Chapter-2

Fractional order thermoelasticity

[1]2.1 Boundary integral equations formulation for fractional order thermoelasticity

2.1.1 Introduction

The present section is devoted to formulate the boundary integral equations for the solutions of equations in context of fractional order thermoelasticity in a three dimensional Euclidean space. It is well accepted that most often numerical techniques are employed as an alternative tool to solve practical engineering problems that are intractable to solve by any analytical method. Moreover, the advent of high-speed computers in today's time has drawn the attention towards versatile and accurate numerical methods in engineering analysis. In the recent years, the Boundary Element Method (BEM) or Boundary Integral Equation Method (BIEM) has been playing a very crucial role for solving linear partial differential equations due to its efficiency with respect to the computer time and storage, its simplicity and the ease of its implementation as compared to other numerical techniques. As mentioned by Brebbia (1978) , Brebbia and Walker (1980), it becomes very methodical as compared to other numerical methods like the Finite Element Method (FEM) for obtaining the solutions of the same accuracy. Particularly, BIEM/BEM method is easily applicable to solve the elasticity problem in an infinite region. BIEM can be applied in many more areas of engineering and sciences including fluid mechanics, acoustics, electromagnetics and fracture mechanics. The first numerical treatment of the BIE method was formulated by Jawson (1963) and Symm (1963). Rizzo and Shippi (1977) introduced the boundary element method for steady state

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thermoelasticity and showed numerical results for three dimensional linear homogeneous isotropic medium. Chen and Dargush (1995) reported the BIE formulations for the dynamic coupled poroelasticity and thermoelasticity with relaxation times by using a unified approach. Work carried out by the researchers like Cruse and Rizzo (1985), Banerjee and Butterfield (1981), Brebbia *et al.* (1984), Ziegler and Irschik (1987) are also worth mentioning in this direction. The application of the boundary element method for three dimensional problems of coupled thermo elasticity has been analyzed by Tanaka *et al.* (1995). Anwar and Sherief (1988) has presented the boundary integral equation formulation for generalized thermoelasticity with relaxation times. Subsequently, several researchers like Kogl and Gaul (2003), El-Karamany and Ezzat (2004, 2004) El-Karamany (2004), Prasad *et al.* (2013), Semwal and Mukhopadhyay (2014) have reported BIE formulations in various thermoelasticity theories.

In recent years, fractional calculus is playing a crucial role in developing several models and it has been verified that the use of fractional order derivatives/integrals lead to the formulation of certain physical problems which are more economical and appropriate than the classical approach. Furthermore, fractional calculus has also been proved to be very useful in the areas of diffusion, heat conduction, viscoelasticity, continuum mechanics, electromagnetism etc. Povstenko (2005) has developed a quasi-static uncoupled thermoelastic model based on the heat conduction equation with fractional order time derivatives. He has used the Caputo fractional derivative (Caputo (1967)) and obtained the stress components corresponding to the fundamental solution of a Cauchy problem for the fractional order heat conduction equation in both the one-dimensional and two-dimensional cases. Remillat, Hassan and Scarpa (2007) have given viscoelastic testing and fractional derivative modeling to describe the thermally induced transformation. In 2010, a new theory of thermoelasticity in the frame of a new consideration of the heat conduction equation with fractional order time derivatives has been proposed by Youssef

(2010, 2012). The uniqueness of the solution has also been proved in the same work. Youssef and Al-Lehaibi (2010) have studied a problem on an elastic half-space using this theory. Subsequently, Sherief *et al.* (2010) has introduced a theory of fractional order thermoelasticity that is based on a new form of the heat conduction model in the frame work of the CV model. The heat conduction law is proposed in the form

$$-K\overrightarrow{\nabla}T = \overrightarrow{q}(x,t) + \tau\frac{\partial^{\alpha}\overrightarrow{q}}{\partial t^{\alpha}} \tag{2.1.1}$$

where $(0 < \alpha < 1)$ is the fractional order parameter and τ corresponds to thermal relaxation parameter.

Here, Caputo's definition of fractional order derivative is employed here.

The main objective of the present work is to formulate the boundary integral equations for the solutions of equations under fractional order thermoelasticity as proposed by Sherief *et al.* (2010) in a three dimensional Euclidean space. We consider a mixed initial-boundary value problem and derive the expressions of fundamental solutions of the corresponding coupled and time-fractional order differential equations in the Laplace transform domain. We formulate the boundary integral equations on the basis of our fundamental solutions and one reciprocal relation in the present context. This formulation is believed to be helpful for the solution of problems under fractional order thermoelasticity by using the boundary element method.

2.1.2 Mathematical formulation: Basic governing equations

We consider a homogeneous isotropic elastic body occupying the region V and bounded by a smooth surface S. We employ a three dimensional rectangular Cartesian co-ordinate system. The basic governing equations that describe the physical components of the thermoelastic system in the context of fractional order thermoelasticity can therefore be considered as follows (Sherief, 2010):

Equations of motion:

$$\mu u_{i,kk} + (\lambda + \mu)u_{k,ki} + \rho F_i - \gamma\theta_{,i} = \rho\ddot{u}_i \tag{2.1.2}$$

Equation of Energy

$$K\theta_{,kk} = \rho C_e(\dot{\theta} + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}\dot{\theta}) + \gamma T_0(\dot{e} + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}\dot{e}) - \rho C_e(Q + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}Q) \tag{2.1.3}$$

where $(0 < \alpha < 1)$, α being the fractional order parameter.

The constitutive equations:

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e - \gamma\theta)\delta_{ij} \tag{2.1.4}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.1.5}$$

where i, j, k varies from 1 to 3.

In the above equations, superposed dot and comma notations are used for time derivative and the material derivative, respectively. Summation convention has been used here and δ_{ij} denotes Kronecker delta. We assume that all the functions are to be the function of x and t, where $x = (x_1,x_2,x_3)$.

2.1.3 Boundary conditions

We assume that (S_1,S_2) and (S_3,S_4) are the partitions of the surface S such that

$$S_1 \cup S_2 = S = S_3 \cup S_4 \tag{2.1.7}$$

$$S_1 \cap S_2 = S_3 \cap S_4 = \phi \tag{2.1.8}$$

Now, we assume the following boundary conditions:

Mechanical Loading Conditions:

We consider that the traction vector component

$$p_i = \sigma_{ij} n_j \tag{2.1.9}$$

is specified on the part S_1 of S. u_i is specified on S_2 . Here n_j are the components of the outward normal n on the surface S.

These conditions can be written as:

$$\sigma_{ij} n_j = p_{i0}(x,t) on S_1 \tag{2.1.10}$$

$$u_i = u_{i0}(x,t) on S_2 \tag{2.1.11}$$

Thermal conditions:

Thermal conditions are taken as

$$\theta = \theta_0(x,t) on S_3 \tag{2.1.12}$$

$$\theta_{,n} = \theta_{n0}(x,t) on S_4 \tag{2.1.13}$$

We consider that the initial conditions are homogeneous.

Clearly, using equation (2.1.4), the components of the traction vector on the surface S is obtained in the form

$$p_i(x,t) = [2\mu e_{ij} + (\lambda e - \gamma\theta)\delta_{ij}] n_j(x) = \mu n_j(x) u_{i,j} + \mu n_j(x) u_{j,i} + \lambda n_i(x) u_{j,j} - \gamma\theta n_i(x) \tag{2.1.14}$$

2.1.4 Governing equations in Laplace transform domain

The Laplace transform of a function $f(t)$ is given by

$$\bar{f}(p) = \int_0^{+\infty} e^{-pt} f(t) dt \tag{2.1.15}$$

Applying the Laplace transform on both sides of equation (2.1.2) - (2.1.4), we get

$$\mu\bar{u}_{i,kk} + (\lambda+\mu)\bar{u}_{k,ki} + \rho\bar{F}_i - \gamma\bar{\theta},i = \rho p^2\bar{u}_i \tag{2.1.16}$$

$$K\bar{\theta},_{kk} = \rho C_e p(1+\tau_0 p^\alpha)(\bar{\theta} + \frac{\gamma T_0}{\rho C_e}\bar{u},_{kk} - \frac{\bar{Q}}{p}) \tag{2.1.17}$$

$$\bar{\sigma}_{ij} = 2\mu\bar{e}_{ij} + (\lambda\bar{u}_{k,k} - \gamma\bar{\theta})\delta_{ij} \tag{2.1.18}$$

Now, we introduce Helmholtz decomposition of the displacement and body force vectors in the following way:

$$u_i = \phi,_i + \epsilon_{ijk}\psi_{k,j} \tag{2.1.19}$$

where,

$$\psi_{i,i} = 0 \tag{2.1.20}$$

$$F_i = X,_i + \epsilon_{ijk}Y_{k,j} \tag{2.1.21}$$

where,

$$Y_{i,i} = 0 \tag{2.1.22}$$

In equations (2.1.19) and (2.1.21), ϕ, X are scalar potentials and ψ_k and Y_k are vector potentials. Therefore, by substituting (2.1.19) and (2.1.21) in (2.1.16) and (2.1.17), we get

$$\triangle_1^2\bar{\phi} - m\bar{\theta} = -\frac{\bar{X}}{c_1^2} \tag{2.1.23}$$

$$\triangle_2^2\bar{\psi}_i = -\frac{\bar{Y}_i}{c_2^2} \tag{2.1.24}$$

$$D\bar{\theta} - ap(1+\tau_0 p^\alpha)\nabla^2\bar{\phi} = -\frac{\rho C_e}{K}(1+\tau_0 p^\alpha)\bar{Q} \tag{2.1.25}$$

where, we have introduced the notations:

$m = \frac{\gamma}{(\lambda+2\mu)}$, $c_1^2 = \frac{(\lambda+2\mu)}{\rho}$, $c_2^2 = \frac{\mu}{\rho}$, $a = \frac{\gamma T_0}{K}$; $\triangle_i^2 \equiv \nabla^2 - p^2/c_i^2$ for $i = 1, 2, 3$;

and $D \equiv [\nabla^2 - \frac{\rho C_e}{K} p(1+\tau_0 p^\alpha)]$

2.1.5 Fundamental solutions in Laplace transform domain

In order to describe the action of body force, and heat source of very large magnitude that act for a very short period of time upon the body, we shall consider the following two cases:

Case 1: We assume that an instantaneous source of heat located at $x_i = y_i$ where $y \in (V \cup S)$ is acting upon an elastic body in the absence of the body forces i.e. $Q = \delta(x-y)\delta(t)$, $F_i = 0$

Let us denote the corresponding fundamental solutions by primes. Now, under the above assumptions, the equations (2.1.23) - (2.1.25) reduce to

$$\triangle_1^2 \bar{\phi}' - m\bar{\theta}' = 0 \tag{2.1.26}$$

$$\triangle_2^2 \bar{\psi}_i' = 0 \tag{2.1.27}$$

$$D\bar{\theta}' - ap(1+\tau_0 p^\alpha)\nabla^2 \bar{\phi}' = -\frac{\rho C_e}{K}(1+\tau_0 p^\alpha)\delta(x-y) \tag{2.1.28}$$

From equation (2.1.27), we can conclude that

$$\bar{\psi}_i' = 0 \tag{2.1.29}$$

Decoupling equations (2.1.26) and (2.1.28) , we arrive at

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)\bar{\phi}' = -m\frac{\rho C_e}{K}(1+\tau_0 p^\alpha)\delta(x-y) \tag{2.1.30}$$

where k_1^2 , k_2^2 are the solutions of characteristic equation

$$Kv - [(\rho C_e + maK)p(1+\tau_0 p^\alpha) + p^{\alpha+1}\frac{K}{c_1^2}]v^2 + p^{\alpha+2}\rho C_e \frac{(1+\tau_0 p^\alpha)}{c_1^2} = 0 \tag{2.1.31}$$

