Chapter-5

Investigation on magneto thermolastic disturbances under thermoelasticity with dual phase-lags

[1]5 Investigation on magneto-thermoelastic disturbances induced by thermal shock in an elastic half space having finite conductivity under dual phase-lag heat conduction

5.1 Introduction

The present chapter seeks to investigate the magneto-thermoelastic interactions in a finitely conducting elastic half-space in contact with vacuum under dual phase-lag thermoelasticity theory. Due to the advancement of short-pulse laser technologies and their huge applications to modern micro-fabrication technologies serious attention of researchers is being paid to high rate heating on thin films (Tzou (1995a)). The fact has been noticed that laser pulses can be made shorter to the range of femtoseconds (10^{-15}s) and when the response time is shorter, the non-equilibrium thermodynamic transition and the microscopic effects in the energy exchange during heat transport procedure become prominent. In view of recent experiments, the heat conduction theory of Cattaneo and Vernotte also fails in some cases, specially for heating of thin films (Brorson et al. (1987). In order to take into account the microscopic effects in heat transport mechanism, some models have been developed such as phonon-scattering model (Joseph and Preziosi (1989, 1990), Guyer and Krumhansl (1966)), phonon-electron interaction model (Brorson *et al.* (1987), Anisimov *et al.* (1974) and Fujimoto *et al.* (1984)) and microscopic two-step model

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(Qiu and Tien (1992, 1993)). Subsequently, Tzou (1995(a,b)) has incorporated the effect of micro-structural interactions in the fast transient process of heat transport phenomenon and proposed a more generalized law of heat conduction, known as dual phase-lag model, in the form

$$\overrightarrow{q}(\overrightarrow{r}, t+\tau_q) = -K\overrightarrow{\nabla}T(\overrightarrow{r}, t+\tau_t) \qquad 1.9$$

Here τ_q, τ_t are two delay times where τ_q represents the phase-lag of the heat flux vector and it captures the thermal wave behavior, a small-scale response in time for heat flux and τ_t is the phase lag of the temperature gradient and it captures the effect of phonon-electron interactions, a micro scale response in space. Thus, the dual phase lag concept is capable of capturing the small-scale response in both space and time. The phase-lags τ_q and τ_t are assumed to be positive and they are the intrinsic properties of medium (Tzou (1997)). Equation (1.9) represents a universal model which is a good explanation of all fundamental behaviors in diffusion, thermal wave, phonon-electron scattering associated with the shortening of the response time. This model establishes that either the temperature gradient may dominate the heat flux or that the heat flux may dominate the temperature gradient. Later on, the dual phase-lag heat conduction model has been extended to the theory of thermoelasticity with dual phase-lags by Tzou (1997). Subsequently, this theory of dual phase-lag thermoelasticity has been illustrated along with the formulation of basic governing equations by Chandrashekharaiah (1998a). This theory is now referred to as dual-phase-lag thermoelasticity theory that accounts for the second sound effect. Roychoudhuri (2007b) investigated a problem of one-dimensional waves in an elastic half space with its plane subjected to some boundary conditions to analyze the effect of phase-lags. Some qualitative analysis on dual phase-lag thermoelasticity have been reported by Quintanilla (2003b) and Quintanilla and Racke (2006a, 2006b). Prasad *et al.* (2010) have investigated the propagation of harmonic plane waves by obtaining the dispersion relation for an isotropic and homogeneous medium in the context of this theory. In this respect, we also refer the work reported by Al-Nimr and Al-Huniti (2000), Chen *et al.* (2002), Lee and Tsai (2008) and Abdallah (2009) etc.

Now, we aim to investigate the propagation of magneto-thermoelastic disturbances produced by a thermal shock in a finitely conducting elastic half-space in contact with vacuum. Normal load has been applied on the boundary of the existing media that is supposed to be permeated by a primary uniform magnetic field. We employ both the parabolic type (dual phase-lag magneto-thermoelasticity of type I (MTDPL-I)) and hyperbolic type (dual phase-lag magneto-thermoelasticity of type II (MTDPL-II)) dual phase-lag heat conduction models to account for the interactions among the magnetic, elastic and thermal fields. Integral transform technique is applied to solve the present problem and the analytical results of both the cases have been obtained separately. A detailed analysis of results has been done in order to understand the nature of waves propagating inside the medium and the effects of the phase-lag parameters. The effect of the presence of magnetic field has been highlighted. Numerical results have also been obtained to analyze the effect of magnetic field on the behavior of the solution more clearly and a detailed analysis of the results predicted by two models has been presented. It has been noted that in some cases, there are significant differences in the solution obtained in the contexts of MTDPL-I and MTDPL-II theory of magneto-thermoelasticity.

5.2 Formulation of the problem: governing equations

In our problem, we have taken a homogeneous and isotropic finitely conducting elastic half space permeated by a primary uniform magnetic field such that a normal load and thermal shock have been applied on the boundary $x_1 = 0$, due to which magneto-thermoelastic disturbances have been initiated and allowed to propagate through the medium $x_1 \geq 0$.

To formulate our problem, we need to consider all the basic governing equations of magnetic, thermal and mechanical fields which are mutually interacting each other. Firstly, we consider the following equations given by Maxwell (1867):

$$\vec{\nabla} \times \vec{E} = -\frac{\mu_0}{C} . \frac{\partial \vec{h}}{\partial t} \tag{5.1}$$

$$\vec{\nabla} \times \vec{h} = \frac{4\pi}{C} \vec{j} \tag{5.2}$$

$$\vec{\nabla} . \vec{h} = 0 \tag{5.3}$$

Generalized Ohm's law is given as

$$\vec{j} = \lambda_0 [\vec{E} + \frac{\mu_0}{C} (\dot{\vec{u}} \times \vec{H_0})] \tag{5.4}$$

where $\vec{E}$, $\vec{h}$, $\vec{H_0}$ represent the electric field, perturbed magnetic field and the initial constant magnetic field, respectively. μ_0 and λ_0 are magnetic permeability and electrical conductivity of the medium, respectively and C is the velocity of light. $\vec{j}$ is the current density vector and $\vec{u}$ is the displacement vector.

Equation of motion including electromagnetic field is given by (Kaliski and Nowacki (1962))

$$\mu\nabla^2\vec{u} + (\lambda+\mu)\vec{\nabla}(\vec{\nabla}.\vec{u}) + \frac{\mu_0}{4\pi}[(\vec{\nabla}\times\vec{h})\times\vec{H}_0] - \gamma\vec{\nabla}\theta = \rho\ddot{\vec{u}} \tag{5.5}$$

where, θ is the temperature above uniform reference temperature T_0.

By employing dual phase-lag heat conduction model-II (DPL-II; (see Tzou (1997) and Chandrashekharaiah (1998))), we take the heat conduction equation for our study in the form

$$K(1+\tau_t\frac{\partial}{\partial t})\nabla^2\theta = (1+\tau_q\frac{\partial}{\partial t}+\tau_q^2\frac{\partial^2}{\partial t^2})(\rho C_v\dot{\theta}+\gamma T_0\dot{u}_{i,i}) \tag{5.6}$$

Special case: In the above equation, substituting $\tau_q^2 = 0$, we achieve following heat conduction equation under dual phase-lag model-I (DPL-I)[(1997),(1998)]:

$$K(1+\tau_t\frac{\partial}{\partial t})\nabla^2\theta = (1+\tau_q\frac{\partial}{\partial t})(\rho C_v\dot{\theta}+\gamma T_0\dot{u}_{i,i}) \tag{5.7}$$

The above two heat conduction models are known as hyperbolic type dual phase-lag magneto-thermoelastic model-II (MTDPL-II) and parabolic type dual phase-lag magneto-thermoelastic model-I (MTDPL-I), respectively. Here 'Thomson effect' has been neglected due to its very small value. C_v is the specific heat at constant strain and ρ is the density.

From equations (1)-(4), after elimination of $\vec{E}$ and $\vec{j}$, we achieve the following relation:

$$\nabla^2\vec{h} - \beta\dot{\vec{h}} = -\beta\vec{\nabla}\times(\dot{\vec{u}}\times\vec{H}_0) \tag{5.8}$$

where $\beta = \frac{4\pi\lambda_0\mu_0}{C^2}$

For simplification, we assume that the magneto-thermoelastic waves propagated in the medium $x_i \geq 0$ depend on one direction i.e. x_1 and the time t. Furthermore, it has been assumed that the initial magnetic field vector is applicable towards x_3 axis i.e. $\vec{H_0} = (0, 0, H_3)$, where H_3 is a constant.

Therefore, equations[(5.1)-(5.4)] lead to

$$\vec{j} = \frac{C}{4\pi}(0, -\frac{\partial h_3}{\partial x_1}, 0) \tag{5.9}$$

$$\dot{\vec{h}} = -\frac{C}{\mu_0}(0, 0, \frac{\partial E_2}{\partial x_1}) \tag{5.10}$$

$$\vec{j} = \lambda_0[0, (E_2 - \frac{\mu_0 H_3 \dot{u}_1}{C}), 0] \tag{5.11}$$

Since, wave is propagating in x_1 direction and we have assumed that the magnetic field has been applied in x_3 direction, then consequently electric field is given by $\vec{E} = (0, E_2, 0)$. where

$$E_2 = -\frac{C}{4\pi\lambda_0}.\frac{\partial h_3}{\partial x_1} + \frac{\mu_0 H_3 \dot{u}_1}{C} \tag{5.12}$$

In view of the above assumptions, equations (5.5)-(5.8) reduce to the forms

$$(\lambda + 2\mu)\frac{\partial^2 u_1}{\partial x_1^2} - \frac{\mu_0 H_3}{4\pi}.\frac{\partial h_3}{\partial x_1} - \gamma\frac{\partial \theta}{\partial x_1} = \rho.\frac{\partial^2 u_1}{\partial t^2} \tag{5.13}$$

$$K(1 + \tau_t\frac{\partial}{\partial t})\nabla^2\theta = (1 + \tau_q\frac{\partial}{\partial t} + \tau_q^2\frac{\partial^2}{\partial t^2})(\rho C_v\frac{\partial \theta}{\partial t} + \gamma T_0\frac{\partial^2 u_1}{\partial x_1 \partial t}) \tag{5.14}$$

$$K(1 + \tau_t\frac{\partial}{\partial t})\nabla^2\theta = (1 + \tau_q\frac{\partial}{\partial t})(\rho C_v\frac{\partial \theta}{\partial t} + \gamma T_0\frac{\partial^2 u_1}{\partial x_1 \partial t}) \tag{5.15}$$

$$\frac{\partial^2 h_3}{\partial x_1^2} - \beta\frac{\partial h_3}{\partial t} = \beta H_3\frac{\partial^2 u_1}{\partial x_1 \partial t} \tag{5.16}$$

For simplicity, in what follows we will use the notations $u_1 = u$, $x_1 = x$.

