q-j} f(0) \frac{t^{q-j}}{\Gamma(q-j+1)}, \; n-1 < q < n, \quad (1.24) $$

which is not useful in real phenomena. To overcome these drawbacks, M. Caputo and F Mainardi (1971) proposed a new definition of the fractional derivative, namely Caputo derivative which permits the construction of initial conditions for fractional IVPs in a form including only the limit values of integer order derivatives at the lower terminal [41].

1.6.1.2 Caputo Fractional Derivative

The definition and properties of the Caputo fractional derivative [39] are given as follows;

Definition 1.4: Let $f \in C^n[a,b]$ and $n-1 < q < n$, then

$$ {}^{C}D_a^q f(x) = \frac{1}{\Gamma(n-q)} \int_a^x (x-u)^{n-q-1} \left(\frac{d^n f(u)}{du^n}\right) du, \quad a < x < b. \tag{1.25}$$

Properties: (i) ${}^{C}D_a^q C=0$, C is a constant. (1.26)

(ii) $\lim_{q \to n} {}^{C}D_a^q f(x) = \frac{d^n f(x)}{dx^n}$. (1.27)

Lemma 1.1: [39] Let $f(x) \in R$ be a differentiable function. Then for any $x \geq x_0$,

$$ \frac{1}{2} {}^{C}D_a^q f^2(x) \leq f(x) {}^{C}D_a^q f(x), \forall\, q \in (0,1). \tag{1.28}$$

Lemma 1.2: Let $f \in C^n[a,b]$ and $n-1 < q < n$, then,

$$ {}^{RL}D_a^q f(x) = {}^{C}D_a^q f(x) + \sum_{k=0}^{n-1} \frac{f^k(a^+)}{\Gamma(1+k-q)} (x-a)^{k-q}. \tag{1.29}$$

From the above lemmas, we can conclude the following results:

(i) If $q = n \in N$, then ${}^{RL}D_a^q f(x) = {}^{C}D_a^q f(x) = D^n f(x)$.

(ii) If $f^k(a) = 0$ for $k = 0,1,\dots,n-1$, then ${}^{RL}D_a^q f(x) = {}^{C}D_a^q f(x)$.

(iii) If $0 < q < 1$, $0 < q < 1$, then ${}^{RL}D_a^q f(x) = {}^{C}D_a^q f(x) + \frac{f(a)}{\Gamma(1-q)} (x-a)^{-q}$.

Theorem 1.1: Let $f \in C^n[a,b]$ and $n-1 < q < n$, then,

$$ J_a^q {}^{C}D_a^q f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^k(a^+)}{\Gamma(1+k)} (x-a)^k, x \geq a. \tag{1.30}$$

Proof: [39]

1.6.2 Leibniz rule

If $f, g \in C^{\infty}[a, b]$, then the Leibnitz rule for the fractional derivative is given by [39],

$${}^{C}D_{a}^{q}\big(f(x)g(x)\big) = \sum_{k=0}^{\infty}\binom{q}{k} g^{k}(x)\,{}^{C}D_{a}^{q-k}f(x) - R_{n}^{q}(x), \tag{1.31}$$

where, $R_{n}^{q}(x) = \frac{1}{n!\,\Gamma(-q)}\int_{a}^{x}\frac{f(t)}{(x-t)^{q+1}}dt\int_{t}^{x} g^{n+1}(u)(t-u)^{n}du.$

1.6.3 Generalized Fractional Derivatives

Here, first we state the definition of the K and A/B-operators and then define the FIDEs in terms of B operator. The operators namely K and A/B operators as presented recently in [42] are defined on integrable functions $h(\xi)$ as follows:

$$(K_{P}^{\alpha}h)(\xi) = r\int_{a}^{\xi}\omega_{\alpha}(\xi,\eta)\,h(\eta)d\eta + s\int_{\xi}^{b}\omega_{\alpha}(\eta,\xi)\,h(\eta)d\eta\,,\ \alpha > 0, \tag{1.32}$$

where, $\xi \in \Im = [a, b]$, $P = \langle a, \xi, b, r, s\rangle$ is a set of parameters, and $\omega_{\alpha}(\xi,\eta)$ be defined on $\Im \times \Im$. We assume $h(\xi)$ and $\omega_{\alpha}(\xi,\eta)$ are the square integrable functions such that right side of the Eq. (1) exists. It follows the linearity property, i.e. for any two integrable functions $h_1(\xi)$ and $h_2(\xi)$,

$$\big(K_{P}^{\alpha}(h_1 + h_2)\big)(\xi) = (K_{P}^{\alpha}h_1)(\xi) + (K_{P}^{\alpha}h_2)(\xi)\,. \tag{1.33}$$