The solution $\bar{\phi}'$ of equation (2.1.30) is given by

$$\bar{\phi}'(x,p) = -m\rho C_e \frac{(1+\tau_0 p^\alpha)}{(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)} \delta(x-y) \tag{2.1.32}$$

By using the Helmholtz equation

$$\frac{1}{(\nabla^2 - k^2)} \delta(r) = -\frac{1}{4\pi r} e^{-kr} \tag{2.1.33}$$

and by using equation (2.1.32), we obtain

$$\bar{\phi}'(x,p) = \frac{m\rho C_e}{4\pi K(k_1^2 - k_2^2)} (1+\tau_0 p^\alpha)(e^{-k_1 r} - e^{-k_2 r}) \tag{2.1.34}$$

Equations (2.1.19), (2.1.20) yield

$$\bar{u'}_i(x,p) = \bar{\phi}'_{,i}(x,p) \tag{2.1.35}$$

Now, using $r = \sqrt{(x_i - y_i)(x_i - y_i)}$,

$$r_{,i} = \frac{(x_i - y_i)}{r} \tag{2.1.36}$$

$$\bar{u}'_i(x,p) = -\frac{\rho C_e}{Kr} \frac{m(1+\tau_0 p^\alpha)}{4\pi(k_1^2 - k_2^2)} [e^{-k_1 r}(1 + \frac{k_1}{r}) - e^{-k_2 r}(1 + \frac{k_2}{r})] r_{,i} \tag{2.1.37}$$

In view of equation (2.1.26), we find

$$\bar{\theta}'(x,p) = \frac{1}{m}(\nabla^2 - \frac{p^2}{c_1^2}) \bar{\phi}'(x,p) \tag{2.1.38}$$

Hence, we can obtain $\bar{\theta}'(x,p)$ as

$$\bar{\theta}'(x,p) = \frac{\rho C_e}{Kr} \frac{m(1+\tau_0 p^\alpha)}{4\pi(k_1^2 - k_2^2)} [e^{-k_1 r}(k_1^2 - \frac{p^2}{c_1^2}) - e^{-k_2 r}((k_2^2 - \frac{p^2}{c_2^2})] \tag{2.1.39}$$

Taking Laplace transform of traction vector, we get from equation (2.1.14)

$$\bar{p}_i(x,t) = \mu n_j(x) \bar{u}_{i,j} + \mu n_j(x) \bar{u}_{j,i} + \lambda n_i(x) \bar{u}_{j,j} - \gamma \bar{\theta} n_i(x) \tag{2.1.40}$$

Now, equation (2.1.37) yields

$$\bar{u}'_{i,j} = -\frac{\rho C_e}{K}[\frac{r_{,i}}{r} g_{1,j} - \frac{(r_{,j}\, r_{,i}\, g_1 - r r_{,ij}\, g_1)}{r^2}] \tag{2.1.41}$$

where

$$g_1 = \frac{m(1+\tau_0 p^\alpha)}{4\pi(k_1^2 - k_2^2)}[e^{-k_1 r}(1+\frac{k_1}{r}) - e^{-k_2 r}(1+\frac{k_2}{r})] \tag{2.1.42}$$

and

$$g_{1,j} = \frac{m(1+\tau_0 p^\alpha)}{4\pi(k_1^2 - k_2^2)}[-k_1 e^{-k_1 r} r_{,j}\,(1+\frac{1}{r^2}) - k_1^2 e^{-k_1 r}\frac{r_{,j}}{r} + k_2 e^{-k_2 r} r_{,j}\,(1+\frac{1}{r^2}) - k_2^2 e^{-k_2 r}\frac{r_{,j}}{r}] \tag{2.1.43}$$

Equation (2.1.41) can be simplified as

$$\bar{u}'_{i,j} = \frac{m(1+\tau_0 p^\alpha)}{4\pi(k_1^2 - k_2^2)}[r_{,i}\, r_{,j}\,\frac{g_3}{r} - \frac{1}{r^2}(\delta_{ij} - 3 r_{,i}\, r_{,j}\,) g_1] \tag{2.1.44}$$

where

$$g_3 = \frac{(1+\tau_0 p^\alpha)}{4\pi(k_1^2 - k_2^2)}[k_1^2 e^{-k_1 r} - k_2^2 e^{-k_2 r}] \tag{2.1.45}$$

Therefore from equations (2.1.40) and (2.1.44), we get

$$\bar{p}'_l(x,p) = \frac{\rho C_e}{K r}\{[2\mu n_k[r_{,l}\, r_{,k}\,(g_3 + 3\frac{g_1}{r})\frac{\delta_{lk}}{r}] + n_l[\lambda r_{,k}\, r_{,k}\,(g_3 + 3\frac{g_1}{r}) - 3\frac{\lambda}{r}]\} \tag{2.1.46}$$

Here Case 1 is completed.

Case 2: In this case, we assume that in absence of heat source i.e. when $Q = 0$, an instantaneous concentrated body force is acting at the point $x_i = y_i$ in the direction of x_j axis. Therefore, we take

$\bar{F}_i = \bar{F}_i^{(j)} = \delta_{ij}\delta(x-y)$. Let $\bar{u}_i^{(j)}, \bar{\theta}^{(j)}$ denote the corresponding fundamental solutions.

We use Helmholtz Resolution for the vectors $u_i^{(j)}$ and $F_i^{(j)}$ and so we can write

$$\bar{u}_i^{(j)} = \bar{\phi}_i^{(j)} + \epsilon_{ilk}\bar{\psi}_{l,k}^{(j)} \tag{2.1.47}$$

$$\bar{F}_i^{(j)} = \bar{P}_i^{(j)} + \epsilon_{ilk}\bar{R}_{l,k}^{(j)} \tag{2.1.48}$$

The potentials in R.H.S. of above equations satisfy the equations

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)\bar{\phi}^{(j)} = -\frac{1}{c_1^2}[\nabla^2 - \frac{\rho C_e}{K}p(1+\tau_0 p^\alpha)]\bar{P}^{(j)} \tag{2.1.49}$$

$$\bar{\theta}^{(j)} = \frac{1}{m}[(\nabla^2 - \frac{p^2}{c_1^2})\bar{\phi}^{(j)} + \frac{\bar{P}^{(j)}}{c_1^2}] \tag{2.1.50}$$

$$(\nabla^2 - \frac{p^2}{c_1^2})\bar{\psi}_l^{(j)} = -\frac{1}{c_2^2}\bar{R}_i^{(j)} \tag{2.1.51}$$

where k_1^2 and k_2^2 are the solutions of the same characteristic equation as given by (2.1.31). In view of the body forces as chosen above, the corresponding Helmholtz decomposition leads to

$$\bar{P}^{(j)} = -\frac{1}{4\pi}(\frac{\delta_{ij}}{r})_{,i} \tag{2.1.52}$$

$$\bar{R}_l^{(j)} = \frac{1}{4\pi}\epsilon_{ilk}(\frac{\delta_{ij}}{r})_{,k} \tag{2.1.53}$$

By using Helmholtz equation, solution of equation (2.1.49) can be written as

$$\bar{\phi}^{(j)} = \frac{1}{4\pi c_1^2}\frac{r_{,i}}{r^2}\frac{\delta_{ij}}{(k_1^2 - k_2^2)}[\sum_{n=1}^{2}(-1)^{n-1}E(1+k_n r)e^{-k_n r}] + \delta_{ij}\frac{r_i}{r^2}\frac{1}{4\pi p^2} \tag{2.1.54}$$

where

$$E = \frac{[k_n^2 - \frac{\rho C_e p(1+\tau_0 p^\alpha)}{K}]}{k_n^2} \tag{2.1.55}$$

$$\bar{\psi}_l^{(j)} = \epsilon_{ijl}(\frac{r_{,i}}{4\pi p^2 r^2})[(1+\frac{pr}{c_2})e^{-\frac{pr}{c_2}} - 1] \tag{2.1.56}$$

$$\bar{u}_l^{(j)} = \frac{U_1\delta_{ij}}{r} + \frac{U_2 r_{,i} r_{,j}}{r} \tag{2.1.57}$$

where

$$U_1 = [\frac{1}{4\pi p^2}(\frac{p^2}{c_2^2} + \frac{p}{rc_2} + \frac{1}{r^2})e^{-\frac{pr}{c_2}} + \sum_{n=1}^{2}(-1)^{n-1}\frac{B_n}{r^2}(1+k_n r)e^{-k_n r}] \tag{2.1.58}$$

$$U_2=[-\frac{1}{4\pi p^2}(\frac{p^2}{c_2^2}+3\frac{p}{rc_2}+3\frac{1}{r^2})e^{-\frac{pr}{c_2}}]+\sum_{n=1}^{2}(-1)^{n-1}\frac{B_n}{r^2}(k_n^2+3\frac{k_n}{r}+\frac{3}{r^2})]e^{-k_n r} \tag{2.1.59}$$

$$B_n=\frac{1}{k_n^2c_1^2}\frac{1}{(k_1^2-k_2^2)}(k_n^2-\frac{\rho C_e p(1+\tau_0 p^\alpha)}{K}) \tag{2.1.60}$$

With the help of equations (2.1.25) and (2.1.49) with the condition that the source of heat is absent, we find

$$\bar{\theta}^{(j)}=G\sum_{n=1}^{2}(-1)^{n-1}e^{-k_n r}(1+k_n r)\frac{r_{,i}}{r^2} \tag{2.1.61}$$

where $G=\frac{\delta_{ij}}{4\pi c_1^2}\frac{p(1+\tau_0p^\alpha)}{(k_2^2-k_1^2)}$

Here, we achieve the conclusion that the temperature for the Case 2 is related with the expression of the displacement for Case 1 as

$$\bar{\theta}^{(j)}=\frac{\rho p\epsilon}{m\gamma}u'_j \tag{2.1.62}$$

where $\epsilon=\frac{mak}{\rho C_e}$

Then, the component of the traction vector can be obtained in a similar way as in Case 1. Thus, we get

$$\bar{p}_l^{(j)}(x,p)=\mu n_k[(\bar{u}_{l,k}^{(j)}+\bar{u}_{k,l}^{(j)})+n_l(\lambda\bar{u}_{k,k}^{(j)}-\gamma\bar{\theta}^{(j)})] \tag{2.1.63}$$

where,

$$\bar{u}_{i,k}^{(j)}=\frac{U_{1,k}}{r}\delta_{ij}-\frac{U_1\delta_{ij}}{r^2}+\frac{U_{2,k}}{r^2}r_{,i}r_{,j}-\frac{U_2}{r^2}r_{,i}r_{,j}r_{,k}+\frac{U_2}{r}r_{,ik}r_{,j}+\frac{U_2}{r}r_{,i}r_{,jk} \tag{2.1.64}$$

$$U_{1,k}=\sum_{n=1}^{2}(-1)^{n-1}B_ne^{-k_nr}\frac{r_{,k}}{r}[-k_n^2-2\frac{(1+k_nr)}{r^2}]+\frac{r_{,k}}{4\pi p^2}e^{\frac{-pr}{c_2}}[-\frac{p^3}{c_2^3}-2\frac{p}{r^2c_2}-\frac{2}{r^3}-\frac{p^2}{c_2^2r}] \tag{2.1.65}$$

$$U_{2,k}=-\frac{r_{,k}}{4\pi p^2}e^{\frac{-pr}{c_2}}[-\frac{p^3}{c_2^3}-6\frac{p}{r^2c_2}-\frac{6}{r^2}-3\frac{p^2}{c_2^2r}]+\sum_{n=1}^{2}(-1)^{n-1}B_ne^{-k_nr}r_{,k}[-k_n^2-6\frac{k_n}{r^2}-3\frac{k_n^2}{r}-\frac{6}{r^3}] \tag{2.1.66}$$

U_1,U_2 are given by equations (2.1.58) and (2.1.59), respectively. This completes Case 2.