Now, since the medium has been assumed to be in contact with vacuum, above equations need to be added to the electrodynamic equations in vacuum. In vacuum, the system of equations of electrodynamics reduce to the following forms:

$$(\frac{\partial^2}{\partial x'^2} - \frac{1}{C^2}\frac{\partial^2}{\partial t^2})\tilde{h}_3 = 0, \quad (\frac{\partial^2}{\partial x'^2} - \frac{1}{C^2}\frac{\partial^2}{\partial t^2})\tilde{E}_2 = 0, \quad \dot{\tilde{\vec{h}}} = C(0, 0, \frac{\partial \tilde{E}_2}{\partial x'}),$$

$$\dot{\tilde{\vec{E}}} = C(0, \frac{\partial \tilde{h_3}}{\partial x'}, 0) \tag{5.17}$$

where $x^{'} = -x$

For our present study, the equations (5.13), (5.14), (5.16)-(5.17) constitute the system under MTDPL-II model and the equations (5.13), (5.15)-(5.17) constitute the system under MTDPL-I model. We will study both the systems separately.

5.3 Initial and boundary conditions

In general, Maxwell electro-magnetic stress tensor T_{ij} in CGS unit is defined as $T_{ij} = \frac{\mu_0}{4\pi}(h_i H_j + h_j H_i - \delta_{ij} h_k H_k) \quad i,\, j,\, k = 1{,}2{,}3$

Therefore, by using this relation, we have taken for our problem

$$T_{11} = -\frac{\mu_0 h_3 H_3}{4\pi}, \;\; \tilde{T}_{11} = -\frac{\tilde{h}_3 H_3}{4\pi} \tag{5.18}$$

where T_{11} and $\tilde{T}_{11}$ are the components of Maxwell stress tensor in the elastic medium and in vacuum, respectively.

The normal stress in the elastic medium is given by

$$\sigma_{11} = (\lambda + 2\mu)\frac{\partial u}{\partial x} - \gamma\theta \tag{5.19}$$

Therefore, we assume the following boundary conditions as

$$\sigma_{11} + T_{11} - \tilde{T}_{11} = \sigma_0 H(t) \quad on \;\; x = x' = 0 \tag{5.20}$$

$$E_2 = \tilde{E}_2, \quad h_3 = \tilde{h}_3 \quad on \;\; x = x' = 0 \tag{5.21}$$

where $H(t)$ is the Heaviside unit function and σ_0 is a constant stress.

The thermal boundary condition has been taken in the following manner:

$$\theta(x,t) = \theta_0 H(t) \quad on \;\; x = x' = 0 \tag{5.22}$$

where θ_0 is a constant.

The initial conditions for MTDPL-II model are assumed to be homogeneous and they are taken as

$$u(x,0) = 0, \quad \theta(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0$$

$$\frac{\partial \theta(x,0)}{\partial t} = 0, \quad \frac{\partial^2 u(x,0)}{\partial t^2} = 0, \quad \frac{\partial^2 \theta(x,0)}{\partial t^2} = 0$$

In the similar way, the initial conditions for MTDPL-I are considered as

$$u(x,0) = 0, \quad \theta(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad \frac{\partial \theta(x,0)}{\partial t} = 0 \tag{5.23}$$

5.4 Solution of the problem

In order to simplify the solution of the problem, we introduce the following notations and non dimensional quantities:

$\xi = \frac{C_0 x}{k}$, $t' = \frac{C_0^2 t}{k}$, $u' = \frac{C_0(\lambda+2\mu+\alpha_0^2\rho)u}{k\gamma T_0}$, $\theta' = \frac{\theta}{T_0}$, $k = \frac{K}{\rho C_v}$, $\epsilon = \frac{\gamma^2 T_0}{C_e(\lambda+2\mu+\alpha_0^2\rho)}$;

$C_e = \rho C_v$, $h_3 = h$, $\eta_1 = \frac{\mu_0 H_3}{4\pi\gamma T_0}$, $\eta_2 = \frac{1}{k\beta}$, $\eta_3 = \frac{H_3\gamma T_0}{\rho C_0^2}$, $\eta_4 = \frac{C_0^2}{4\pi\lambda_0 k}$, $\eta_5 = \frac{\mu_0\gamma H_3 T_0}{\rho C^2}$;

$C_1^2 = \frac{\lambda+2\mu}{\rho}$, $C_0^2 = C_1^2 + \alpha_0^2$, $\alpha = \frac{C_0}{C}$, $\alpha_0^2 = \frac{\mu_0 H_3^2}{4\pi\rho}$, $\tau_q^{'} = \frac{C_0^2}{k}\tau_q$, $\tau_t^{'} = \frac{C_0^2}{k}\tau_t$

Using above notations and non dimensional quantities on boundary conditions [(5.20)-(5.22)], we get the following simplified forms of our boundary conditions:

$$\frac{C_1^2}{C_0^2}u_{,\xi} - \theta + \eta'\tilde{h} = \frac{\sigma_0}{\gamma T_0}H(\tau) \quad on\ \xi = \xi' = 0 \tag{5.24}$$

$$h = \tilde{h} \quad on\ \xi = \xi' = 0 \tag{5.25}$$

$$-\eta_4 h_{,\xi t} + \eta_5 u_{,tt} + \tilde{h}_{,\xi} = 0 \quad on\ \xi = \xi' = 0 \tag{5.26}$$

$$\theta = \frac{\theta_0}{T_0}H(\tau) \quad on\ \xi = \xi' = 0 \tag{5.27}$$

where we have used the notation $\eta' = \frac{(1-\mu_0)H_3}{4\pi\gamma T_0}$. Here primes have been removed from the quantities t',u', θ' for the shake of clarity.

Further, the total stress σ_1 in the elastic half space is given by

$$\sigma_1 = \sigma_{11} + T_{11} \tag{5.28}$$

From above equation, we get the dimensionless form of total stress as σ, where $\sigma = \frac{\sigma_1}{\gamma T_0}$.

Now, the initial conditions for MTDPL-II model become

$$u(\xi,0) = 0, \quad \theta(\xi,0) = 0, \quad \frac{\partial u'}{\partial t}(\xi,0) = 0$$

$$\frac{\partial \theta}{\partial t}(\xi,0) = 0, \quad \frac{\partial^2 u}{\partial t^2}(\xi,0) = 0, \quad \frac{\partial^2 \theta}{\partial t^2}(\xi,0) = 0$$

Similarly, the initial conditions for MTDPL-I model are reduce to

$$u(\xi,0) = 0, \quad \theta(\xi,0) = 0, \quad \frac{\partial u}{\partial t}(\xi,0) = 0, \quad \frac{\partial \theta}{\partial t}(\xi,0) = 0 \tag{5.29}$$

Equation given by (5.17) get the form

$$\tilde{h}_{,\xi\xi} - \alpha^2 \tilde{h}_{,tt} = 0, \quad \tilde{E}_{2,\xi'\xi'} - \alpha^2 \tilde{E}_{2,tt} = 0 \tag{5.30}$$

where $\xi' > 0$ and $\xi' = -\xi$, $\tilde{h_3} = \tilde{h}$

Now, we study the present problem in the contexts of two different models-MTDPL-I and MTDPL-II separately.

5.4.1 Case-I: Magneto-thermoelastic dual phase lag model-II (MTDPL-II)

First, applying the above non dimensional quantities in equations (5.13), (5.14) and (5.16), we obtain the following equations:

$$\frac{C_1^2}{C_0^2} u_{,\xi\xi} - \eta_1 h_{,\xi} - \theta_{,\xi} - u_{,tt} = 0, \quad \xi > 0 \tag{5.31}$$

$$(1 + \tau_t^{'} \frac{\partial}{\partial t})\theta_{,\xi\xi} - (1 + \tau_q^{'} \frac{\partial}{\partial t} + \tau_q^{'2} \frac{\partial^2}{\partial t^2})(\theta_{,t} + \epsilon u_{,\xi t}) = 0, \quad \xi > 0 \tag{5.32}$$

$$\eta_2 h_{,\xi\xi} - h_{,t} = \eta_3 u_{,\xi t}, \quad \xi > 0 \tag{5.33}$$

On setting $\tau_t^{'} = 0$ and $\tau_q^{'2} = 0$ in above equations (5.31)-(5.33), we find that the resulting equations are in agreement with the equations of (Roychoudhuri and Banerjee (Mukhopadhyay) (1996)). Above equations given by (5.31)-(5.33) constitute the set of coupled partial differential equations with coupled boundary conditions in three variables u, θ and h. Since, it is very difficult to solve this system analytically, therefore for the purpose of simplification, it is assumed that the perturbed magnetic field h varies very slowly with distance so that $\frac{\partial^2 h}{\partial \xi^2} \approx 0$. Then, equations (5.33)and (5.31) reduce to

$$h = -\eta_3 u_{,\xi}, \quad \xi > 0 \tag{5.34}$$

$$u_{,\xi\xi} - \theta_{,\xi} - u_{,tt} = 0, \quad \xi > 0 \tag{5.35}$$

In order to solve the problem, we first apply Laplace transform to both sides of equation (5.35) and we obtain the following equation:

$$(\frac{\partial^2}{\partial \xi^2} - s^2)\overrightarrow{u} - \frac{\partial \theta}{\partial \xi} = 0, \quad \xi > 0 \tag{5.36}$$

where, the over-headed bars represent the fields in the Laplace transform domain.