Now we consider A and B-operators [42],

$$(A_{P}^{\alpha}h)(\xi) = \mathfrak{D}^{m}(K_{P}^{m-\alpha}h)(\xi), \tag{1.34}$$

$$= \mathfrak{D}^{m}\left(r\int_{a}^{\xi}\omega_{m-\alpha}(\xi,\eta)h(\eta)d\eta + s\int_{\xi}^{b}\omega_{m-\alpha}(\eta,\xi)h(\eta)d\eta\right),\ \alpha > 0,$$

$$(B_P^{\alpha} h)(\xi) = (K_P^{m-\alpha} \mathfrak{D}^m h)(\xi),$$

$$= r \int_a^{\xi} \omega_{m-\alpha}(\xi, \eta) \mathfrak{D}^m h(\eta) d\eta + s \int_{\xi}^{b} \omega_{m-\alpha}(\eta, \xi) \mathfrak{D}^m h(\eta) d\eta, \ \alpha > 0, \tag{1.35}$$

where, $m-1 < \alpha < m$, m is an integrer and $P = \langle a, \xi, b, r, s \rangle$ and $\mathfrak{D}$ denote the differential operator. In the definition of B-operator, we assume that $\mathfrak{D}^m h(\xi)$ is once integrable on the domain. More details on these operators can be found in [42].

1.7 Fractional Integro-Differential Equation

Fractional differential equations have been developed as a significant object of exploration in recent years motivated by their several applications to the problems arising in physics, mechanics, and other fields. Fractional integro-differential equation (FIDEs) are the type of IDEs in which fractional derivative is defined for the unknown function instead of ordinary derivative. According to the literature, FIDEs can be defined in several ways. The most of the authors defined FIDEs of Volterra type in the following ways [43],

$$\mathfrak{D} w(x) = g(x) + f(x) w(x) + \int_0^x \kappa(x, u) \mathcal{G}\big(w(u)\big) du. \tag{1.36}$$

Here $\mathfrak{D}$ denotes some fractional derivative (RL or Caputo derivative) of w with respect to x and $\mathcal{G}$ is some linear or nonlinear operator and $\kappa(x, u)$ may be smooth or weakly singular defined by,

$$\kappa(x, u) = a(x, u)(x - u)^{-\mu}, \qquad 0 < \mu < 1, \tag{1.37}$$

where $a(x, u)$ is a smooth function.

1.7.1 Literature Review of Fractional Integro-Differential Equations

Mathematical modelling of most of the real-life problems frequently described in form of fractional differential equations (FDEs) and FIDEs. In particular, these equations arise in a lot of practical problems, for examples, electromagnetic waves, heat conduction, dielectric polarization, radiative equilibrium, viscoelasticity, elasticity, fracture mechanics and diffusion equations [44-50] etc. It isn't convincible to tackle all FIDEs analytically, thus we require to create approximations procedure in sense of numerical techniques to solve FIDEs. Recently, fractional calculus has pulled in numerous analysis effectively in various areas by scientists and engineers. The benefit of using fractional integral and derivative is that these are not local property of the function.

The existence and uniqueness of the solution of FDEs are explored in [44, 48]. In 2002, Diethelm [48] published a paper investigating the results of existence, uniqueness, and stability of the solution of nonlinear FDEs considering the Riemann– Liouville differential operators and the initial conditions were taken in Caputo's sense, thus permitting the explanation in a physical manner. In particular, it was investigated how the solution depended on the order and the initial conditions. Local and global existence and uniqueness of the solution FIDEs have been discussed in [51, 52].

Kubo [53] investigated the presence of singular FIDEs in traditional Brownian motion using fractional Langevin equation. There are numerous numerical methods in literature to approximate the solution of singular FIDEs additionally for IDEs. Kamrani [54] solved a stochastic form of FIDEs by Galerkin method and discussed its convergence. A special form of FIDEs has been solved by polynomial spline collocation method [55]. Piecewise polynomial collocation method has been used to approximate the solution of linear FIDEs with weakly singular kernel [56]. Legendre and Chebyshev wavelets have been used to

obtain the approximate solution of FIDEs [57, 58, 59]. Some more methods for the numerical treatment of linear FIDEs such as discrete collocation [41] and collocation method based on hybrid function [60] have been discussed to solve linear FIDEs. In [61], Saadatmandi and Dehghan applied the Legendre collocation method for numerical solution of the FIDEs. Kumar et. al. [62] presented the comparison of three schemes namely linear, quadratic and quadratic-linear and their convergence results for solving the FIDEs. The solution of nonlinear FIDEs has been obtained in using collocation and hybrid functions [63, 64]. In [65], spectral collocation method has been studied for solving FIDEs. The approach is based on Jacobi-Gauss quadrature formula. In [66, 67], the system of FIDEs has been studied by the authors. Galerkin method and wavelet Galerkin method based numerical methods are studied by the authors respectively in [68] and [69] for FIDEs. In [70, 71], authors presented the second kind Chebyshev wavelet-based approximation and CAS wavelet methods respectively for the FIDEs. Some other methods such as least squares method [72], tau approximation method [73] and novel matrices method [74] are discussed by the authors in past few years. More recently in 2015, Tohidi and co-authors [75] presented the Euler function based operational matrix approach for such problems.