2.1.6 Reciprocity theorem

When a body is under the action at two different thermoelastic loadings, the reciprocal relation states the relation between two sets of thermoelastic loadings and corresponding thermoelastic configurations.

In order to find out the integral representations of the displacement and temperature distributions in terms of boundary values, we will now employ a reciprocity theorem for the linear theory of fractional order thermo elasticity by using above two cases of thermoelastic loading. For this, we consider the body with volume V subjected to two different systems of thermoelastic loadings $L = (F_i^{(\beta)}, Q^{(\beta)}, p_i^{(\beta)}, u_{i0}^{(\beta)}, \theta_0^{(\beta)}, \theta_{n0}^{(\beta)})$, for $\beta = 1$, 2 and the corresponding thermoelastic configurations are denoted as $I^\beta = (u_i^\beta, \theta^\beta)$; $\beta = 1, 2$.

Now by following Sherief *et al.* (2010), we can derive the following reciprocal relation:

$$\frac{\rho T_0 p(1+\tau_0 p^\alpha)}{k}\int_V [\bar{F}_i^{(1)}\bar{u}_i^{(2)} - \bar{F}_i^{(2)}\bar{u}_i^{(1)}]dV - \frac{\rho C_e(1+\tau_0 p^\alpha)}{k}\int_V [\bar{Q}^{(1)}\bar{\theta}^{(2)} - \bar{Q}^{(2)}\bar{\theta}^{(1)}]dV =$$

$$\int_{S_3} [\bar{\theta}^{(1)}\bar{\theta},_n^{(2)} - \bar{\theta}^{(2)}\bar{\theta},_n^{(1)}]dS + \int_{S_4} [\bar{\theta}^{(1)}\bar{\theta},_n^{(2)} - \bar{\theta}^{(2)}\bar{\theta},_n^{(1)}]dS$$

$$-\frac{T_0 p(1+\tau_0 p^\alpha)}{k}\{\int_{S_1} [\bar{\sigma}_{ij}^{(1)}\bar{u}_i^{(2)} - \bar{\sigma}_{ij}^{(2)}\bar{u}_i^{(1)}]n_j dS + \int_{S_2} [\bar{\sigma}_{ij}^{(1)}\bar{u}_i^{(2)} - \bar{\sigma}_{ij}^{(2)}\bar{u}_i^{(1)}]n_j dS\} \tag{2.1.67}$$

Clearly, for an infinite isotropic medium, the body forces, heat sources act only in a bounded region and the surface integral in equation (2.1.67) will be absent.

2.1.7 Boundary integral equations

In order to obtain the integral representation of the transformed displacement and temperature inside the bounded region V in terms of the prescribed functions $\bar{p}_{i0}$,$\bar{u}_{i0}$,$\bar{\theta}_0$,$\bar{\theta}_{n0}$ on the surface S, the fundamental solutions $\bar{u}_i'$,$\bar{\theta}'$,$\bar{u}_i^{(j)}$,$\bar{\theta}^{(j)}$ in the infinite region and their values $\bar{u}_{i0}'$,$\bar{\theta}_0'$,$\bar{u}_{i0}^{(j)}$,$\bar{\theta}_0^{(j)}$ on the surface S, we consider the following two cases:

First, as we have considered earlier $\bar{F}_i = 0$ and $\bar{Q} = \delta(x-y)$ where $y \in (V \cup S)$ thus equation (2.1.67) becomes

$$(1+\tau_0 p^\alpha)\Delta(x)\bar{\theta}(x,p) = \frac{K}{\rho C_e}\{\int_{S_3}[\bar{\theta}'_0\bar{\theta},_n - \bar{\theta}_0\bar{\theta}'_{n0}]dS + \int_{S_4}[\bar{\theta}'\bar{\theta}_{n0} - \bar{\theta}\bar{\theta}',_n]dS\}$$
$$-\frac{T_0}{\rho C_e}p(1+\tau_0 p^\alpha)\{\int_{S_1}[\bar{p_{i0}}\bar{u}'_i - \bar{p}'_{i0}\bar{u}_i]dS + \int_{S_2}[\bar{\sigma}_{ij}\bar{u}'_{i0} - \bar{\sigma}'_{ij}\bar{u}_{i0}]n_j dS\}$$
$$-\frac{T_0}{C_e}(p+\tau_0 p^\alpha)\int_V \bar{F}_i\bar{u}'_i dV + (1+\tau_0 p^\alpha)\int_V \bar{Q}\bar{\theta}' dV \qquad (2.1.68)$$

where $\bar{u}'_i$,$\bar{\theta}'$ are the fundamental solutions obtained previously in Case 1 and we denote

$\int_V \delta(x-y)dV(y) = \Delta(x)$, where $\Delta(x) = 1, x \in V$

$\Delta(x) = 0, if \notin (V \cup S)$

$\Delta(x) = \frac{1}{2}, x \in S$

Next, we assume

$\bar{F}_i^{(j)} = \delta_{ij}\delta(x-y)$ and $\bar{Q} = 0$;

Therefore, from equation (2.1.66) we get as previously

$$p(1+\tau_0 p^\alpha)\Delta(x)\bar{u}_j(x,p) = \frac{K}{\rho T_0}\{\int_{S_3}[\bar{\theta}_0\bar{\theta},_n^{(j)} - \bar{\theta}_0^{(j)}\bar{\theta},_n]dS$$

$$+\int_{S_4}[\bar{\theta}\bar{\theta}_{n0}^{(j)} - \bar{\theta}^{(j)}\bar{\theta}_{n0}]dS\} + \frac{1}{\rho}p(1+\tau_0 p^\alpha)\{\int_{S_1}[\bar{p}_{i0}\bar{u}_i^{(j)} - \bar{p}_{i0}^{(j)}\bar{u}_{i0}]dS$$

$$+\int_{S_2}[\bar{\sigma}_{il}\bar{u}_{i0}^{(j)} - \bar{\sigma}_{il}^{(j)}\bar{u}_{i0}]n_j dS\} - p(1+\tau_0 p^\alpha)\int_V \bar{F}_i\bar{u}_i^{(j)}dV - \frac{C_e}{T_0}(1+\tau_0 p^\alpha)\int_V \bar{Q}\bar{\theta}^{(j)}dV \tag{2.1.69}$$

where, $\bar{u}^j$,$\bar{\theta}^{(j)}$ denote the fundamental solutions which have been obtained previously in Case 2.

Now, applying the inverse Laplace transform to equation (2.1.68) and using the Convolution theorem of Laplace transform as

$L^{-1}(\bar{F}_1(p)\bar{F}_2(p)) = \int_0^t F_1(\tau)F_2(\tau)d\tau$

we arrive at

$$\Delta(x)[\theta(x,t) + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}\theta(x,t)] = M_1(x,t) \tag{2.1.70}$$

where,

$$M_1(x,t) = \frac{K}{\rho C_e}\int_0^t\{\int_{S_3}[\theta_0'(y,t-\tau)\theta,_n(y,x,\tau) - \theta_0(y,x,\tau)\theta',_n(y,t-\tau)]dS$$

$$+\int_{S_4}[\theta'(y,t-\tau)\theta_{n0}(y,x,\tau) - \theta(y,x,\tau)\theta_{n0}'(y,t-\tau)]dS\}d\tau$$

$$+\frac{T_0}{\rho C_e}\int_0^t\{\int_{S_1}[p_{i0}(y,x,\tau)(\frac{\partial}{\partial\tau} + \tau_0\frac{\partial^{\alpha+1}}{\partial\tau^{\alpha+1}})u_i'(y,t-\tau)]dS$$

$$\int_{S_2}[\sigma_{ij}(y,t-\tau)(\frac{\partial}{\partial\tau} + \tau_0\frac{\partial^{\alpha+1}}{\partial\tau^{\alpha+1}})u_{i0}'(y,x,\tau)]dS\}d\tau$$

$$-\frac{T_0}{\rho C_e}\int_0^t\{\int_{S_1}u_i(y,x,\tau)(\frac{\partial}{\partial\tau} + \tau_0\frac{\partial^{\alpha+1}}{\partial\tau^{\alpha+1}})p_{i0}'(y,t-\tau)dS$$

$$+\int_{S_2}u_{i0}(y,t-\tau)(\frac{\partial}{\partial\tau} + \tau_0\frac{\partial^{\alpha+1}}{\partial\tau^{\alpha+1}})\sigma_{ij}'(y,x,\tau)dS\}d\tau$$

$$+\frac{T_0}{C_e}\int_0^t\int_V F_i(y,t-\tau)(\frac{\partial}{\partial\tau}+\frac{\partial^{\alpha+1}}{\partial\tau^{\alpha+1}})u_i'(y,x,\tau)dVd\tau + \int_0^t\int_V Q(y,t-\tau)(1+\frac{\partial^\alpha}{\partial\tau^\alpha})\theta'(y,x,\tau)dVd\tau \tag{2.1.71}$$

Solving equation (2.1.70), we get the expression of temperature in the following manner:

$$\theta(x,t)=\frac{1}{\alpha\Delta x\tau_0}[\int_0^t E_{\alpha,\alpha}(-\frac{1}{\tau_0}\tau^\alpha)M_1(x,t-\tau)d\tau] \tag{2.1.72}$$

where, $E_{\alpha,\alpha}(z)$ is known as "Mittag-Leffler" function which is defined as (Podlubny (1999)) :

$$E_{\alpha,\beta}(z)=\sum_{k=1}^{2}\frac{z^k}{\Gamma(\alpha k+\beta)} \tag{2.1.73}$$

Hence, equations (2.1.72) and (2.1.73) yield the solution for temperature distributions as

$$\theta(x,t)=\frac{1}{\alpha\Delta x\tau_0}[\int_0^t e^{-\frac{1}{\tau_0}\tau^\alpha}\tau^{\alpha-1}M_1(x,t-\tau)d\tau] \tag{2.1.74}$$

Similarly, from equation (2.1.69), we obtain

$$\Delta(x)[u_j(x,t)+\tau_0\frac{\partial^\alpha}{\partial t^\alpha}u_j(x,t)]=M_2(x,t) \tag{2.1.75}$$

where,

$$\begin{aligned}
M_2(x,t)=\frac{K}{\rho T_0}\int_0^t\{\int_{S_3}[\theta_0(y,x,\tau)\frac{\partial u_j'(y,t-\tau)}{\partial n}-u_j'(y,t-\tau)\frac{\partial\theta(y,x,\tau)}{\partial n}]dS \\
+\int_{S_4}[\theta(y,x,\tau)\frac{\partial u_j'(y,t-\tau)}{\partial n}-u_j'(y,t-\tau)\theta_{n0}(y,x,\tau)]dS\}d\tau+ \\
\frac{1}{\rho}\int_0^t\{\int_{S_1}[p_{i0}(y,x,\tau)(1+\tau_0\frac{\partial^\alpha}{\partial\tau^\alpha})u_i^{(j)}(y,t-\tau)dS] \\
+\int_{S_2}[\sigma_{il}(y,t-\tau)(1+\tau_0\frac{\partial^\alpha}{\partial\tau^\alpha})u_{i0}^{(j)}(y,x,\tau)n_l dS]\}d\tau \\
-\frac{1}{\rho}\{\int_0^t\int_{S_1}u_i(y,x,\tau)(1+\frac{\partial^\alpha}{\partial\tau^\alpha})p_{i0}^{(j)}(y,t-\tau)dS \\
+\int_{S_2}u_{i0}(y,t-\tau)(1+\frac{\partial^\alpha}{\partial\tau^\alpha})\sigma_{il}^{(j)}(y,x,\tau)n_l dS\}d\tau \\
+\int_V F_i(y,t-\tau)(1+\tau_0\frac{\partial^\alpha}{\partial\tau^\alpha})u_i^{(j)}(y,x,t-\tau)dVd\tau
\end{aligned}$$

$$
-\int_0^t \int_V Q(y,t-\tau)(1+\tau_0 \frac{\partial^\alpha}{\partial \tau^\alpha}) u_j'(y,x,t-\tau) dV d\tau \tag{2.1.76}
$$

In a similar way as we have done previously, the solution of equation (2.1.76) yields displacement component as

$$
u_j(x,t) = \frac{1}{\alpha \Delta x \tau_0}[\int_0^t e^{-\frac{1}{\tau_0}\tau^\alpha} \tau^{\alpha-1} M_2(x,t-\tau) d\tau] \tag{2.1.77}
$$

Taking the limit $x \to \xi$, where ξ is a point on the boundary S, we get from equations (2.1.74) and (2.1.77) as-

$\theta(\xi,t) = \frac{2}{\alpha\tau_0}[\int_0^t e^{-\frac{1}{\tau_0}\tau^\alpha} \tau^{\alpha-1} M_1(\xi,t-\tau) d\tau]$

$u_j(\xi,t) = \frac{2}{\alpha\tau_0}[\int_0^t e^{-\frac{1}{\tau_0}\tau^\alpha} \tau^{\alpha-1} M_2(\xi,t-\tau) d\tau]$. This completes our formulation. The above two equations together with the prescribed boundary conditions which are taken in the beginning and the limiting behaviour of fundamental solutions as $r \to 0$ can be used to set up linear equations of the boundary integral equation method.