Equations (5.32) and [(5.24)-(5.27)] reduce to

$$((1 + \tau_t^{'} s^2)\frac{\partial^2}{\partial \xi^2} - (1 + \tau_q^{'} s^2 + \tau_q^{'2} s^3)s)\bar{\theta} - \epsilon s(1 + \tau_q^{'} s^2 + \tau_q^{'2} s^3)\frac{\partial \bar{u}}{\partial \xi} = 0, \quad \xi > 0 \tag{5.37}$$

$$\frac{C_1^2}{C_0^2}\frac{\partial \vec{u}}{\partial \xi} - \bar{\theta} + \eta' \bar{\bar{h}} = \frac{\sigma_0}{\gamma T_0}\frac{1}{s} \quad on \ \xi = \xi' = 0 \tag{5.38}$$

$$\bar{h} = \bar{\bar{h}} \quad on \ \xi = \xi' = 0 \tag{5.39}$$

$$\eta_5 s^2 \bar{u} + (1 - \eta_4 s)\frac{\partial \bar{h}}{\partial \xi} = 0 \quad on \ \xi = \xi' = 0 \tag{5.40}$$

$$\bar{\theta} = \frac{\theta_0}{T_0}.\frac{1}{s} \quad on \ \xi = \xi' = 0 \tag{5.41}$$

Elimination of $\bar{u}$ from equation (5.36) and equation (5.37) yields

$$[(1 + \tau_t^{'} s^2)\frac{\partial^4}{\partial \xi^4} - (s(1 + \tau_q^{'} s^2 + \tau_q^{'2} s^3) + s^2(1 + \tau_t^{'} s^2) + \epsilon s(1 + \tau_q^{'} s^2 + \tau_q^{'2} s^3))\frac{\partial^2}{\partial \xi^2}$$

$$+ s^3(1 + \tau_q^{'} s^2 + \tau_q^{'2} s^3)]\bar{\theta} = 0 \tag{5.42}$$

The general solution of the above equation, vanishing at $\xi \to \infty$ is given by

$$\bar{\theta} = Ae^{-\lambda_1 \xi} + Be^{-\lambda_2 \xi} \tag{5.43}$$

where A and B are arbitrary constants and λ_1^2 and λ_2^2 are the roots of the equation given below:

$$[(1+\tau_t^{'}s^2)\lambda^4-(s(1+\tau_q^{'}s^2+\tau_q^{'2}s^3)+s^2(1+\tau_t^{'}s^2)+\epsilon s(1+\tau_q^{'}s^2+\tau_q^{'2}s^3))\lambda^2+s^3(1+\tau_q^{'}s^2+\tau_q^{'2}s^3)]=0 \tag{5.44}$$

A, B are determined from equations (5.38), (5.41) and (5.43) as

$A=\frac{(k_1\lambda_2^2(1+\tau_t^{'}s^2)-k_2s-k_2\tau_q^{'}s^3-k_2\tau_q^{'2}s^4)}{s\eta''(\lambda_2^2-\lambda_1^2)(1+\tau_t^{'}s^2)}$, $\quad B=\frac{-(k_1\lambda_1^2(1+\tau_t^{'}s^2)-k_2s-k_2\tau_q^{'}s^3-k_2\tau_q^{'2}s^4)}{s\eta''(\lambda_2^2-\lambda_1^2)(1+\tau_t^{'}s^2)}$;

where, $k_1=\frac{\theta_0\eta''}{T_0}$, $k_2=\frac{\theta_0\eta''}{T_0}+\frac{\theta_0\epsilon}{T_0}+\frac{\sigma_0\epsilon}{\gamma T_0}$

By using equations (5.34), (5.37) and (5.43), we achieve the following analytical solution of displacement and perturbed magnetic field and total stress in the Laplace transform domain in terms of A and B as

$$\bar{u}(\xi,s)=\frac{1}{\epsilon s(1+\tau_q^{'}s^2+\tau_q^{'2}s^3)}[A\{\frac{(1+\tau_q^{'}s^2+\tau_q^{'2}s^3)s-(1+\tau_t^{'}s^2)\lambda_1^2}{\lambda_1}\}e^{-\lambda_1\xi}$$

$$+B\{\frac{(1+\tau_q^{'}s^2+\tau_q^{'2}s^3)s-(1+\tau_t^{'}s^2)\lambda_2^2}{\lambda_2}\}e^{-\lambda_2\xi}] \tag{5.45}$$

$$\bar{h}(\xi,s)=-\frac{\eta_3}{\epsilon s(1+\tau_q^{'}s^2+\tau_q^{'2}s^3)}[A\{-(1+\tau_q^{'}s^2+\tau_q^{'2}s^3)s+(1+\tau_t^{'}s^2)\lambda_1^2\}e^{-\lambda_1\xi}$$

$$+B\{-(1+\tau_q^{'}s^2+\tau_q^{'2}s^3)s+(1+\tau_t^{'}s^2)\lambda_2^2\}e^{-\lambda_2\xi}] \tag{5.46}$$

$$\bar{\sigma}(\xi,s)=\frac{1}{\epsilon s(1+\tau_q^{'}s^2+\tau_q^{'2}s^3)}[A\{-(1+\tau_q^{'}s^2+\tau_q^{'2}s^3)s+(1+\tau_t^{'}s^2)\lambda_1^2\}e^{-\lambda_1\xi}$$

$$+B\{-(1+\tau_q^{'}s^2+\tau_q^{'2}s^3)s+(1+\tau_t^{'}s^2)\lambda_2^2\}e^{-\lambda_2\xi}]-[Ae^{-\lambda_1\xi}+Be^{-\lambda_2\xi}]$$

5.4.2 Case-II: Magneto-thermoelastic dual phase-lag model-I (MTDPL-I)

The case when we apply $\tau_q^{'2}=0$ in the above solutions of case-I represent the case of MTDPL-I. Further, applying $\tau_q^{'2}=0$ in equation (5.44), we get the following equation for magneto-thermoelastic dual phase-lag model-I (MTDPL-I):

$$[(1+\tau_t^{'}s^2)\lambda'^4-(s(1+\tau_q^{'}s^2)+s^2(1+\tau_t^{'}s^2)+\epsilon s(1+\tau_q^{'}s^2))\lambda'^2+s^3(1+\tau_q^{'}s^2)]=0 \quad (5.48)$$

We denote the roots of above equation (5.48) as $\lambda_1'^2$ and $\lambda_2'^2$ so that the solutions in the context of MTDPL-I can be obtained from equations (5.43), (5.45), (5.46) and (5.47) by replacing λ_1 and λ_2 with $\lambda_1^{'}$, $\lambda_2^{'}$, respectively.

5.5 Short-time approximation

It is clear from equations (5.44) and (5.48) that the roots of both equations are dependent on Laplace transform parameter s. The closed form analytical solutions of the above system in physical domain is therefore a formidable task. However, phase-lag effects are short-lived; therefore, we attempt to understand the behavior of waves propagating through the medium by deriving the solutions applicable for very small values of time. Hence, in this section we concentrate our attention on small-time approximated analytical solutions for both the cases. For our analysis, we obtain our results for MTDPL-I and MTDPL-II theories separately.

5.5.1 Case-I: Magneto-thermoelastic dual phase-lag model-II

Assuming s to be very large, we obtain the solution of equation (5.44) for large s as

$$\lambda_1=\sqrt{\frac{a_4}{2\tau_t^{'}}}s+\frac{a_5}{2\sqrt{2a_4\tau_t^{'}}}+(\frac{{a_4}^{\frac{1}{2}}}{8\sqrt{2\tau_t^{'}}}(\frac{-a_5^2}{a_4^2}+\frac{4a_6}{a_4})-\frac{\sqrt{a_4}}{2\sqrt{2\tau_t'^3}})\frac{1}{s}+O(s^{-2}) \quad (5.49)$$

$$\lambda_2=\sqrt{\frac{a_7}{2\tau_t^{'}}}s+\frac{a_8}{2\sqrt{2a_7\tau_t^{'}}}+(\frac{{a_6}^{\frac{1}{2}}}{8\sqrt{2\tau_t^{'}}}(\frac{-a_7^2}{a_6^2}+\frac{4a_8}{a_6})-\frac{\sqrt{a_6}}{2\sqrt{2\tau_t'^3}})\frac{1}{s}+O(s^{-2}) \quad (5.50)$$

where

$a_1 = ((1+8\epsilon)\tau_q^{'2} + \tau_t^{'})^2 - 4\tau_q^{'2}\tau_t^{'}, \quad a_2 = 2\tau_q^{'3}(1+\epsilon)^2 - 2\tau_q^{'}\tau_t^{'};$

$a_3 = (1+\epsilon)^2\tau_q^{'2} + 2\tau_t^{'} + 2\tau_q^{'2}(\epsilon - 1), \quad a_4 = a_1^{\frac{1}{2}} + \tau_t^{'} + \tau_q^{'2}(1+\epsilon), \quad a_5 = (1+\epsilon)\tau_q^{'} + \frac{a_2}{2\sqrt{a_1}};$

$a_6 = 1 + \frac{a_1^{\frac{1}{2}}}{8}(\frac{4a_3}{a_1} - \frac{a_2^2}{a_1^2}), \quad a_7 = a_1^{\frac{1}{2}} + \tau_t^{'} - \tau_q^{'2}(1+\epsilon), \quad a_8 = (1+\epsilon)\tau_q^{'} - \frac{a_2}{2\sqrt{a_1}}$

For the sake of convenience, we write λ_1 and λ_2 in the following form:

$$\lambda_1 = \frac{s}{v_1} + B_1 + D_1(\frac{1}{s}) + O(s^{-2}) \tag{5.51}$$

$$\lambda_2 = \frac{s}{v_2} + B_2 + D_2(\frac{1}{s}) + O(s^{-2}) \tag{5.52}$$

where different notations in the above equations are given by

$B_1 = \frac{a_5}{2\sqrt{2a_4\tau_t^{'}}}, \quad \frac{1}{v_1} = \sqrt{\frac{a_4}{2\tau_t^{'}}};$

$D_1 = (\frac{a_4^{\frac{1}{2}}}{8\sqrt{2\tau_t^{'}}}(\frac{-a_5^2}{a_4^2} + \frac{4a_6}{a_4}) - \frac{\sqrt{a_4}}{2\sqrt{2\tau_t^{'3}}});$

$B_2 = \frac{a_8}{2\sqrt{2a_7\tau_t^{'}}}, \quad \frac{1}{v_2} = \sqrt{\frac{a_7}{2\tau_t^{'}}};$

$D_2 = (\frac{a_6^{\frac{1}{2}}}{8\sqrt{2\tau_t^{'}}}(\frac{-a_7^2}{a_6^2} + \frac{4a_8}{a_6}) - \frac{\sqrt{a_6}}{2\sqrt{2\tau_t^{'3}}})$

5.5.1.1 Solution in Laplace transform domain

Substituting values of A and B and λ_1 and λ_2 in the expressions of temperature, displacement and perturbed magnetic field given by (equations (5.43), (5.45), and (5.46), we achieve the solutions in terms of increasing powers of $\frac{1}{s}$ for MTDPL-II in the following forms:

$$\bar{\theta}(\xi, s) = \frac{v_1^2 v_2^2}{\eta'' \tau_t'(v_1^2 - v_2^2)}[(\frac{N_1}{s} + \frac{N_2}{s^2})e^{-(B_1+\frac{s}{v_1})\xi} - (\frac{N_1'}{s} + \frac{N_2'}{s^2})e^{-(B_2+\frac{s}{v_2})\xi}] \tag{5.53}$$

$$\bar{u}(\xi, s) = \frac{v_1^3 v_2^2}{\eta'' \epsilon \tau_q^{'2}\tau_t'(v_1^2 - v_2^2)}[M_1\frac{1}{s^2} + M_2\frac{1}{s^3}]e^{-(B_1+\frac{s}{v_1})\xi} - \frac{v_2^3 v_1^2}{\eta'' \epsilon \tau_q^{'2}\tau_t'(v_1^2 - v_2^2)}[M_1'\frac{1}{s^2} + M_2'\frac{1}{s^3}]e^{-(B_2+\frac{s}{v_2})\xi} \tag{5.54}$$

$$\bar{h}(\xi,s)=\frac{\eta_3 v_1^2 v_2^2}{\eta''\epsilon(v_1^2-v_2^2)}\{[Q_1\frac{1}{s}+Q_2\frac{1}{s^2}]e^{-(B_1+\frac{s}{v_1})\xi}-[Q_1'\frac{1}{s}+Q_2'\frac{1}{s^2}]e^{-(B_2+\frac{s}{v_2})\xi}\} \quad (5.55)$$

where, all constants used above are given by

$N_1=\frac{k_1}{v_2^2}+k_1\tau_t^{'}B_2^2-2v_1v_2\frac{(v_1B_2-v_2B_1)}{(v_1^2-v_2^2)}(-k_2\tau_q^{'}+\frac{2k_1B_2\tau_t^{'}}{v_2})$