1.8 Projection Method:

We describe the projection method [76] for the second kind integral equation,

$$\gamma u(x) - \int_I K(x,t)u(t)dt = f(x), \quad x \in I. \tag{1.38}$$

A function u_n is chosen from a finite dimensional subspace of functions that is supposed to be the best approximation of exact solution $u(x)$. The preferred solution u_n is chosen such that it satisfy (1.38) approximately. There are various senses in which u_n satisfy

(1.38) approximately and these lead to so many methods. The most widespread of these are Galerkin methods and collocation methods, and they are defined below. When we formulate these methods in abstract context with functional analysis, they all need necessary usage of projection operators.

1.8.1 General Theory

The operator form of integral equation (1.38) can be written as,

$$(\gamma - K)u(x) = f(x). \tag{1.39}$$

Here K is supposed to be a compact operator from a Banach space X to X defined by,

$$Ku(x) = \int_I K(x,t)u(t)dt. \tag{1.40}$$

The most popular choices for X are $C(I)$ and $L^2(I)$ .For Galerkin's method and its generalizations, Sobolev spaces $H^0(I)$ are also used commonly, with $H^0(I) = L^2(I)$.

We consider a family of subspaces $X_n \subset X, n \geq 1$, with finite dimension $n+1$. Let X_n has a basis $S = \{\theta_0, \theta_1, \ldots \theta_n\}$ in X. A function $u_n \in X_n$, which is assumed to be the best approximation of u, can be written as,

$$u_n(x) = \sum_{j=0}^{n} c_j \theta_j(x), \quad x \in I. \tag{1.41}$$

Substituting this in Eq. (1.39),

$$\gamma\, u_n(x) - \int_I K(x,t)\, u_n(t)dt = f(x). \tag{1.42}$$

The coefficients $\{c_j | j = 0,1, \ldots, n\}$ are approximated by making the equation to be exact. For further use, we assume,

$$\tau_n(x) = \gamma\, u_n(x) - \int_I K(x,t)\, u_n(t)dt - f(x). \quad (1.43)$$

The term $\tau_n(x)$ is called the residual and it is obtained by replacing u by u_n.

Mathematically,

$$\tau_n(x) = (\gamma - K)\, u_n(x) - f(x),$$

$$= \sum_{j=0}^{n} c_j(\gamma - K)\theta_j(x) - f(x). \quad (1.44)$$

The coefficients $\{c_j | j = 0,1, \ldots, n\}$ are selected by making $\tau_n(x)$ to be nearly zero [76].

1.8.2 Collocation Method:

We choose n nodes $x_0, x_1, \ldots, x_n \in I$ and require,

$$\tau_n(x_i) = 0, \qquad i = 0,1,2, \ldots . n, \quad (1.45)$$

$$\Longrightarrow (\gamma - K)\, u_n(x_i) - f(x_i) = 0,$$

$$\Longrightarrow \gamma \sum_{j=0}^{n} c_j \theta_j(x_i) - \sum_{j=0}^{n} c_j \int_I K(x_i,t)\theta_j(t)dt - f(x_i) = 0. \quad (1.46)$$

This directs to a system of linear equations in unknowns $\{c_j\}$ [76].

Now the question arises, is this system possess a solution, and if so, will it be unique?

1.8.3 Existence Uniqueness of the Solution Obtained by Collocation Method:

For this discussion, an operator named projection operator P_n is introduced such that

$P_n: X \overrightarrow{onto} X_n$.

For $u \in X$, we define $P_n u$ to be that element of X_n that interpolates u at the points $x_0, x_1, \ldots, x_n \in I$. That is,

$$P_n u(x) = \sum_{j=0}^{n} a_j \, \theta_j(x), \tag{1.47}$$

where the coefficients a_j will be calculated by the linear system,

$$\sum_{j=0}^{n} a_j \, \theta_j(x_i) = u(x_i), \qquad i = 0,1,2,\ldots.n. \tag{1.48}$$

The unique values of a_j is obtained if

$$det\big[\theta_j(x_i)\big] \neq 0. \tag{1.49}$$

Hereafter in the whole argument, we consider this is true when we discuss the collocation method. From condition $det\big[\theta_j(x_i)\big] \neq 0$, it is followed that $\{\theta_0, \theta_1, \ldots \theta_n\}$ is the linearly independent set over I [76].