2.1.8 Example

Now, we will consider an example in order to illustrate the present formulation. We consider a formulation, in which we determine the primary variables $u_i(x,t)$ and $\theta(x,t)$ as the solution of the field equations (2.1.2) and (2.1.3), subjected to the homogeneous initial and boundary conditions as follows:

$$
\sigma_{ij}(x_B,t)n_j(x_B) = p_{i0}(x_B,t) = 0 \tag{2.1.78}
$$

$$
\theta_{,n}(x_B,t) = \theta_{n0}(x_B,t) = 0 \tag{2.1.79}
$$

where x_B is a point on $S_1 = S_4$

and

$$
\theta(x_B,t) = \theta_0(x_B,t) \tag{2.1.80}
$$

$$u_i(x_B,t) = u_{i0}(x_B,t) \tag{2.1.81}$$

where x_B is a point on $S_2 = S_3$

Now, using equations (2.1.68) and (2.1.69), we achieve

$$\bar{\theta}(x,p) = -\frac{K}{\rho C_e(1+\tau_0 p^\alpha)}\{\int_{S_3}[\bar{\theta}_0(y,x,p)\bar{\theta}',_n(y,p) - \bar{\theta}'_0(y,p)\bar{\theta},_n(y,x,p)]dS$$

$$+\int_{S_4}[\bar{\theta}(y,x,p)\bar{\theta}'_{n0}(y,p) - \bar{\theta}'(y,p)\bar{\theta}_{n0}(y,x,p)]dS\}$$

$$+\frac{T_0 p}{\rho C_e}\{\int_{S_1}[\bar{p_{i0}}'(y,x,p)\bar{u}_i(y,p) - \bar{p}_{i0}(y,p)\bar{u}'_i(y,x,p)]dS+$$

$$\int_{S_2}[\bar{\sigma}'_{ij}(y,p)\bar{u}_{i0}(y,x,p) - \bar{\sigma}_{ij}(y,x,p)\bar{u}'_{i0}(y,p)]n_j dS\}$$

$$-\frac{T_0 p}{C_e}\int_V \bar{F}_i(y,p)\bar{u}'_i(y,x,p)dV + \int_V \bar{Q}(y,p)\bar{\theta}'(y,x,p)dV \tag{2.1.82}$$

$$p\bar{u}_j(x,p) = \frac{K}{\rho T_0(1+\tau_0 p^\alpha)}\{\int_{S_3}[\bar{\theta}_0(y,x,p)\bar{\theta},_n^{(j)}(y,p) - \bar{\theta}_0^{(j)}(y,p)\bar{\theta},_n(y,x,p)]dS$$

$$+\int_{S_4}[\bar{\theta}(y,p)\bar{\theta}_{n0}^{(j)}(y,x,p) - \bar{\theta}^{(j)}(y,x,p)\bar{\theta}_{n0}(y,p)]dS\} - \frac{p}{\rho}\{\int_{S_1}[\bar{p}_{i0}^{(j)}(y,p)$$

$$\bar{u}_i(y,x,p) - \bar{p}_{i0}(y,x,p)\bar{u}_i^{(j)}(y,p)]dS + \int_{S_2}[\bar{\sigma}_{ij}^{(j)}(y,x,p)\bar{u}_{i0}(y,p) - \bar{\sigma}_{ij}(y,x,p)\bar{u}_{i0}^{(j)}(y,p)]n_j dS\}$$

$$-p\int_V \bar{F}_i(y,p)\bar{u}_i^{(j)}(y,x,p)dV + \int_V \bar{Q}(y,p)\bar{\theta}^{(j)}(y,x,p)dV \tag{2.1.83}$$

Here, we have taken $\Delta(x) = 1$. In view of (2.1.80)-(2.1.81), the functions $\theta(x_B,t)$ and $u_i(x_B,t)$ are unknowns on the part $S_1 = S_4$ of the surface S. We further assume that the fundamental solutions satisfy the conditions

$\bar{\theta}'_0(x_B,t) = \bar{\theta}_0^{(j)}(x_B,t) = 0$ on $S_2 = S_3$and $\bar{u}'_{i0}(x_B,t) = \bar{u}_{i0}^{(j)}(x_B,t) = 0$ on $S_2 = S_3$.

Therefore, by using these conditions and taking $x \to \xi$ we get the equations (2.1.82) and (2.1.83) as the system of two Fredholm's integral equations:

$$0 = -\frac{K}{\rho C_e(1+\tau_0 p^\alpha)}\{\int_{S_3}[\bar{\theta}_0(y,p)\frac{\partial}{\partial n'(\xi)}\bar{\theta}',_n(y,\xi,p)]dS$$

$$+\int_{S_4}[\bar{\theta}(y,p)\frac{\partial}{\partial n'(\xi)}\bar{\theta}'_{n0}(y,\xi,p)]dS\}+\frac{T_0p}{\rho C_e}\{\int_{S_4}[\bar{p_{i0}}'(y,p)\frac{\partial}{\partial n'(\xi)}\bar{u}_i(y,\xi,p)]dS+$$

$$\int_{S_3}[\bar{\sigma}'_{ij}(y,p)\frac{\partial}{\partial n'(\xi)}\bar{u}_{i0}(y,\xi,p)]n_jdS\}-\frac{T_0p}{C_e}\int_V\bar{F}_i(y,p)\frac{\partial}{\partial n'(\xi)}\bar{u}'_i(y,\xi,p)dV$$

$$+\int_V\bar{Q}(y,p)\frac{\partial}{\partial n'(\xi)}\bar{\theta}'(y,\xi,p)dV \qquad (2.1.84)$$

$$p\bar{u}_j(\xi,p)=\frac{K}{\rho T_0(1+\tau_0p^\alpha)}\{\int_{S_3}[\bar{\theta}_0(y,\xi,p)\bar{\theta},_n^{(j)}(y,p)]dS$$

$$+\int_{S_4}[\bar{\theta}(y,p)\bar{\theta}_{n0}^{(j)}(y,\xi,p)]dS\}-\frac{p}{\rho}\{\int_{S_4}[\bar{p}_{i0}^{(j)}(y,p)$$

$$\bar{u}_i(y,\xi,p)]dS+\int_{S_3}[\bar{\sigma}_{ij}^{(j)}(y,\xi,p)\bar{u}_{i0}(y,p)]n_jdS\}$$

$$-p\int_V\bar{F}_i(y,p)\bar{u}_i^{(j)}(y,\xi,p)dV+\int_V\bar{Q}(y,p)\bar{\theta}^{(j)}(y,\xi,p)dV \qquad (2.1.85)$$

Where $n'(\xi)$ is the outer normal to the surface S_4. By employing suitable numerical techniques, the integrals involved in equations (2.1.84) and (2.1.85) can be discretized and the problem then reduces to finding the solution of a system of linear equations.The final solution can therefore be determined by using the suitable numerical method of Laplace inversion.

2.1.9 Conclusions

In the present section of the thesis, fundamental solutions have been established in the Laplace transform domain for fractional order thermoelasticity. Then by employing a suitable reciprocal relation, we formulate boundary integral equations for a mixed boundary initial value problem.

At last, we have given an example which represents a better explanation for our formulation. We believe that the present formulation will help to find the numerical solution of a concrete problem under fractional order thermoelasticity by the BEM / BIEM method.

There are many numerical methods for solving the BIE system like the Point collocation method etc. In the point collocation method, the boundary is discretized at some points (collocation points). This method is believed to be accurate for several one and two-dimensional problems and the convergence rate of the point collocation method can be applied for better results.

[1]2.2 On harmonic plane wave propagation under fractional order thermoelasticity: An analysis of fractional order heat conduction equation

2.2.1 Introduction

In the present section, we employ fractional order theory of thermoelasticity (Ezzat and El Karamany (2011)) and make an attempt to investigate the effects of fractional order parameter on the propagation of plane wave propagating through a thermoelastic medium. Plane wave can be defined as the wave whose wavefronts are infinitely long straight lines. Such waves travel in the direction perpendicular to the wavefronts. Several researchers have shown their interest in the study of plane waves propagating through elastic medium in the context of coupled thermoelastcity theory. It is worth to mention that Chadwick and Sneddon (1958) studied the propagation of plane waves in classical thermoelasticity. In generalized thermoelasticity, the propagation of plane waves with one relaxation time is analyzed by Nayfeh and Nemat-Nasser (1971) and subsequently by Puri (1973). Later on, Puri (1976) studied the plane wave propagation in infinite rotating elastic medium. Agarwal (1979) established the whitham-stability of harmonic plane waves in TRDTE media. Haddow and Wegner (1996) re-checked the propagation of wave in Green Lindsay model and Lord Shulman model. Suh and Burger (1998) achieved the order of magnitude of thermal relaxation time in TRDTE. Chandrasekaraiah (1996) reported the investigation of plane waves in the context of thermoelasticity of Green and Naghdi of type- II. Puri and Jordan (2004) and Kothari and Mukhopadhyay (2012) investigated the propagation of plane waves in type-III thermoelastic media.

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Sherief *et al.* (2010), Ezzat and El KAramany (2011) have developed two different models in generalized thermoelasticity by using fractional order time-derivatives. Recently, some investigations in the context of fractional order thermoelasticity are carried out by some authors including Kumar and Gupta (2015) and Atwa (2011).

The main objective of the present work is to analyze the fractional order heat conduction law as compared to the classical Fourier law of heat conduction and the Cattaneo-Vernotte law of heat conduction that includes one thermal relaxation parameter. For this, the present study investigates the influence of fractional order parameter α (which lies between 0 and 1) on the propagation of harmonic plane wave and compare the results predicted by fractional order thermoelasticity theory with respect to the classical coupled thermoelasticity theory as well as the generalized thermoelasticity theory. After formulating the problem, we find associated dispersion relation and with the help of dispersion relation solution, we have analyzed the effects of α on the behavior of harmonic plane wave. This study brings to light some specific features of the proposed heat conduction model that involves fractional order derivative with fractional parameter, α.