$N_2=(\frac{2k_1B_2}{v_2}-k_2)-2v_1v_2\frac{(v_1B_2-v_2B_1)}{(v_1^2-v_2^2)}(\frac{k_1}{v_2^2}+k_1\tau_t^{'}B_2^2)$

$N_1'=\frac{k_1}{v_1^2}+k_1\tau_t^{'}B_1^2-2v_1v_2\frac{(v_1B_2-v_2B_1)}{(v_1^2-v_2^2)}(-k_2\tau_q^{'}+\frac{2k_1B_1\tau_t^{'}}{v_1})$

$N_2'=(\frac{2k_1B_1}{v_1}-k_2)-2v_1v_2\frac{(v_1B_2-v_2B_1)}{(v_1^2-v_2^2)}(\frac{k_1}{v_1^2}+k_1\tau_t^{'}B_1^2)$

$M_1=(\tau_q^{'2}-\frac{\tau_t^{'}}{v_1^2})(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_2^2})$

$M_2=\{(B_1v_1(\tau_q^{'2}-\frac{\tau_t^{'}}{v_1^2})+\tau_q^{'}-2B_1\tau_t^{'})+(\tau_q^{'2}-\frac{\tau_t^{'}}{v_1^2})(\frac{2v_1v_2(v_1B_2-v_2B_1)}{(v_1^2-v_2^2)}+\frac{1}{\tau_q^{'}}\}$

$M_1'=(\tau_q^{'2}-\frac{\tau_t^{'}}{v_2^2})(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_1^2})$

$M_2'=\{(B_2v_2(\tau_q^{'2}-\frac{\tau_t^{'}}{v_2^2})+\tau_q^{'}-2B_2\tau_t^{'})+(\tau_q^{'2}-\frac{\tau_t^{'}}{v_2^2})(\frac{2v_1v_2(v_1B_2-v_2B_1)}{(v_1^2-v_2^2)}+\frac{1}{\tau_q^{'}}\}$

$Q_1=(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_1})(-\frac{1}{\tau_t^{'}}+\frac{1}{v_1^2\tau_q^{'2}}),\ Q_1'=(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_2})(-\frac{1}{\tau_t^{'}}+\frac{1}{v_2^2\tau_q^{'2}})$

$Q_2=(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_1})(-\frac{1}{\tau_t^{'}}+\frac{1}{v_1^2\tau_q^{'2}})+(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_2^2})(\frac{2B_1}{v_1\tau_q^{'2}}-\frac{1}{\tau_q^{'}\tau_t^{'}})-\frac{2v_1v_2(v_1B_2-v_2B_1)}{(v_1^2-v_2^2)})(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_1})(-\frac{1}{\tau_t^{'}}+\frac{1}{v_1^2\tau_q^{'2}})$

$Q_2'=(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_2})(-\frac{1}{\tau_t^{'}}+\frac{1}{v_2^2\tau_q^{'2}})+(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_1^2})(\frac{2B_2}{v_2\tau_q^{'2}}-\frac{1}{\tau_q^{'}\tau_t^{'}})-\frac{2v_1v_2(v_1B_2-v_2B_1)}{(v_1^2-v_2^2)}(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_2})(-\frac{1}{\tau_t^{'}}+\frac{1}{v_2^2\tau_q^{'2}})$

From equation (5.47), we further obtain the non dimensional total stress in the half space in Laplace transform domain as

$$\bar{\sigma}(\xi,s)=\frac{v_1^2v_2^2}{\eta''\epsilon\tau_q^{'2}\tau_t'(v_1^2-v_2^2)}\{[P_1\frac{1}{s}+P_2\frac{1}{s^2}]e^{-(B_1+\frac{s}{v_1})\xi}-[P_1'\frac{1}{s}+P_2'\frac{1}{s^2}]e^{-(B_2+\frac{s}{v_2})\xi}\}$$
$$-\frac{v_1^2v_2^2}{\eta''\tau_t'(v_1^2-v_2^2)}[(\frac{N_1}{s}+\frac{N_2}{s^2})e^{-(B_1+\frac{s}{v_1})\xi}-(\frac{N_1'}{s}+\frac{N_2'}{s^2})e^{-(B_2+\frac{s}{v_2})\xi}] \quad (5.56)$$

where

$P_1=(-\tau_q^{'2}+\frac{\tau_t^{'}}{v_1^2})(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_2^2}),\ P_1'=(-\tau_q^{'2}+\frac{\tau_t^{'}}{v_2^2})(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_1^2})$

$P_2=\{-(-\tau_q^{'2}+\frac{\tau_t^{'}}{v_1^2})(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_2^2})(\frac{1}{\tau_q^{'}}+\frac{2v_1v_2(v_1B_2-v_2B_1)}{(v_1^2-v_2^2)})+(-\tau_q^{'2}+\frac{\tau_t^{'}}{v_1^2})(-k_2\tau_q^{'}+\frac{2k_1B_2\tau_t^{'}}{v_2})+(-\tau_q^{'2}+\frac{\tau_t^{'}}{v_1^2})(-k_2\tau_q^{'2}+\frac{k_1\tau_t^{'}}{v_2^2})\}$

$P_2' = \{-(-\tau_q'^2+\frac{\tau_t'}{v_2^2})(-k_2\tau_q'^2+\frac{k_1\tau_t'}{v_1^2}(\frac{1}{\tau_q'}+\frac{2v_1v_2(v_1B_2-v_2B_1)}{(v_1^2-v_2^2)})+(-\tau_q'^2+\frac{\tau_t'}{v_2^2})(-k_2\tau_q'+\frac{2k_1B_1\tau_t'}{v_1})+$
$(-\tau_q'^2+\frac{\tau_t'}{v_2^2})(-k_2\tau_q'^2+\frac{k_1\tau_t'}{v_1^2})\}$

This completes the solution in Laplace transform domain for the case of MTDPL-II model.

5.5.1.2 Solution in physical domain

The solutions obtained in the previous section are given in Laplace transform domain. The solution of different fields in physical domain can be derived by inverting the Laplace transforms involved in the expressions given by equations [(5.53)-(5.56)] for MTDPL-II case. By applying suitable formulae of Laplace inversion, we finally obtain the solution in physical domain for the case of MTDPL-II as follows:

$$\theta(\xi,t) = \frac{v_1^2v_2^2}{\eta''(v_1^2-v_2^2)}[e^{-B_1\xi}(N_1H(t-\frac{\xi}{v_1})+N_2(t-\frac{\xi}{v_1})H(t-\frac{\xi}{v_1}))-$$
$$e^{-B_2\xi}(N_1'H(t-\frac{\xi}{v_2})+N_2'(t-\frac{\xi}{v_2})H(t-\frac{\xi}{v_2}))] \tag{5.57}$$

$$u(\xi,t) = \frac{v_1^3v_2^2}{\eta''\epsilon\tau_q'^2(v_1^2-v_2^2)}e^{-B_1\xi}[F_1(t-\frac{\xi}{v_1})H(t-\frac{\xi}{v_1})+F_2(t-\frac{\xi}{v_1})^2H(t-\frac{\xi}{v_1})]$$
$$-\frac{v_1^3v_2^2}{\eta''\epsilon\tau_q'^2(v_1^2-v_2^2)}e^{-B_2\xi}[M_1'(t-\frac{\xi}{v_1})H(t-\frac{\xi}{v_1})+M_2'(t-\frac{\xi}{v_1})^2H(t-\frac{\xi}{v_1})] \tag{5.58}$$

$$h(\xi,t) = \frac{\eta_3v_1^2v_2^2}{\eta''\epsilon(v_1^2-v_2^2)}\{[Q_1H(t-\frac{\xi}{v_1})+Q_2(t-\frac{\xi}{v_1})H(-\frac{\xi}{v_1}))]e^{-B_1\xi}$$
$$-[Q_1'H(t-\frac{\xi}{v_2})+Q_2'(t-\frac{\xi}{v_2})H(t-\frac{\xi}{v_2})]e^{-B_2\xi}\} \tag{5.59}$$

$$\sigma(\xi,t) = \frac{v_1^2v_2^2}{\eta''\epsilon\tau_q'^2\tau_t'(v_1^2-v_2^2)}\{[P_1H(t-\frac{\xi}{v_1})+P_2(t-\frac{\xi}{v_1})H(t-\frac{\xi}{v_1})]e^{-B_1\xi}$$
$$-[P_1'H(t-\frac{\xi}{v_2})+(t-\frac{\xi}{v_2})H(t-\frac{\xi}{v_2})]e^{-B_2\xi}\}-\frac{v_1^2v_2^2}{\eta''\tau_t'(v_1^2-v_2^2)}\{[N_1H(t-\frac{\xi}{v_1})+N_2(t-\frac{\xi}{v_1})H(t-\frac{\xi}{v_1})]e^{-B_1\xi}$$

$$-[N_1' H(t-\frac{\xi}{v_2}) + N_2'(t-\frac{\xi}{v_2})H(t-\frac{\xi}{v_2})]e^{-B_2\xi}\} \tag{5.60}$$

5.5.2 Case-II: Magneto-thermoelastic dual phase-lag model-I (MTDPL-I)

In a similar way like MTDPL-II, assuming s to be very large, we obtain the solution of equation (5.48) for large s as

$$\lambda_1' = C_2 + s + \frac{B_4}{s} + O(s^{-2}) \tag{5.61}$$

$$\lambda_2' = (\frac{G_1}{\sqrt{s}} + \frac{\sqrt{s}}{c_2}) + O(s^{\frac{-3}{2}}) \tag{5.62}$$

where the different constants which are independent of s are given by
$B_4 = 16(2+\tau_t^{'}) + 8\tau_q^{'2}(1+2(1+\epsilon)^2) - \frac{1}{2\tau_t^{'}} - \frac{\epsilon^2\tau_q^{'2}}{16\tau_t^{'2}}$, $C_2 = \frac{\epsilon\tau_q^{'}}{4\tau_{t'}}$;
$G_1 = \frac{B_2\tau_q^{'}}{2(2+\epsilon)\tau_t^{'}}$, $\frac{1}{c_2} = \sqrt{\frac{(2+\epsilon)\tau_q^{'}}{2\tau_t^{'}}}$