To see more clearly that P_n is linear, and to give a more explicit formula, we introduce a new set of basis functions. For each $i, 1 \leq i \leq n+1$,, let $l_i \in X_n$ be that element that satisfies the interpolation conditions,

$$l_i\big(t_j\big) = \delta_{ij}, \qquad j = 0,1, \ldots, n+1. \tag{1.50}$$

By (1.48), there is a unique such l_i, and the set $\{l_i : 1 \leq i \leq n+1\}$ is a new basis for X_n. With polynomial interpolation, such functions l_i are called Lagrange basis functions, and we will use this name with all types of approximating subspaces X_n. With this new basis, we can write,

$$P_n u(x) = \sum_{j=0}^{n+1} u\big(t_j\big) l_j(t), \tag{1.51}$$

Clearly, P_n is linear and finite rank. In addition, as an operator on $C(I)$ to $C(I)$,

$$\| P_n \| = \max_{t \in I} \sum_{j=1}^{n} |l_j(t)| \tag{1.52}$$

We note that,

$P_n u(x) = 0$, if and only if $u(x_i) = 0, \qquad \forall\, i = 0,1,2,\ldots.n,$

Now Eq. (1.45) asserts that,

$$P_n \tau_n = 0,$$

or,

$$P_n(\gamma - K)\, u_n(x) = P_n f(x), \qquad \forall\, u_n \in X_n. \tag{1.53}$$

1.8.4 Galerkin Method:

To discuss this method, we assume X to be a Hilbert space, for example, $X = L^2(I)$ and let $\langle\, . \,|\, . \,\rangle$ be the inner product defined on X. Now, we need τ_n to satisfy

$$\langle\, \tau_n |\, \theta_i \rangle = 0, \qquad \forall\, i = 0,1,2,\ldots.n. \tag{1.54}$$

The term $\langle\, \tau_n |\, \theta_i \rangle$ denotes the Fourier coefficient of τ_n with respect to the θ_i.

To approximate u_n, from Eq. (1.44) and (1.54), we obtain the linear system given by

$$\gamma \langle\, u_n(x) |\, \theta_i(x) \rangle - \left\langle \int_I K(x,t)\, u_n(t) dt \,\middle|\, \theta_i(x) \right\rangle - \langle\, f(x) |\, \theta_i(x) \rangle = 0,$$

$$\gamma \sum_{j=0}^{n} c_j \left\{ \left\langle \theta_j(x) \,\middle|\, \theta_i(x) \right\rangle - \left\langle \int_I K(x,t) \theta_j(x) dt \,\middle|\, \theta_i(x) \right\rangle \right\} - \langle\, f(x) |\, \theta_i(x) \rangle = 0, \tag{1.55}$$

This is Galerkin's method for finding an estimated solution to Eq. (1.38). Now the same question arises as before in collocation method, is this system possess a solution, and if so, will it be unique? [76]

1.8.5 Existence and Uniqueness of the Solution Obtained by Collocation Method:

For existence and uniqueness of the solution obtained by Galerkin method, again we introduce projection operator,

$P_n: X \overrightarrow{onto} X_n$,

defined as $P_n u$ the solution of the following problem,

$$\| u - P_n u \| = \min_{w \in X_n} \| u - w \|. \tag{1.56}$$

Since the dimension of X_n is finite, it is easy to show that the above problem has a solution; and as X_n is an inner product space, uniqueness of the solution can be proved [76].

To obtain the better understanding of P_n, we define $P_n u$ explicitly by introducing the orthonormal basis $\{\vartheta_0, \vartheta_1, \ldots \vartheta_n\}$ for X_n, which is constructed by using the Gram-Schmidt process from $\{\theta_0, \theta_1, \ldots \theta_n\}$. The element ϑ_i is a linear combination of $\{\theta_0, \theta_1, \ldots \theta_n\}$. Moreover,

$$\langle \vartheta_i | \vartheta_j \rangle = \delta_{ij}, \qquad \forall\, i, j = 0,1,2,\ldots.n. \tag{1.57}$$

With $\{\vartheta_0, \vartheta_1, \ldots \vartheta_n\}$, it is clear that,

$$P_n u = \sum_{i=0}^{n} \langle u | \vartheta_j \rangle \vartheta_j. \tag{1.58}$$

We note that,

$P_n u(x) = 0$, if and only if $\langle u | \vartheta_j \rangle = 0, \qquad \forall\, i = 0,1,2,\ldots.n.$

From the above equation as, Eq. (1.54) can be written as

$$P_n \tau_n = 0,$$

or,

$$P_n(\gamma - K)\, u_n(x) = P_n f(x), \qquad \forall\, u_n \in X_n. \tag{1.59}$$

Note the similarity to Eq. (1.53).

For more study on Projection method see [76-80].