2.2.2 Basic governing equations: Problem formulation

We employ the thermoelasticity theory with fractional order heat conduction. Therefore, the equations that represent the displacement and thermal fields in isotropic elastic medium in the absence of body forces and heat sources are written as follows:

Equation of motion without body force:

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} - \gamma T_0 \theta_{,i} = \rho \ddot{u}_i \tag{2.2.1}$$

Equation of heat conduction in the context of fractional order thermoelasticity without heat source (Ezzat and El Karamany (2011)):

$$K\theta_{,ii} = \rho C_v(\dot{\theta} + \frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha}\dot{\theta}) + \gamma T_0(\dot{e} + \frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha}\dot{e}) \tag{2.2.2}$$

In the above equations, the superposed dot and comma notations are used for time derivative and the material derivative respectively. δ_{ij} denotes kronecker delta. Summation convention has been used here and i, j, k varies from 1 to 3. $\tau_0$is the thermal relaxation time and α is the fractional order parameter (see Ezzat (2011)).

For solving this problem, it is important to simplify it and for that we introduce the following non dimensional quantities

$u_i' = \frac{u_i}{c_0 t_r}$, $\theta' = \frac{\theta - T_0}{T_0}$, $x_i' = \frac{x_i}{c_0 t_r}$

$t' = \frac{t}{t_r}$, $K' = \frac{K}{\rho C_v c_0^2 t_r}$, $\tau_0' = \frac{\tau_0}{t_r}$, $a_2 = \frac{\gamma}{\rho C_v}$

where $t_r > 0$ is the characteristic response in time for the medium and $c_0 = \sqrt{\frac{(\lambda+2\mu)}{\rho}}$ is the velocity of longitudinal elastic wave. For our analysis, we will consider only longitudinal plane wave solution; since it is found that the transverse plane wave is neither modified nor contributed by the temperature field of the medium.

2.2.3 Dispersion relation solution

By using the above non-dimensional quantities, we transform the governing equations into their non dimensional forms. However, for simplicity we drop primes from the equations which were obtained by introducing the non dimensional quantities in equations (2.2.1) and (2.2.2). Hence, we obtain the following equations:

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} - \gamma T_0 \theta_{,i} = \rho c_0^2 \ddot{u}_i \tag{2.2.4}$$

$$K\theta_{,ii} = (1 + \frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha})(\dot{\theta} + a_2 \dot{u_{i,i}}) \tag{2.2.5}$$

We have taken longitudinal plane wave solution in the form:

$$u_j = Ad_j e^{i(\omega t - \gamma n_i x_i)} \tag{2.2.6}$$

$$\theta = Be^{i(\omega t - \gamma n_i x_i)} \tag{2.2.7}$$

where $\omega > 0$ is the angular frequency of wave, d_j is the unit vector in the direction of displacement, γ is a complex constant, n_j is the unit vector normal to the wavefront and A and B are complex amplitudes. Phase velocity of waves is defined by $\frac{\omega}{Re[\gamma]}$.

Then equations (2.2.6) and (2.2.7) correspond to the dilatational wave for which $\frac{\omega}{2\pi}$ is the frequency and $\frac{2\pi}{Re[\gamma]}$ is the wave length. For considering physically realistic wave, we must take here $Re[\gamma] > 0$ and $Im[\gamma] \leq 0$.

Now, employing equations (2.2.6) and (2.2.7) into equations (2.2.4) and (2.2.5) and after doing some manipulations, we obtain-

$$(\gamma^2 - \omega^2)A - ia_1\gamma B = 0 \tag{2.2.8}$$

$$A(a_2\gamma\omega(1 + \frac{\tau_0^\alpha}{\Gamma(\alpha+1)}(i\omega)^\alpha)) + B(K\gamma^2 + i\omega(1 + \frac{\tau_0^\alpha}{\Gamma(\alpha+1)}(i\omega)^\alpha)) = 0 \tag{2.2.9}$$

where, $a_1 = \frac{\beta T_0}{(\lambda+2\mu)}$. Here we have used the property $d_j n_j = n_j d_j = 1$.
Now, we will find the dispersion relation. For achieving non-trivial solutions, the determinant of the coefficient matrix of the above system of equations (2.2.8) and (2.2.9) will be 0. Hence, we find-

$$\begin{vmatrix} \gamma^2 - \omega^2 & -ia_1\gamma \\ a_2\gamma\omega(1 + \frac{\tau_0^\alpha}{\Gamma(\alpha+1)}(i\omega)^\alpha) & K\gamma^2 + i\omega(1 + \frac{\tau_0^\alpha}{\Gamma(\alpha+1)}(i\omega)^\alpha) \end{vmatrix} = 0$$

After solving the above determinant, we find the dispersion relation as -

$$K\gamma^4+\gamma^2[-\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\omega^{1+\alpha}sin[\frac{\alpha\pi}{2}](1+a_1a_2)+i\omega(1+\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\omega^\alpha cos[\frac{\alpha\pi}{2}])(1+a_1a_2)-K\omega^2]$$

$$+[\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\omega^{3+\alpha}sin[\frac{\alpha\pi}{2}]-i\omega^3(1+\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\omega^\alpha cos[\frac{\alpha\pi}{2}])]=0 \qquad (2.2.10)$$

We write equation (2.2.10) in the following form-

$$K\gamma^4+(-P+iQ)\gamma^2+(S+iT)=0 \qquad (2.2.11)$$

where, $P=K\omega^2+\epsilon\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\omega^{(1+\alpha)}sin[\frac{\alpha\pi}{2}]+\omega^{(1+\alpha)}\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}sin[\frac{\alpha\pi}{2}]$
$Q=\omega(1+\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\omega^\alpha cos[\frac{\alpha\pi}{2}])+\epsilon\omega(1+\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\omega^\alpha cos[\frac{\alpha\pi}{2}])$
$S=\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}sin[\frac{\alpha\pi}{2}]\omega^{(3+\alpha)}$
$T=-\omega^3(1+\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\omega^\alpha cos[\frac{\alpha\pi}{2}])$
where, $\epsilon=a_1a_2$.
Equation (2.2.11) represents the dispersion relation of plane harmonic wave under fractional order thermoelasticity theory and clearly it shows the influence of fractional order parameter α.

Now the roots of equation (2.2.11) can be obtained in the following manner

$$\gamma_{1,2}^2=\frac{P-iQ\pm\sqrt{D(\omega)}}{2K} \qquad (2.2.12)$$

where $D(\omega)=(-P+iQ)^2-4K(S+iT)$

$$Re[D(\omega)]=P^2-Q^2-4KS \qquad (2.2.13)$$

$$Im[D(\omega)]=-2i(PQ+2KT) \qquad (2.2.14)$$

Here, we will take only two of the four roots of γ given by the equation (2.2.12) which have the negative imaginary parts, because due to these roots we find the negative value of the decay coefficient.
For finding the values of γ, a theorem (Ponnusamy (2001)) on complex analysis has been used that can be explained in the following way:

2.2.4 Theorem

If $Z = x + iy$ and $Y^2 = Z$, where Y is a complex quantity then values of Y are given by
$Y = \pm[\sqrt{\frac{|Z|+x}{2}} + isign(y)\sqrt{\frac{|Z|-x}{2}}]$
where $sign(y) = +1, y \geq 0$
$sign(y) = -1, y < 0$

By using above mentioned theorem, we find the desired values of γ, i.e the values of γ with negative imaginary parts.

Hence, we find the two different modes of longitudinal wave corresponding to these two values of γ. One is pre-dominantly elastic mode wave and the other is predominantly thermal mode wave. We denote the former one by γ_1 and the other one by γ_2. We will investigate both of these modes in details in the context of fractional order heat conduction and compare the results with the same under classical model and generalized model.

2.2.5 Analytical results

From equation (2.2.11), it is very difficult to analyze the behavior of waves. Hence in our analysis, we have considered two different cases, namely the case of high frequency and the case of low frequency values. In both the cases, we find several quantities which contain a great importance during the analysis of plane wave; like 'phase velocity', 'specific loss' and 'penetration depth' of the wave. These quantities are used to characterize the behaviour of such harmonic plane waves over space and time and they are determined by the following formulas:

Phase Velocity-

$$V_{e,,t} = \frac{\omega}{Re[\gamma_{1,2}]} \tag{2.2.15}$$

Specific Loss-

$$(\frac{\Delta W}{W})_{e,t} = 4\pi|\frac{Im[\gamma_{1,2}]}{Re[\gamma_{1,2}]}| \tag{2.2.16}$$

Penetration Depth-

$$D_{e,t} = \frac{1}{|Im[\gamma_{1,2}]|} \tag{2.2.17}$$

In above formulae we used the subscripted notations e and t to denote the wave components for the elastic mode longitudinal wave and the thermal mode longitudinal wave, respectively. Now, we analyze the behavior of plane wave based on the desired roots of γ (in which the imaginary parts of the roots are negative). For both the cases, we investigate the wave of high frequency as well as the wave of low frequency.

2.2.5.1 Analytical expressions of results in high frequency case

In this case we consider $\omega >> 1$ and after detailed lengthy manipulations, we find the expressions of γ from equation (2.2.12) as-

$$\gamma_1 \sim [\omega[1 + \frac{a}{\omega^{1-\alpha}}] + i\frac{B_1}{2}\omega^{\alpha}[1 + \frac{a'}{\omega^{2(1-\alpha)}}]], (\omega \to \infty) \tag{2.2.18}$$

$$\gamma_2 \sim [c\omega^{\frac{1+\alpha}{2}}[1+\frac{d}{\omega^{1-\alpha}}] + ic'\omega^{\frac{1+\alpha}{2}}[1-\frac{d'}{\omega^{1-\alpha}}]], (\omega \to \infty) \qquad (2.2.19)$$

where,

$a = \frac{1}{8}(\frac{\tau_0^\alpha}{\Gamma(\alpha+1)} sin[\frac{\pi\alpha}{2}])(1-\epsilon)$

$B_1 = \frac{1}{2K}(\frac{n_1}{2K} - \frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\epsilon cos[\frac{\pi\alpha}{2}])$, $B_2 = \frac{1}{2K}(\frac{n_1}{2K} - (1+\epsilon))$

$n_1 = 2K\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}(2-\epsilon)cos[\frac{\pi\alpha}{2}]$, $n_2 = 2K(2-\epsilon)$

$a' = \frac{B_2^2}{B_1^2}$, $a_6 = (\frac{\tau_0^\alpha}{\Gamma(\alpha+1)} sin[\frac{\pi\alpha}{2}])^2(\epsilon-1)^2 + \frac{1}{4}(\frac{-a_3}{\hat{a_1}} + \frac{a_5}{2})^2$

$\hat{a_1} = K^2$, $a_3 = -(\frac{\tau_0^\alpha}{\Gamma(\alpha+1)})^2(1+\epsilon)^2 cos[\pi\alpha]$, $a_5 = \frac{2\hat{a_1}a_3 + \hat{a_2}^2}{\hat{a_1}^2}\frac{4K^2(\frac{\tau_0^\alpha}{\Gamma(\alpha+1)})^2(1+\epsilon)^2 cos[\frac{\pi\alpha}{2}]^2}{\hat{a_1}^2}$

$a_7 = \frac{1}{4}(\frac{a_3}{\hat{a_1}} + \frac{a_5}{2})$, $d = \frac{1}{\sqrt{g}\sqrt{(\sqrt{g} + \frac{\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}(1+\epsilon)sin[\frac{\pi\alpha}{2}]}{2})}}$

$c' = \sqrt{(\sqrt{g} - \frac{\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}(1+\epsilon)sin[\frac{\pi\alpha}{2}]}{2})}$, $d' = \frac{\hat{a}K}{(\sqrt{g} - \frac{\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}(1+\epsilon)sin[\frac{\pi\alpha}{2}]}{2})(\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}(1+\epsilon)sin[\frac{\pi\alpha}{2}])}$