5.5.2.1 Solution in Laplace transform domain

From the same pattern as in case 5.1.1, applying $\tau_q^{'2} = 0$ and replacing λ_1 with λ_1' and λ_2 with λ_2' in equations [(5.43),(5.45) and (5.46)] and in the expressions of A and B we obtain the following results for MTDPL-I:

$$\bar{\theta}(\xi,s) = \frac{1}{\eta''\tau_t'(-1+\frac{2G_1}{c_2})}[\frac{S_1}{s^2} + \frac{S_2}{s^3}]e^{-(s+C_2)\xi}$$

$$-\frac{1}{\eta''\tau_t'(-1+\frac{2G_1}{c_2})}[\frac{S_1'}{s} + \frac{S_2'}{s^2}]e^{-(\frac{\sqrt{s}}{c_2}+\frac{G_1}{\sqrt{s}})\xi} \tag{5.63}$$

$$\bar{u}(\xi,s) = \frac{1}{\eta''\tau_q'\tau_t'\epsilon}[\frac{L_1}{s^2} + \frac{L_2}{s^3}]e^{-(C_2+s)\xi} - \frac{c_2}{\eta''\tau_q'\tau_t'\epsilon}[\frac{L_1'}{s^{\frac{3}{2}}} + \frac{L_2'}{s^{\frac{5}{2}}}]e^{-(\frac{\sqrt{s}}{c_2}+\frac{G_1}{\sqrt{s}})\xi} \tag{5.64}$$

$$\bar{h}(\xi,s) = \frac{1}{\eta''\tau_q'\tau_t'\epsilon(1-\frac{2G_1}{c_2})}[R_1\frac{1}{s} + R_2\frac{1}{s^2}]e^{-(C_2+s)\xi}$$

$$-\frac{1}{\eta''\tau_q'\tau_t'\epsilon(1-\frac{2G_1}{c_2})}[R_1'\frac{1}{s}+R_2'\frac{1}{s^2}]e^{-(\frac{\sqrt{s}}{c_2}+\frac{G_1}{\sqrt{s}})\xi} \tag{5.65}$$

where different notations are given by

$S_1 = (k_1\tau_t^{'} - \frac{k_2\tau_q^{'}}{c_2^2})$, $S_2 = (\frac{-2G_1k_1k_2\tau_t^{'}\tau_q^{'}}{c_2} - \frac{(-2C_2+\frac{1}{c_2^2})}{(-1+\frac{2G_1}{c_2})}(k_1\tau_t^{'} - \frac{k_2\tau_q^{'}}{c_2^2}))$;

$S_1' = k_1\tau_t'$, $S_2' = \{(-k_2\tau_q^{'} + 2C_2k_1\tau_t^{'}) + k_1\tau_t^{'}\frac{(-2C_2+\frac{1}{c_2^2})}{(-1+\frac{2G_1}{c_2})}\}$;

$L_1 = \tau_t^{'}(\frac{k_1\tau_t^{'}}{c_2^2} - k_2\tau_q^{'})$, $L_2 == \frac{2k_1\tau_t^{'}\tau_q^{'}G_1}{c_2} + (\frac{\tau_t^{'}}{c_2} - \tau_q^{'} - C_2\tau_t^{'})(-k_2\tau_q^{'} + \frac{k_1\tau_t^{'}}{c_2^2})$;

$L_1' = (-\tau_q^{'} + \frac{\tau_t^{'}}{c_2^2})(-k_2\tau_q^{'} + 2k_1C_2\tau_t^{'})$, $L_2' = k_1\tau_t^{'}(2C_2 - \frac{1}{c_2})(\tau_q^{'} - \frac{\tau_t^{'}}{c_2^2}) - c_2G_1(\tau_q^{'} - \frac{\tau_t^{'}}{c_2})$;

$R_1 = \tau_t^{'}(\frac{-k_1\tau_t^{'}}{c_2^2} - k_2\tau_t^{'})$, $R_2 = [\frac{2G_1k_1\tau_t^{'2}}{c_2^2} + (2C_2\tau_t^{'} - \tau_q^{'})(-k_2\tau_q^{'} + \frac{k_1\tau_t^{'}}{c_2^2})]$;

$R_1' = k_1\tau_t'(-\tau_q^{'} + \frac{\tau_t^{'}}{c_2^2})$, $R_2' = [\frac{2G_1k_1\tau_t^{'2}}{c_2^2}(-\tau_q^{'} + \frac{\tau_t^{'}}{c_2^2})(-k_2\tau_q^{'} + 2k_1C_2\tau_t^{'})]$

Similarly, we obtain the non dimensional total stress in Laplace transform domain (from eq. (5.47)) for MTDPL-I model is given by

$$\bar{\sigma}(\xi,s) = \frac{1}{\eta''\epsilon\tau_t^{'}\tau_q^{'}}[\{\frac{b_1}{s}+\frac{b_1'}{s^2}\}e^{-(C_2+s)\xi} - \{\frac{d}{s}+\frac{d'}{s^2}\}e^{-(\frac{G_1}{\sqrt{s}}+\frac{\sqrt{s}}{c_2})}]$$

$$-\frac{1}{\eta''\tau_t^{'}(-1+\frac{2G_1}{c_2})}[\frac{S_1}{s^2}+\frac{S_2}{s^3}]e^{-(s+C_2)\xi} + \frac{1}{\eta''\tau_t^{'}(-1+\frac{2G_1}{c_2})}[\frac{S_1'}{s}+\frac{S_2'}{s^2}]e^{-(\frac{\sqrt{s}}{c_2}+\frac{G_1}{\sqrt{s}})\xi} \tag{5.66}$$

where

$b_1 = 2C_2\tau_t^{'}(\frac{-2G_1}{c_2} + (-2C_2 + \frac{1}{c_2}))(-k_2\tau_q^{'} + \frac{1}{c_2^2})$

$b_1' = (\frac{-2G_1}{c_2} + (-2C_2 + \frac{1}{c_2}))(-k_2\tau_q' + \frac{1}{c_2^2})(\frac{1}{\tau_q'}(-k_2\tau_q^{'} + \frac{1}{c_2^2}) + \frac{2G_1}{c_2}(-2C_2 + \frac{1}{c_2}))$

$d = (2C_2 - \frac{1}{c_2})k_1(1+\tau_t^{'}) + 2C_2 - \frac{1}{c_2}\frac{\tau_t^{'}}{c_2^2}(-k_2\tau_q^{'} + 2k_1C_2\tau_t^{'}) + (-k_2\tau_q^{'} + 2k_1C_2\tau_t^{'})(\frac{2G_1\tau_t^{'}}{c_2} - \tau_q^{'})$

$d' = (\frac{1}{\tau_q'}(-k_2\tau_q^{'} + 2k_1C_2\tau_t^{'}) + k_1(1+\tau_t^{'})(-2C_2 + \frac{1}{c_2}))(\frac{\tau_t'}{c_2^2}) + (\frac{2G_1\tau_t'}{c_2} - \tau_q^{'})(2C_2 - \frac{1}{c_2^2})(-k_2\tau_q^{'} +$
$2k_1C_2\tau_t^{'})$

This completes the solution in Laplace transform domain for the case of MTDPL-I model.

5.5.2.2 Solution in physical domain

In a similar way like the case of MTDPL-II, the solution of different fields in physical domain for this case can be derived by inverting the Laplace transform involved in the expressions given by equations [(5.63)-(5.66)]. By using suitable formulae of Laplace inversion, we obtain the solution in physical domain for the present case as follows:

$$\theta(\xi,t) = \frac{1}{\eta''\tau_t'(-1+\frac{2G_1}{c_2})} e^{-C_2\xi}[S_1(t-\xi)H(t-\xi) + S_2(t-\xi)^2 H(t-\xi)]$$

$$-\frac{1}{\eta''\tau_t'(-1+\frac{2G_1}{c_2})}[S_1' Erfc(\frac{\xi}{2c_2\sqrt{t}}) + S_2' 4t\, i^2 Erfc(\frac{\xi}{2c_2\sqrt{t}})] \tag{5.67}$$

$$u(\xi,t) = \frac{1}{\eta''\tau_t'\tau_q'\epsilon} e^{-C_2\xi}[L_1(t-\xi)H(t-\xi) + L_2(t-\xi)^2 H(t-\xi)] - \frac{c_2}{\eta''\tau_t'\tau_q'\epsilon}$$

$$[F_1(4t)^{\frac{1}{2}} i\, Erfc(\frac{\xi}{2c_2\sqrt{t}}) + F_2(4t)^{\frac{3}{2}} i^3 Erfc(\frac{\xi}{2c_2\sqrt{t}})] \tag{5.68}$$

$$h(\xi,t) = \frac{1}{\eta''\tau_t'\tau_q'\epsilon(1-\frac{2G_1}{c_2})}[R_1 H(t-\xi) + R_2(t-\xi)H(t-\xi)]e^{-C_2\xi}$$

$$-\frac{1}{\eta''\tau_t'\tau_q'\epsilon(1-\frac{2G_1}{c_2})}[R_1' Erfc(\frac{\xi}{2c_2\sqrt{t}}) + R_2' 4t\, i^2 Erfc(\frac{\xi}{2c_2\sqrt{t}})] \tag{5.69}$$

$$\sigma(\xi,t) = \frac{1}{\eta''\epsilon\tau_t'\tau_q'}[e^{-C_2\xi}\{b_1 H(t-\xi) + b_1'(t-\xi)H(t-\xi)\}$$

$$-\{d\, Erfc(\frac{\xi}{2c_2\sqrt{t}}) + d' 4t\, i^2 Erfc(\frac{\xi}{2c_2\sqrt{t}})\}] - \frac{1}{\eta''\epsilon\tau_t'(-1+\frac{2G_1}{c_2})} e^{-C_2\xi}$$

$$[S_1(t-\xi)H(t-\xi)+S_2(t-\xi)^2 H(t-\xi)] + \frac{1}{\eta''\epsilon\tau_t'(-1+\frac{2G_1}{c_2})}[S_1' Erfc(\frac{\xi}{2c_2\sqrt{t}}) + S_2' 4t i^2 Erfc(\frac{\xi}{2c_2\sqrt{t}})] \tag{5.70}$$

where $i^n Erfc(z) = \int_z^\infty i^{n-1} erf(x)dx$, $i^0 Erfc(z) = Erfc(z)$, $i^{-1}Erfc(z) = \frac{2}{\sqrt{\pi}} e^{-z^2}$

5.6 Analysis of analytical results

The solution obtained in the sections 5.5.1.2 and 5.5.2.2 for different fields in the physical domain indicate some significant informations predicted by two different models of dual phase-lags. From the short time approximated solutions given by equations (5.57)-(5.60) in the context of MTDPL-II model and the solutions given by (5.67)-(5.70) in the case of MTDPL-I, we can observe that the solution of each field consists of two parts under both the theories of magneto-thermoelasticity (MTDPL-I and MTDPL-II); one part of solution is modified elastic and the other one is modified thermal in nature. In the case of MTDPL-I model, the terms containing $H(t-\xi)$ represent modified elastic wave propagating with speed unity. Hence, in this case we conclude that the non dimensional speed of elastic wave is finite and equal to 1, which imply that non dimensional speed of elastic wave is not effected by any of the phase-lag parameters $\tau_q^{'}$ and $\tau_t^{'}$ and it is also independent from the effect of magnetic field. We further note that the modified elastic wave in this case is propagating with an attenuation and C_2 is the attenuating coefficient of this wave. From the expression of C_2, it is evident that the attenuation coefficient is clearly dependent on phase-lag parameters $\tau_q^{'}$ and $\tau_t^{'}$ in such a manner that on increasing $\tau_q^{'}$, attenuation coefficient increases and when we increase $\tau_t^{'}$, the value of attenuation coefficient decreases. Attenuation coefficient is also dependent on the magnetic field (see eq. (5.61)). It is observed that the value of attenuation coefficient decreases when magnetic field increases. The other part of solutions of each field in case of MTDPL-I theory is not wave type, but a diffusive type and imply that the speed of thermal wave is not finite in this case. The solution is however influenced by the presence of magnetic field.