$c = \sqrt{\frac{(\frac{\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}(1+\epsilon)sin[\frac{\pi\alpha}{2}]}{2} + \sqrt{g})}{4K}}$, $g = (\frac{\tau_0^\alpha}{\Gamma(\alpha+1)})^2(1+\epsilon)^2(sin[\frac{\pi\alpha}{2}])^2 + (\frac{\hat{a}}{2} - (1+\epsilon)\frac{\tau_0^\alpha}{\Gamma(\alpha+1)} sin[\frac{\pi\alpha}{2}])^2]$

$\hat{a} = \frac{(1+\epsilon)^2(\frac{\tau_0^\alpha}{\Gamma(\alpha+1)})^2 sin[\frac{\pi\alpha}{2}] + (\frac{\tau_0^\alpha}{\Gamma(\alpha+1)})^2(3+\epsilon)^2 cos[\frac{\pi\alpha}{2}]^2 + (\frac{\tau_0^\alpha}{\Gamma(\alpha+1)})^2(1+\epsilon)^2 cos[\pi\alpha]}{2}$, $\hat{a_2} = -4K\frac{\tau_0^\alpha}{\Gamma(\alpha+1)} sin[\frac{\pi\alpha}{2}]$

2.2.5.2 Analytical results for low frequency asymptotes

Here, we have taken $\omega << 1$. In this case the roots of equation (2.2.11) can be expressed as:

$$\gamma_1 \sim a_0[D\omega[1+\hat{c}\omega^\alpha] + iE\omega^{1+\alpha}[1+\hat{c'}\omega^\alpha]], (\omega \to 0) \qquad (2.2.20)$$

$$\gamma_2 \sim a_0[F\omega^{\frac{1}{2}}[1+\hat{d}\omega^\alpha] + iG\omega^{\frac{1}{2}}[1+\hat{d'}\omega^\alpha]], (\omega \to 0) \qquad (2.2.21)$$

where, $a_0 = \frac{1}{\sqrt{2K}}$, $D = \sqrt{\frac{b_4+b_8}{2}}$, $b_8 = \sqrt{(b_4^2 + b_6^2)}$

$b_4 = (1+\epsilon)\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\sqrt{1-cos[\frac{\pi\alpha}{2}]}(1+(1+\epsilon)\frac{\tau_0^\alpha}{\Gamma(\alpha+1)} sin[\frac{\pi\alpha}{2}])$, $b_6 = \frac{1}{2}(1+\epsilon)\frac{\tau_0^\alpha}{\Gamma(\alpha+1)} cos[\frac{\pi\alpha}{2}]$

$\hat{c} = \frac{b_8 b_9}{2(b_4+b_8)}$, $b_9 = \frac{b_7 b_6^2}{(b_4^2+b_6^2)}$, $b_7 = \frac{b_4}{b_3 - \frac{\tau_0^\alpha}{\Gamma(\alpha+1)} cos[\frac{\pi\alpha}{2}]]}$

$E = \sqrt{\frac{-b_4+b_8}{2}}$, $\hat{c'} = \frac{b_8 b_9}{2(-b_4+b_8)}$
$a_0 = \frac{1}{\sqrt{2K}}$, $F = (1+\epsilon)^{\frac{1}{2}}$, $\hat{d} = \frac{c_1}{4(1+\epsilon)}$, $c_1 = b_1 + \frac{\tau_0^\alpha}{\Gamma(\alpha+1)}(1+\epsilon)sin[\frac{\pi\alpha}{2}]$
$b_1 = (1+\epsilon)\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\sqrt{1-cos[\frac{\pi\alpha}{2}]}$, $G = (1+\epsilon)^{\frac{1}{2}}$, $\hat{d'} = -\frac{1}{4}(b_1 + \frac{\tau_0^\alpha}{\Gamma(\alpha+1)}(1+\epsilon)sin[\frac{\pi\alpha}{2}])$

2.2.6 Analytical expressions of various components of plane waves

With the help of equations (2.2.18)-(2.2.21) and by using the formulae (2.2.15)-(2.2.17), we evaluate the asymptotic expressions of phase velocity, specific loss, penetration depth for both the elastic and thermal mode waves in both cases (high and low frequency cases).

2.2.6.1 High frequency case (For $0 < \alpha < 1$)

With the help of equation (2.2.18) and (2.2.19) and using the formula of phase velocity given by (2.2.15) in case of large values of frequency, ω, we get the desired expressions for both the elastic and thermal mode waves as follows:

Phase velocity for elastic wave:

$$V_e \sim [1 - \frac{a}{\omega^{1-\alpha}}], (\omega \to \infty) \tag{2.2.22}$$

Phase velocity for thermal wave:

$$V_t \sim \frac{1}{c}\omega^{\frac{1-\alpha}{2}}[1 - \frac{d}{\omega^{1-\alpha}}], (\omega \to \infty) \tag{2.2.23}$$

In the similar way, using equations (2.2.18), (2.2.19), and with the help of formula of specific loss we find its asymptotic expressions for both the modes of wave (for large value of ω) as given below:

Specific loss for elastic wave:

$$S_e = \frac{1}{4\pi}(\frac{\Delta W}{W})_e \sim \frac{B_1}{2\omega^{1-\alpha}}[1 + \frac{-a}{\omega^{1-\alpha}}],\ (\omega \to \infty) \tag{2.2.24}$$

Specific loss for thermal wave:

$$S_t = \frac{1}{4\pi}(\frac{\Delta W}{W})_t \sim \frac{c'}{c}[1 - \frac{d + d'}{\omega^{1-\alpha}}],\ (\omega \to \infty) \tag{2.2.25}$$

Now, we find expressions of penetration depth for both elastic and thermal mode wave with the help of equations (2.2.18), (2.2.19) and formula (2.2.17) respectively (for large values of ω).

Penetration depth for elastic wave:

$$D_e \sim \frac{2}{B_1\omega^{\alpha}}[1 - \frac{a'}{\omega^{2(1-\alpha)}}],\ (\omega \to \infty) \tag{2.2.26}$$

Penetration depth for thermal wave:

$$D_t \sim \frac{1}{c'\omega^{\frac{1+\alpha}{2}}}[1 + \frac{d'}{\omega^{1-\alpha}}], (\omega \to \infty) \tag{2.2.27}$$

2.2.6.2 Special case $\alpha = 1$

Expressions of thermal mode wave and elastic mode wave are separately presented for the case $\alpha = 1$.

Thermal mode wave

$V_t = \frac{1}{\tilde{f}}(1 - \frac{\tilde{h}}{\omega^2}), (\omega \to \infty)$

$S_t = \frac{\tilde{m}}{\tilde{f}\omega}(1 + \frac{\tilde{h}-\tilde{n}}{\omega^2}), (\omega \to \infty)$

$D_t = \frac{1}{\tilde{m}}(1 - \frac{\tilde{n}}{\omega^2}), (\omega \to \infty)$

where, we have used the following notations:

$\tilde{f} = (\tilde{a} - K - \tau_0(1+\epsilon)^2)^2$, $\tilde{h} = \frac{1}{4}(\tilde{c^2}\tilde{d}^2(\tilde{c} - (1+\epsilon))^2) + \frac{\tilde{a}\tilde{b}}{\tilde{a}-K-\tau_0(1+\epsilon)}$

$\tilde{a} = \sqrt{(K^4 + 2K\tau_0(\epsilon - 1) + \tau_0^2(1+\epsilon)^2)}\frac{(2K(1+\epsilon)+\tau_0(1+\epsilon)^2+4K)}{2}$

$\tilde{b} = \frac{1}{8}[(2K(1+\epsilon) + \tau_0(1+\epsilon)^2 + 4K)^2 - 4\frac{(1+\epsilon)^2}{K^4+2K\tau_0(\epsilon-1)+\tau_0^2\epsilon^2}]$

$\tilde{c} = \frac{1}{2}(2K(1+\epsilon) + \tau_0(1+\epsilon)^2 + 4K)\sqrt{(K^4 + 2K\tau_0(\epsilon - 1) + \tau_0^2(1+\epsilon)^2)}$

$\tilde{d} = \frac{(1+\epsilon)^4}{2(2K(1+\epsilon)+\tau_0(1+\epsilon)^2+4K)(K^4+2K\tau_0(\epsilon-1)+\tau_0^2(1+\epsilon)^2)}$

$\tilde{m} = \sqrt{((\tilde{a} - K - \tau_0(1+\epsilon)^2))}\frac{\sqrt{(\tilde{c^2}\tilde{d^2}(\tilde{c}-(1+\epsilon))^2)}}{2}$, $\tilde{n} = \frac{\tilde{k}}{\tilde{j}}$

$\tilde{k} = \frac{2(\tilde{c}-(1+\epsilon))^2\tilde{c^2}\tilde{d^2}+\frac{\tilde{a^2}\tilde{b^2}}{(\tilde{a}-K-\tau_0(1+\epsilon)^2)^2}}{4}$, $\tilde{j} = \frac{(\tilde{c}-(1+\epsilon))^2\tilde{c^2}\tilde{d^2}}{2}$

Elastic mode wave

$V_e = \frac{1}{p_1^{\frac{1}{4}}}(1 - \frac{3q_1^2}{32p_1^2\omega^2}), (\omega \to \infty)$

$S_e = \frac{1}{16q_1p_1^{\frac{7}{4}}\omega}(1 - \frac{5q_1^2}{16p_1^2\omega^2}), (\omega \to \infty)$

$D_e = 16q_1p_1^{\frac{3}{2}}(1 - \frac{7q_1^2}{32p_1^2\omega^2}), (\omega \to \infty)$

where,

$p_1 = (\epsilon^2 + 2\tau_0\epsilon + (K-\tau)^2 + 2K\tau_0\epsilon)$, $q_1 = 4K - 2(K + \tau_0(1+\epsilon))(1+\epsilon)$

2.2.6.3 Low frequency case

Similarly, with the help of equations (2.2.20) and (2.2.21) and assuming frequency ω to be very small, we get the asymptotic expressions for different wave components of elastic and thermal mode waves as follows:

Phase velocity for elastic wave:

$$V_e \sim \frac{1}{A_3}[1 - c\omega^{\alpha}], (\omega \to 0) \tag{2.2.28}$$

where $A_3 = a_0 D$

Phase velocity for thermal wave:

$$V_t \sim \frac{1}{A_4}\omega^{\frac{1}{2}}[1 - d\omega^{\alpha}], (\omega \to 0) \tag{2.2.29}$$

where $A_4 = a_0 F$

Specific loss for elastic wave:

$$S_e \sim \frac{E}{D}\omega^{\alpha}[1 + (c' - c)\omega^{\alpha}], (\omega \to 0) \tag{2.2.30}$$

Specific loss for thermal wave:

$$S_t \sim \frac{G}{F}[1 + (d' - d)\omega^{\alpha}], (\omega \to 0) \tag{2.2.31}$$

Penetration depth for elastic wave:

$$D_e \sim \frac{1}{a_0 E\omega^{1+\alpha}}[1 - c'\omega^{\alpha}], (\omega \to 0) \tag{2.2.32}$$

Penetration depth for thermal wave:

$$D_t \sim \frac{1}{a_0 G\omega^{\frac{1}{2}}}[1 - d'\omega^{\alpha}], (\omega \to 0) \tag{2.2.32}$$

2.2.7 Numerical results

In the previous section, we have derived the asymptotic results which determine the limiting nature of longitudinal plane wave. We derived the asymptotic expressions of various components of wave like, phase velocity, specific loss and penetration depth for the thermal mode and elastic mode wave for very low and very high frequency values in the context of fractional order heat conduction model. The limiting behaviour of the wave components for different values of fractional order parameter can be observed from these analytical results. In order to compare the results under fractional order theory with the generalized thermoelasticity theory that includes one thermal relaxation parameter, we also derive the expressions for the special case $\alpha = 1$. In the present section, we make an attempt to find the numerical values of different wave characterizations for the intermediate values of wave frequency and also we verify the analytical results in order to better understand the influence of fractional order theory. For this motive, we assume following non-dimensional values:

$K = 1$

$\epsilon = 0.0168$

$\tau_0 = 0.01$

We have used the software Mathematica (version 6) and solve the dispersion relation given by (2.2.11). Then, with the help of the formulae of various components of plane wave, we compute the numerical values of components of both types of wave for the different values of the frequency ω and for the different values of α. The values of the wave characterizations for the case $\tau_0 = 0$ are also obtained. The results are plotted in different figures (Figures 2.2.1-2.2.12). The cases of plots for low frequency and high frequency are shown in different figures. In all the Figures, the dot-dashed lines represent the case: $\tau_0 = 0$; the thin solid lines are used for the case: α=0.25; thick solid lines represent the case: α= 0.5; thick dashed lines are used for the case: α=0.75; thin dotted lines represent the case: $\alpha = 0.9$ and the thick dotted lines represent the case: α=1.

2.2.8 Analysis of analytical results and numerical results

2.2.8.1 Analysis of phase velocity

Figures 2.2.1 and 2.2.2 show the variations of phase velocity of thermal mode wave for low and high frequency, respectively. Figure 2.2.1 represents that the phase velocity of thermal mode wave is initially an increasing function of frequency ω. Figure 2.2.2 exhibits the relation between phase velocity and frequency ω for higher frequency values. Here, we can examine the significant role of α. We further note that the differences of predictions by fractional order thermoelastic model, the classical model and the generalized model by Lord and Shulman is significant. Specially, there is a significant change in the phase velocity profiles for the cases when $\alpha \leq 0.5$ and the cases when $\alpha > 0.5$. This fact is in good agreement with our analytical results (see equation (2.2.23)) which indicate that in the case when $0 < \alpha < 1$, V_t is an increasing function of frequency, where as V_t tends to a constant limiting value as $\omega \to \infty$ in the case when $\alpha = 1$. Hence, Figure 2.2.2 shows that for the cases when $0.5 < \alpha < 1$, the profile is slowly increasing with the increment of ω . However, when α is very close to 1 for example say 0.9, its nature is very close with the nature of plot when $\alpha = 1$. At $\alpha = 1$, profile achieves a constant limiting value as frequency reaches nearer to 300 and the constant limiting value is nearer to 10. This fact is also very clear from the analytical result of special case. This implies that the effect of α is more prominent on V_t when $\alpha \leq 0.5$. For $\alpha > 0.5$ profiles are very close together. The values of V_t for various cases are also shown in Table 1. We also note the above mentioned facts from this table. When frequency is very high, say nearer to 1000, the phase velocity of thermal mode wave for the cases when $\alpha > 0.5$ reaches to an almost constant limiting value which is dependent on the fractional order parameter. In the case when $\alpha = 0.9$, the speed is nearer to 9.1 which is less than the speed in case of generalized thermoelastic model. Hence, we can conclude that when α is nearer to 1 then the fractional order theory predicts more realistic results as compared to the case when $\alpha \leq 0.5$.

Figures 2.2.3 and 2.2.4 depict the variations of phase velocity of elastic mode wave for low and high frequency, respectively for different values of α. It is evident from Figure 2.2.3 that the phase velocity for elastic mode wave is initially a decreasing function of frequency ω(for low frequency). There is no significant change in the phase velocity profiles of elastic mode wave as the value of α changes. The case of $\tau_0 = 0$ also shows a similar trend. Figure 2.2.4 represents the trend of phase velocity of this mode when frequency tends to very high values. It is noted that the trend of variation of the profiles for all values of α is almost similar. We further observe that as ω increases, the phase velocity V_e reaches to a constant limiting value after showing a local minimum value and that limiting value is very nearer to 1 with a slight difference depending on fractional parameter. This is also verified from the analytical expression of phase velocity for elastic mode wave for high frequency case (see Eq. (2.2.22)).

The variation of phase velocity of thermal wave is more clearly understandable from the following table -

V_t	$\omega = 100$	$\omega = 300$	$\omega = 500$	$\omega = 700$	$\omega = 900$
$\tau_0 = 0$	14.1	24.2	31.6	37.4	42.4
$\alpha = 0.25$	6.6	10.3	12.6	14.3	15.8
$\alpha = 0.50$	5.4	7.3	8.3	9.1	9.7
$\alpha = 0.75$	6.0	7.1	7.7	8.0	8.3
$\alpha = 0.9$	7.5	8.3	8.6	8.8	8.9
$\alpha = 1$	9.1	9.9	9.9	10.0	10.0

Table 2.2.1 The variation of phase velocity of thermal wave

2.2.8.2 Analysis of specific loss

Figures 2.2.5 and 2.2.6 reveal the behavior of specific loss for thermal mode wave for values of low and high frequencies, respectively. In Figure 2.2.5, we can observe that specific loss for thermal mode wave profile is a decreasing function of ω for all values of α but it is constant for the case of classical coupled theory, i.e., when $\tau_0 = 0$. For this wave field, α has a prominent role, although the specific loss of thermal mode wave tends to a constant value which is 4π as $\omega \to 0$ for all α. From this Figure, we can conclude that the maximum value of specific loss is 4π for each α. This result is also in good agreement from analytical expression found out in previous section (see Eq. 2.2.25). Figure 2.2.6 shows the behavior of specific loss with the frequency ω for thermal mode wave for higher values of frequency. Here specific loss profile is still decreasing function of frequency ω. Significant variation occurs among the profiles of specific loss for the cases $\tau_0 = 0$ and $0 < \alpha \leq 1$ but there is significant effect of α and when it increases, a significant variation occurs among the profiles of specific loss. It exhibits the lesser value of specific loss for higher values of α. However, for all values of the fractional parameter, the specific loss tends to a constant limiting value depending on α as $\omega \to \infty$. For $\alpha = 1$ the limiting value is zero, whereas in all other cases it is a non-zero constant value. Here we can understand that like the generalized theory, the fractional order heat conduction model gives better result as compared to the classical theory of thermoelasticity. Figure 2.2.7 and Figure 2.2.8 display the specific loss of elastic mode wave. The numerical results indicated by these Figures are in complete agreement with our analytical asymptotic results (see Eqs. (2.2.24), and (2.2.30)) and as well as the results for the case when $\alpha = 1$. We note that in all cases, specific loss of elastic mode wave tends to zero when $\omega \to \infty$ and also when $\omega \to 0$. Moreover, Figures 2.2.7 and 2.2.8 reveal that this specific loss profile increases rapidly with the increase of frequency and after achieving an extreme value, it starts decreasing and again tends to zero value as $\omega \to \infty$. This extreme value is different for different profiles and it decreases as the value of α increases. For $\tau_0 = 0$ it is nearer to 0.004. For $\alpha > 0.5$, the effect of α is not prominent. However, the trend of variation is almost similar for all the profiles. This result also matches with our analytical results.

2.2.8.3 Analysis of penetration depth

Figure 2.2.9 and Figure 2.2.10 depict the behavior of penetration depth for thermal mode wave for low and high frequency, respectively. From Figures 2.2.9 and 2.2.10, it is evident that penetration depth profile of thermal mode wave is a decreasing function of ω and it approaches to constant limiting value which is nearer to 0 as $\omega \to \infty$ for every α $(0 < \alpha < 1)$ but in the case when $\alpha = 1$, the penetration depth shows a constant limiting value as frequency reaches a high value and this value is near about 0.2. This fact is also evident from the analytical result obtained in special case when $\alpha = 1$and Eq. (2.2.27).

The behavior of penetration depth for elastic mode wave can be observed from Figures 2.2.11 and 2.2.12 and the asymptotic results given by Eqs. (2.2.26) and (2.12.32). For this profile, we note that like the penetration depth of thermal mode wave, the penetration depth of elastic mode wave also decreases rapidly as frequency increases but in Fig. 12, the penetration depth shows constant value for the case $\tau_0 = 0$ and $\alpha = 1$ and this value is near about 120. The effect of fractional order parameter is prominent on this profile. For the cases when $0 < \alpha < 1$, it slowly decreases to zero values as $\omega \to \infty$. This fact is also in good agreement with the analytical results given by eq. (2.2.26).

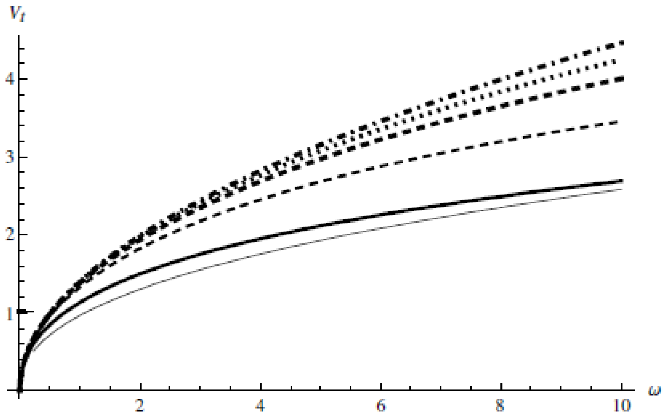

Figure 2.2.1 Variation of phase velocity of thermal mode wave with frequency (low frequency): Thick dot dashed line: $\tau_0 = 0$, Thin line: $\alpha = 0.25$, Thick line: $\alpha = 0.5$, Thin dashed line: $\alpha = 0.75$, Thick dashed line: $\alpha = 0.9$, Thick dotted line: $\alpha = 1$

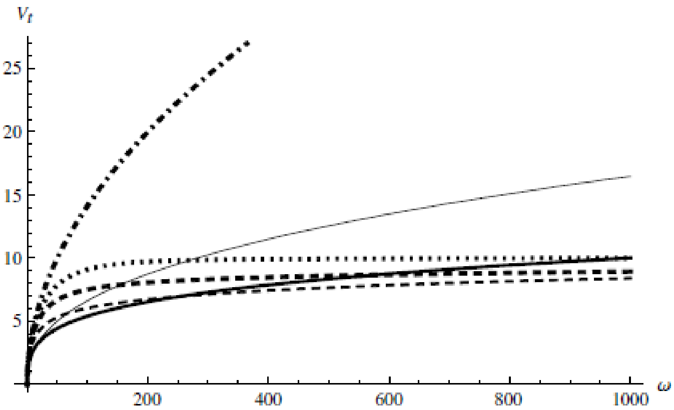

Figure 2.2.2 Variation of phase velocity of thermal mode wave with frequency (high frequency): Thick dot dashed line: $\tau_0 = 0$, Thin line: $\alpha = 0.25$, Thick line: $\alpha = 0.5$, Thin dashed line: $\alpha = 0.75$, Thick dashed line: $\alpha = 0.9$, Thick dotted line: $\alpha = 1$