The solution under MTDPL-II model are completely different in nature. In this case, solution of each field like, displacement, temperature, total stress consists of two different waves propagating with finite speeds and attenuating with distance. Here, v_1 and v_2 are the finite speeds of the modified elastic and modified thermal waves, respectively (since $v_1 < v_2$). Accordingly, in the solutions of temperature, displacement and stress under MTDPL-II, the terms which contain $H(t-\frac{\xi}{v_1})$, represent the contribution of elastic mode wave in the neighborhood of the wavefront $\xi = v_1 t$; similarly the terms containing $H(t-\frac{\xi}{v_2})$ represent the incorporation of thermal mode wave in the neighborhood of wavefront $\xi = v_2 t$. The expressions for the speeds denote that both the speeds are influenced by the two phase lags τ_q' and τ_t' and both are effected by magnetic field too. Furthermore B_1 and B_2 represent attenuation coefficients for the modified elastic wave and modified thermal wave, respectively. B_1 and B_2 both are dependent on two phase lags and also both are dependent on magneto-thermoelastic coupling constant ϵ. This implies that the speed and attenuation of both the waves are influenced by the magnetic field under MTDPL-II model.

Furthermore we observe in case of MTDPL-II that the physical fields such as temperature, stress and perturbed magnetic field have discontinuities with finite jumps at both the elastic and thermal wave fronts but displacement is continuous at both the wave fronts (see equations (5.57)-(5.60)).

However, in the context of MTDPL-I model, we find different results. We observe here that only stress and perturbed magnetic field are discontinuous with finite jumps at elastic wavefront but temperature and displacement are free from any discontinuities (see equations (5.67)-(5.70)). This result is also in contrast with the results of Roychoudhuri and Banerjee (Mukhopadhyay) (1996) in which Lord Shulman model (LS model) has been used and in that model we see that temperature, stress and perturbed magnetic field suffer from discontinuities with finite jumps at both the elastic and thermal wave fronts and only displacement is free from any discontinuities. This indicates a distinct feature of dual-phase-lag model-I.

The finite jump discontinuities at the elastic wave front in MTDPL-I model and the finite jump discontinuities at elastic and thermal wave fronts in case of MTDPL-

II model for different fields are obtained as follows:

Finite jumps under MTDPL-II model:

$[\theta^+ - \theta^-]_{\xi=v_1t} = \frac{v_1^2v_2^2}{\eta''(v_1^2-v_2^2)}[N_1e^{-B_1v_1t}],\ [\theta^+ - \theta^-]_{\xi=v_2t} = \frac{-v_1^2v_2^2}{\eta''(v_1^2-v_2^2)}[N_2e^{-B_2v_2t}]$

$[h^+ - h^-]_{\xi=v_1t} = \frac{\eta_3v_1^2v_2^2}{\eta''\epsilon(v_1^2-v_2^2)}\{[Q_1e^{-B_1v_1t}],\ [h^+ - h^-]_{\xi=v_2t} = \frac{-\eta_3v_1^2v_2^2}{\eta''\epsilon(v_1^2-v_2^2)}\{[Q_1'e^{-B_2v_2t}]$

$[\sigma^+ - \sigma^-]_{\xi=v_1t} = \frac{v_1^2v_2^2}{\eta''\epsilon\tau_q'^2\tau_t'(v_1^2-v_2^2)}[P_1e^{-B_1v_1t}] - \frac{v_1^2v_2^2}{\eta''\tau_t'(v_1^2-v_2^2)}\{[N_1e^{-B_1v_1t}]$

$[\sigma^+ - \sigma^-]_{\xi=v_2t} = -[\frac{v_1^2v_2^2}{\eta''\epsilon\tau_q'^2\tau_t'(v_1^2-v_2^2)}[P_1'e^{-B_2v_2t}] - \frac{v_1^2v_2^2}{\eta''\tau_t'(v_1^2-v_2^2)}[N_1'e^{-B_2v_2t}]]$

Finite jumps in the context of MTDPL-I model:

$[\sigma^+ - \sigma^-]_{\xi=t} = \frac{1}{\beta''\epsilon\tau_q'\tau_t'}[b_1e^{-C_2t}] - \frac{1}{\eta''\epsilon\tau_t'(-1+\frac{2G_1}{c_2})}[S_1e^{-C_2t}]$

$[h^+ - h^-]_{\xi=t} = \frac{1}{\eta''\tau_t'\tau_q'\epsilon(1-\frac{2G_1}{c_2})}[R_1e^{-C_2t}]$

5.7 Numerical results and discussion

In the previous section, we made attempt to derive short-time approximated analytical results predicted by two models MTDPL-I and MTDPL-II that represent the effects of magneto-thermo-elastic interaction and highlighted the effects of phase-lag parameters and presence of magnetic field. However in the present section, we aim to illustrate the problem and instead of short-time approximated solutions, we study the behaviour of numerical values of the physical fields like temperature, displacement and stress and perturbed magnetic field with distance under two different models: MTDPL-I and MTDPL-II model. Here, we employ the numerical method proposed by Bellmen *et al.* (1966) for the inversion of Laplace transforms, and compute the numerical values of these physical quantities by directly solving equations (5.43)-(5.48) numerically. We have used software Mathematica for our computational work. The results are plotted for MTDPL-I and MTDPL-II models separately to show the behavior of the fields at three non-dimensional times 1.21, 0.69 and 0.13. In order to observe the effects of magnetic field, we also plot the corresponding results under DPL-I and DPL-II models by assuming the magnetic

parameters in our solution to be zero. We have chosen the copper material for our numerical computation. We have used the following physical data for the copper material: $k = 1.14cm^2/s$, $T_0 = 20^o c$, $H_3 = 10^4 oersted$;
$\lambda = 1.387 \times 10^{12} dyn/cm^2$, $\mu = 0.448 \times 10^{12} dyn/cm^2$, $\rho = 8.930g/cm^3$;
$\mu_0 = 1$, $\alpha_t = (16.5 \times 10^{-6})^o c^{-1}$
We further assume the following dimensionless values
$\sigma_0' = \frac{\sigma_0}{\gamma T_0} = 1$, $\theta_0' = \frac{\theta_0}{T_0} = 1$, $\tau_q^{'} = 0.2$, $\tau_t^{'} = 0.15$

5.7.1 Behavior of temperature under MTDPL-I and MTDPL-II

It is clear from Figures 5.1, 5.2 and 5.3 that the results of temperature field under DPL-I and MTDPL-I are almost the same. Similarly, the results of DPL-II and MTDPL-II are also in complete agreement. However, the results under DPL-I and MDPL-I are significantly different with the results under DPL-II and MTDPL-II models. This imply that although the effect of magnetic field is not prominent in the behavior of temperature field under MTDPL-I and MTDPL-II models, but there are significant differences among the results of two different theories (MTDPL-I and MTDPL-II). This difference decreases as the time decreases. Although, the behavior of temperature is in decreasing trend in all figures but we observe that the non-dimensional temperature field achieves negative value before approaching to zero value in MTDPL-II but it is always positive in MTDPL-I case.

5.7.2 Behavior of displacement under MTDPL-I and MTDPL-II

Figs. 5.4, 5.5 and 5.6 represent the nature of displacement field at various times $t = 1.21$, 0.69 and 0.13 for MTDPL-I and MTDPL-II models. Here, the effect of magnetic field is prominent in both the cases (MTDPL-I and MTDPL-II) at all three times, although its effect gradually decreases as the distance from boundary increases. The consequence of the presence of magnetic field is in such a manner that the value of displacement becomes less in the presence of magnetic field at higher time but this behavior becomes opposite at smaller time (i.e., at $t = 0.69$ and $\tau = 0.13$). Here,the value of displacement becomes greater when we consider the presence of magnetic field in the medium. This fact is evident for both the theories MTDPL-I and MTDPL-II. Furthermore, the maximum numerical value of displacement is greater in MTDPL-II model in comparison to the results under MTDPL-I model and there is a prominent difference in predictions by MTDPL-I and MTDPL-II models.

5.7.3 Behavior of stress under MTDPL-I and MTDPL-II

Figures 5.7, 5.8 and 5.9 exhibit the nature of stress field at different times 1.21 and 0.69 and 0.13 predicted by two models. It is noted that the influence of magnetic field is significant on the nature of stress field in the contexts of both the models. In the absence of magnetic field, when the distance $\xi = 0$, the value of stress is 1 at all three times in cases of MTDPL-I and MTDPL-II models but in the presence of magnetic field, the value of stress becomes greater than 1 under both the theories. Furthermore, we also observe that the minimum value of stress in case of MTDPL-I is greater than the minimum value of stress in case of MTDPL-II at higher times, i.e., at $t =$1.21 and $t = 0.69$ (see Figures 5.7 and 5.8). However, Fig. 5.9 indicates that at very small time, the minimum value of stress in case of MTDPL-I is less than the minimum value of stress under MTDPL-II model.

Furthermore, the difference between two models decreases as time decreases.

5.7.4 Behavior of perturbed magnetic field under MTDPL-I and MTDPL-II

Figs. 5.10, 5.11 and 5.12 display the nature of perturbed magnetic field at different times 1.21 and 0.69 and 0.13 for two models. Here the difference between the predictions by two models is significant. Minimum value of perturbed magnetic field in MTDPL-II is less than minimum value of perturbed magnetic field for MTDPL-I, however maximum value of perturbed magnetic field in MTDPL-II is greater than maximum value of perturbed magnetic field for MTDPL-I.

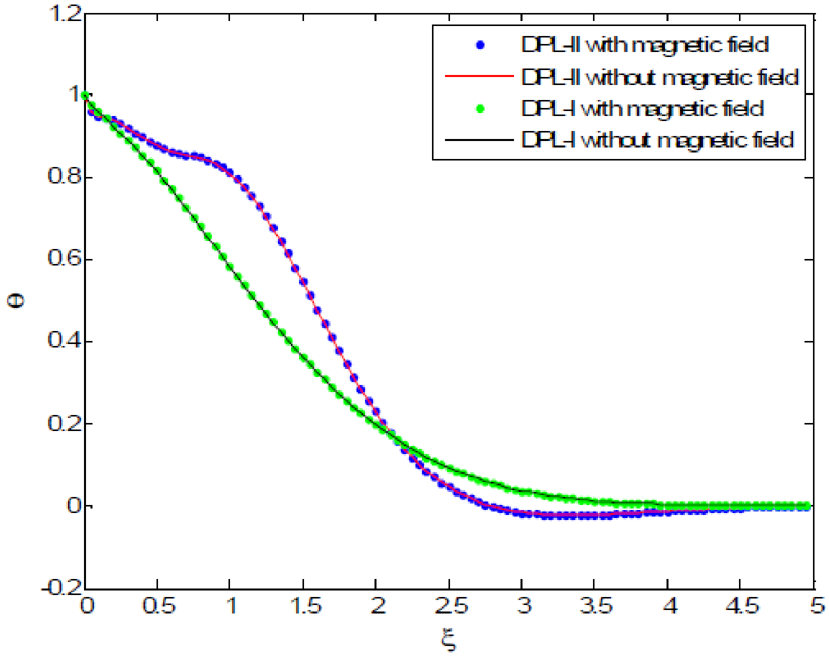

Figure 5.1 Variation of temperature distribution with distance at $t = 1.21$

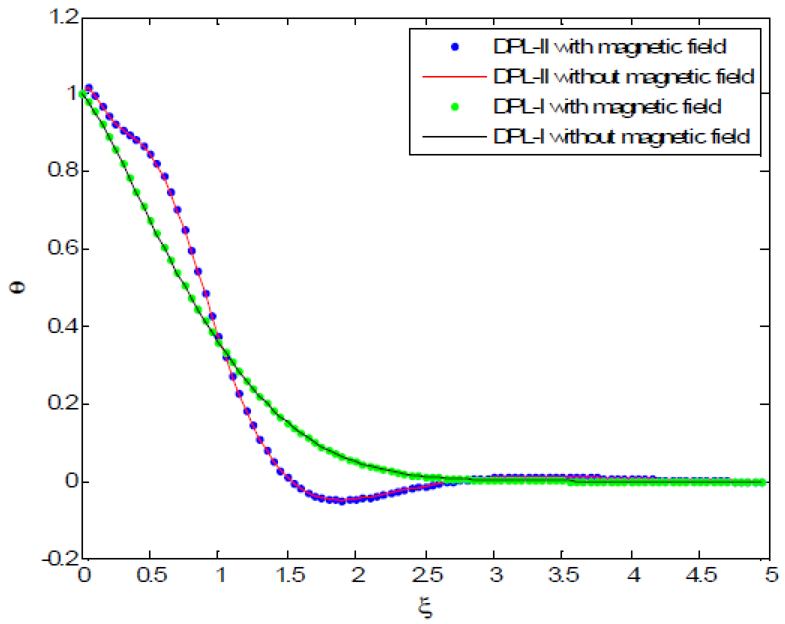

Figure 5.2 Variation of temperature distribution with distance at $t = 0.69$

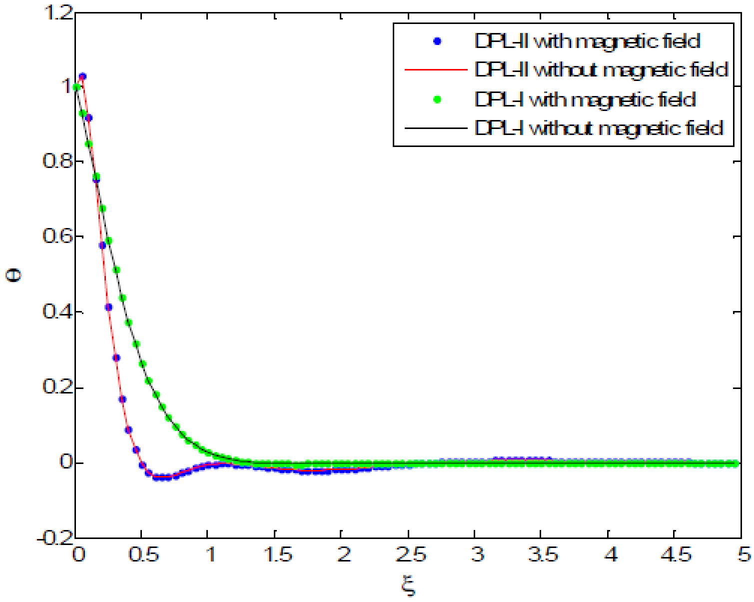

Figure 5.3 Variation of temperature distribution with distance at $t =$ 0.13

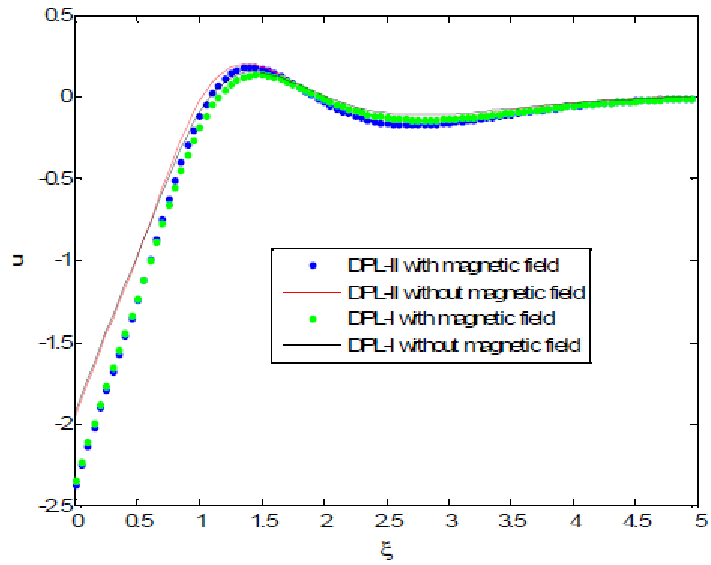

Figure 5.4 Variation of displacement with distance at $t = 1.21$

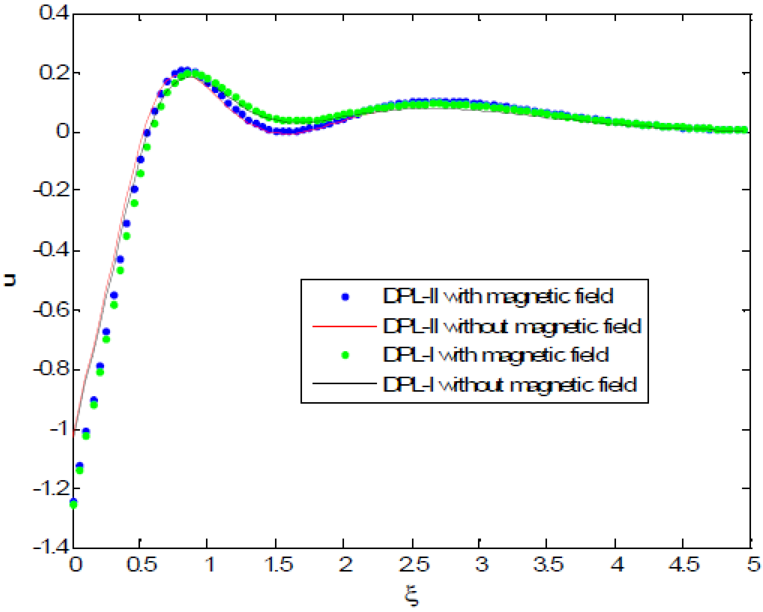

Figure 5.5 Variation of displacement with distance at $t = 0.69$

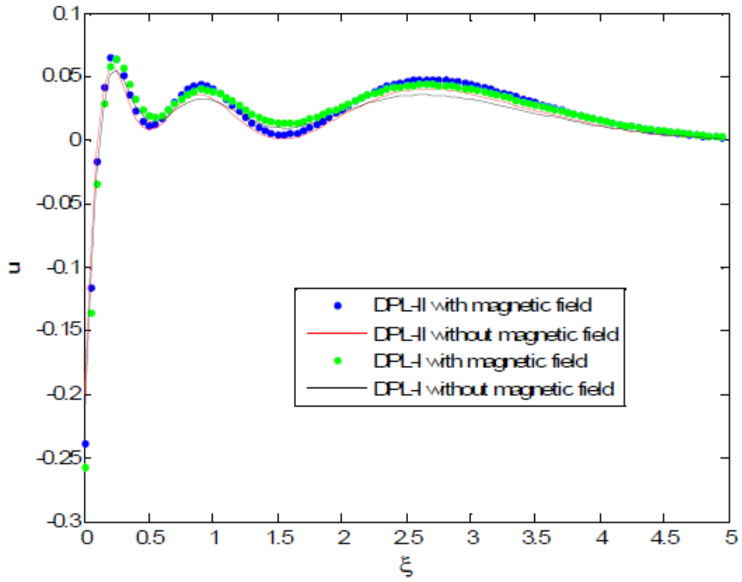

Figure 5.6 Variation of displacement with distance at $t = 0.13$

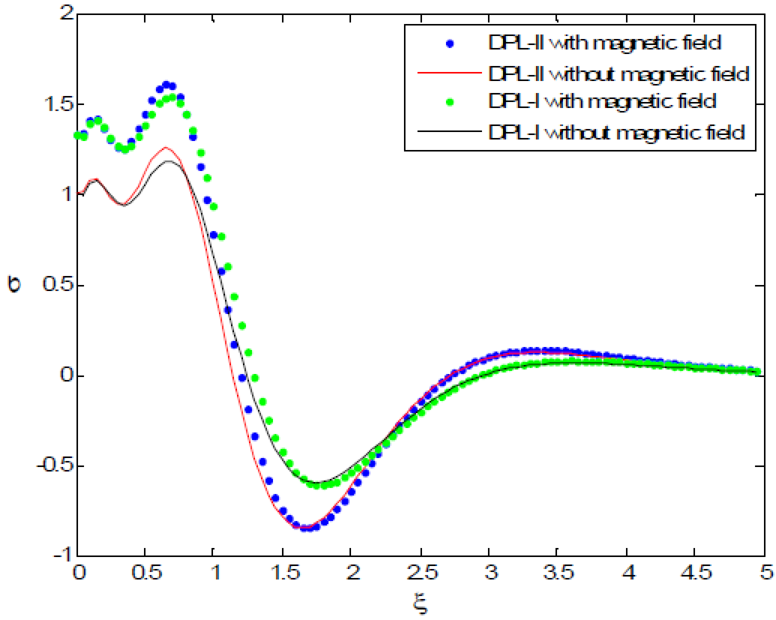

Figure 5.7 Variation of stress with distance at $t = 1.21$

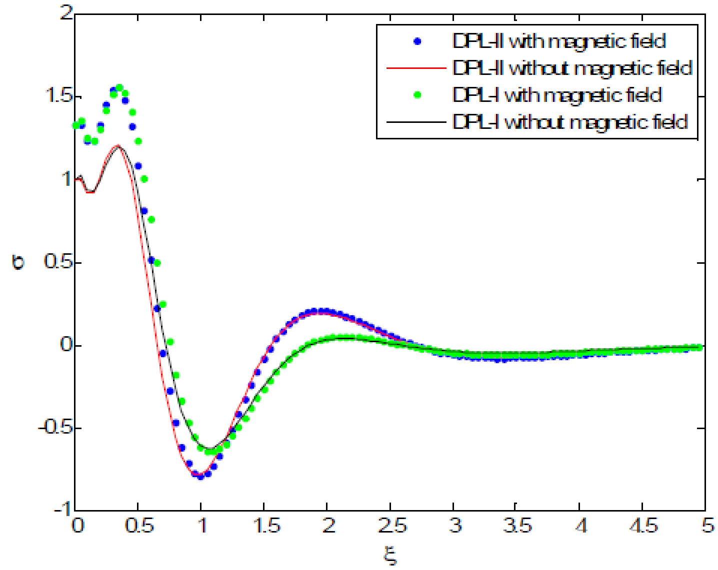

Figure 5.8 Variation of stress with distance at $t = 0.69$

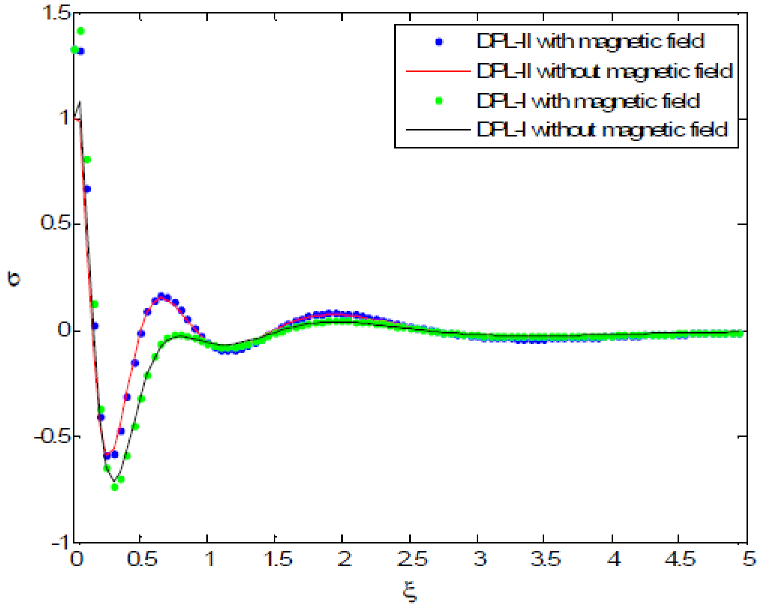

Figure 5.9 Variation of stress with distance at t = 0.13

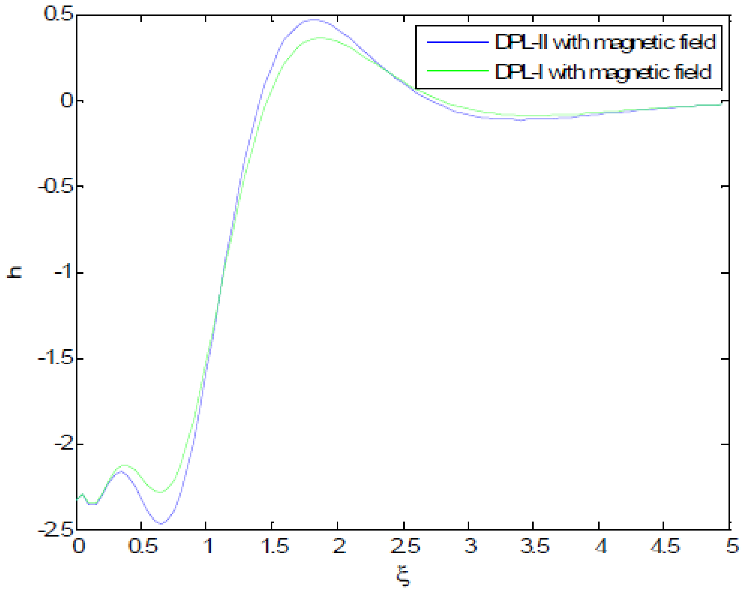

Figure 5.10 Variation of perturbed magnetic field with distance at t = 1.21

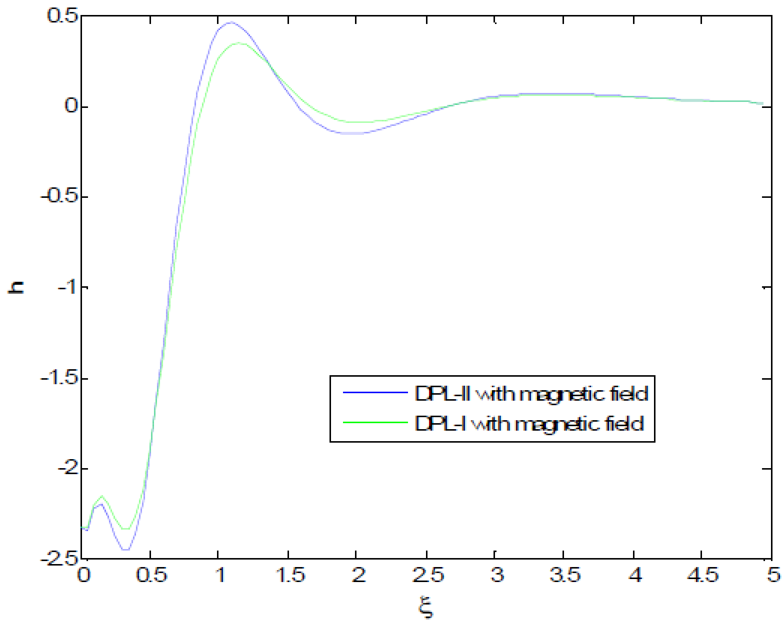

Figure 5.11 Variation of perturbed magnetic field with distance at $t = 0.69$

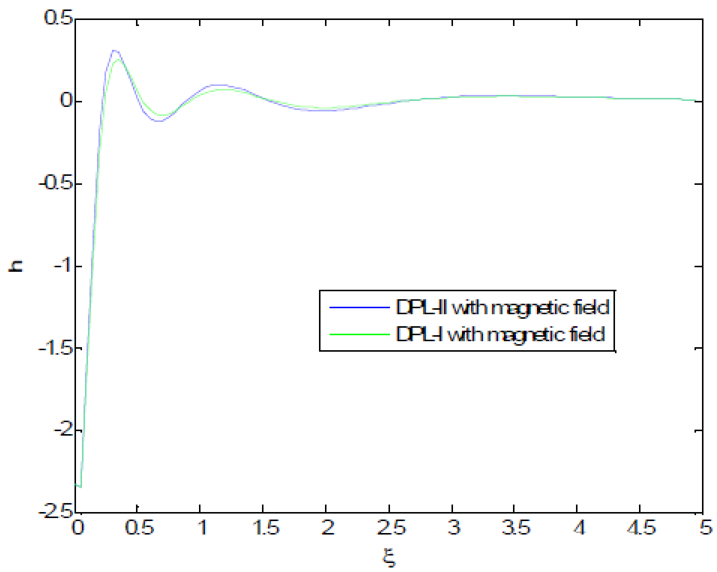

Figure 5.12 Variation of perturbed magnetic field with distance at $t = 0.13$

5.8 Conclusions

In the present work, we employed dual phase-lag magneto-thermoelasticity theory and studied a problem of elastic half space with finite conductivity permeated by a uniform magnetic field. The boundary of the half space is subjected to a normal load and a thermal shock that originate magneto-thermoelastic waves inside the medium.

We have presented a thorough analysis of the effects of magnetic field on wave propagation, and to investigate the nature of distributions of different fields like temperature, displacement, stress and perturbed magnetic field in the media in contexts of two models of magneto-thermoelasticity with dual phase-lags, namely MTDPL-I model and MTDPL-II model. The numerical values of distributions of the physical fields for a suitable material have also been computed and displayed in graphical forms.

Significant differences among the analytical results predicted by two models MTDPL-I and MTDPL-II are observed. In the case of MTDPL-I, we found that solution of each field consists of two parts. The first one is a wave part that is identified as modified elastic wave and the second part is not wave type, but of diffusive type. The non-dimensional speed of elastic wave is found to be finite and equal to 1, i.e. the dimensionless speed of elastic wave is not effected by any of the phase-lag parameters $\tau_q^{'}$ and $\tau_t^{'}$ and it is also independent from the effect of magnetic field. But the attenuation coefficient of modified elastic wave is dependent on phase-lag parameters $\tau_q^{'}$ and $\tau_t^{'}$. However, we note that the solution of each field variable in case of MTDPL-II model consists of two coupled waves: modified elastic and modified thermal wave. The non-dimensional speed of both the waves are finite and dependent on phase-lag parameters $\tau_q^{'}$ and $\tau_t^{'}$ and magnetic field too. Furthermore, we observe that temperature, stress and perturbed magnetic fields have discontinuities with finite jumps at both the elastic and thermal wave fronts and displacement is observed to be continuous in nature at both the wavefronts in the context of MTDPL-II model. However, we obtain different results under MTDPL-I model. We observe here that in this case, only stress and perturbed magnetic field show discontinuities having finite jumps at elastic wave front but the temperature and displacement are free from any discontinuities.

We also observe significant differences in the numerical results predicted by two different models. It is noted that the non dimensional temperature achieves negative value for a region before approaching to zero value in MTDPL-II but it is always positive under MTDPL-I case. While observing the nature of stress, it is found that the minimum value of stress in MTDPL-I is greater than the minimum value of stress in MTDPL-II at higher time, although during initial time of interaction, the minimum value of stress in MTDPL-I is less than the minimum value of stress in MTDPL-II. Furthermore, the maximum numerical value of displacement is greater in case of MTDPL-II as compared to the case of MTDPL-I.

Magnetic field is not prominently effective in the distribution of temperature in the contexts of both the MTDPL-I and MTDPL-II models, however, the stress field and displacement field are effected by the presence of magnetic field under both the models.