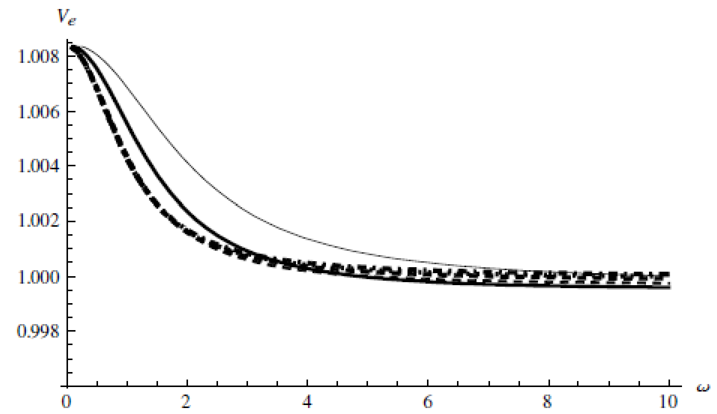

Figure 2.2.3 Variation of phase velocity of elastic mode wave with frequency (low frequency): Thick dot dashed line: $\tau_0 = 0$, Thin line: $\alpha = 0.25$, Thick line: $\alpha = 0.5$, Thin dashed line: $\alpha = 0.75$, Thick dashed line: $\alpha = 0.9$, Thick dotted line: $\alpha = 1$

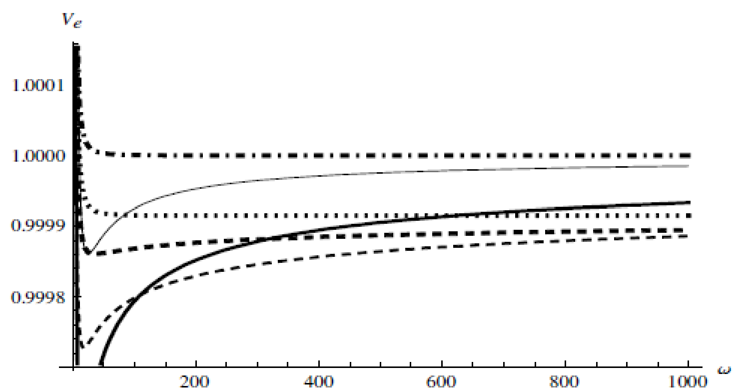

Figure 2.2.4 Variation of phase velocity of elastic mode wave with frequency (high frequency): Thick dot dashed line: $\tau_0 = 0$, Thin line: $\alpha = 0.25$, Thick line: $\alpha = 0.5$, Thin dashed line: $\alpha = 0.75$, Thick dashed line: $\alpha = 0.9$, Thick dotted line: $\alpha = 1$

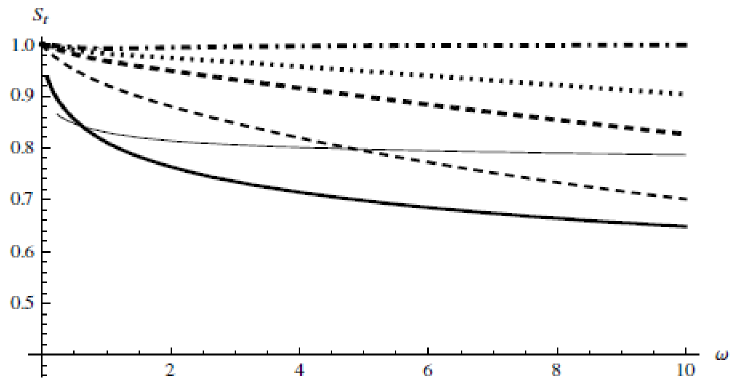

Figure 2.2.5 Variation of specific loss of thermal mode wave with frequency (low frequency): Thick dot dashed line: τ_0= 0, Thin line: $\alpha = 0.25$, Thick line: $\alpha = 0.5$, Thin dashed line: $\alpha = 0.75$, Thick dashed line: $\alpha = 0.9$, Thick dotted line: $\alpha = 1$

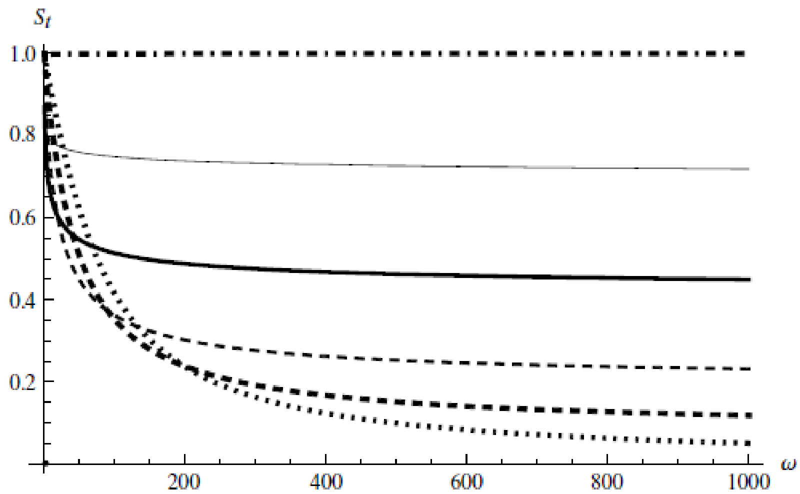

Figure 2.2.6 Variation of specific loss of thermal mode wave with frequency (high frequency): Thick dot dashed line: τ_0= 0, Thin line: $\alpha = 0.25$, Thick line: $\alpha = 0.5$, Thin dashed line: $\alpha = 0.75$, Thick dashed line: $\alpha = 0.9$, Thick dotted line: $\alpha = 1$

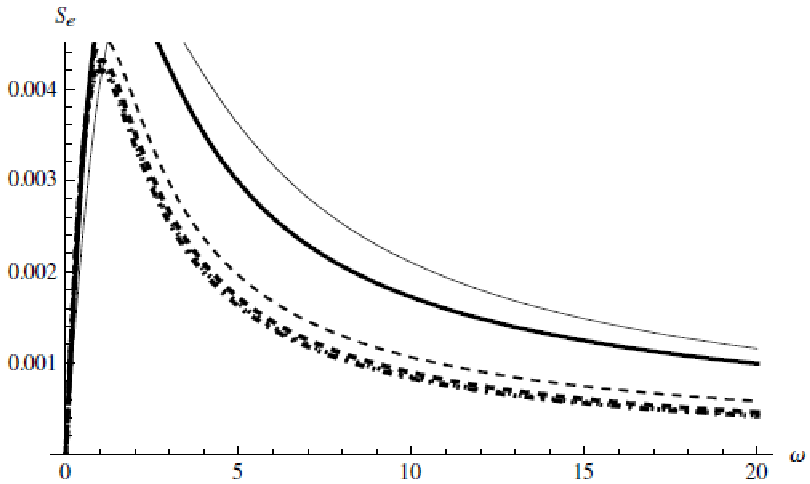

Figure 2.2.7 Variation of specific loss of elastic mode wave with frequency (low frequency): Thick dot dashed line: τ_0= 0, Thin line: α = 0.25, Thick line: α = 0.5, Thin dashed line: α = 0.75, Thick dashed line: α = 0.9, Thick dotted line: α = 1

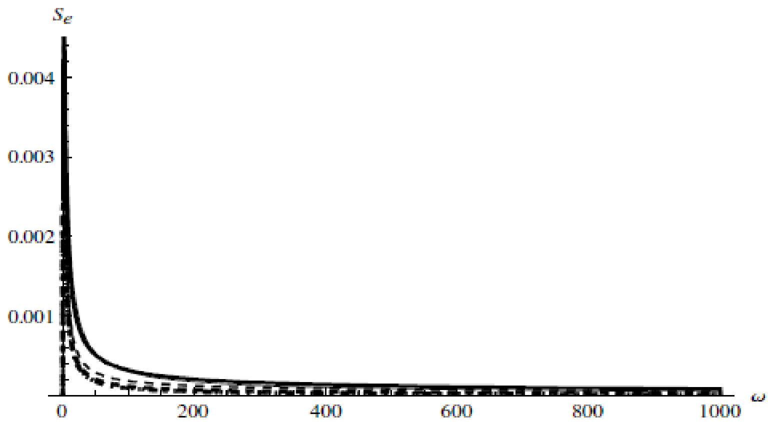

Figure 2.2.8 Variation of specific loss of elastic mode wave with frequency (high frequency): Thick dot dashed line: τ_0= 0, Thin line: α = 0.25, Thick line: α = 0.5, Thin dashed line: α = 0.75, Thick dashed line: α = 0.9, Thick dotted line: α = 1

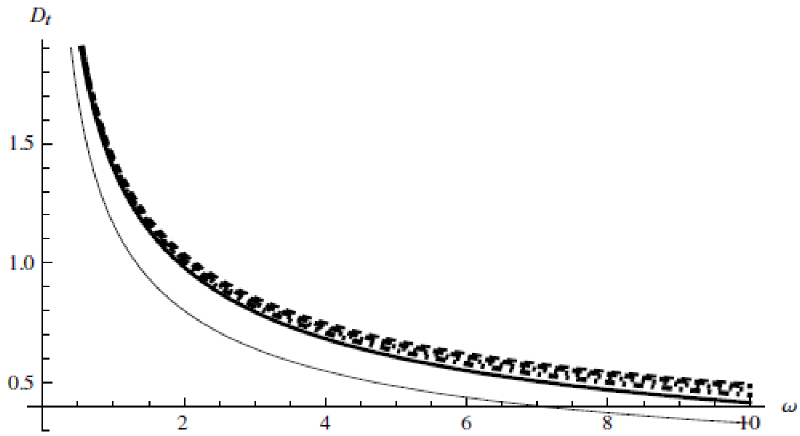

Figure 2.2.9 Variation of penetration depth of thermal mode wave with frequency (low frequency): Black Thick Dot Dashed = τ_0= 0, Black Thin α = 0.25, Black Thick α = 0.5, Black Thin Dashed α = 0.75, Black Thick Dashed α = 0.9, Black Thick Dotted α = 1

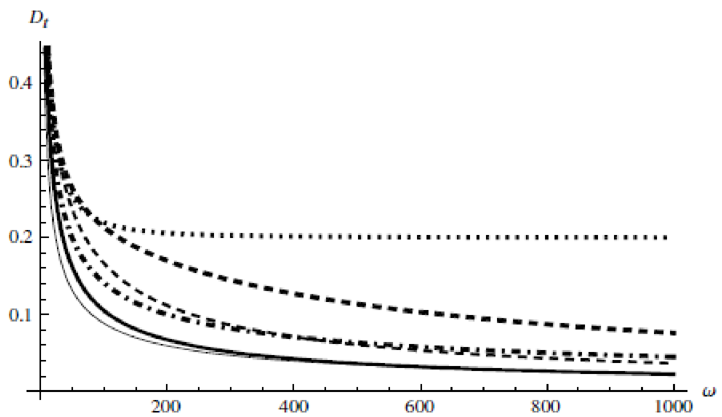

Figure 2.2.10 Variation of penetration depth of thermal mode wave with frequency (high frequency): Black Thick Dot Dashed = τ_0= 0, Black Thin α = 0.25, Black Thick α = 0.5, Black Thin Dashed α = 0.75, Black Thick Dashed α = 0.9, Black Thick Dotted α = 1

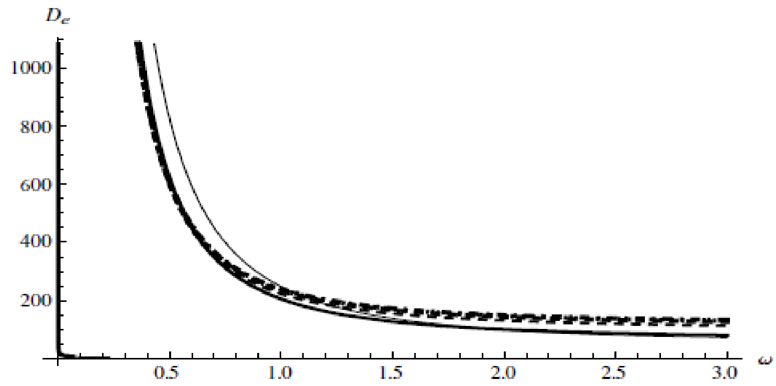

Figure 2.2.11 Variation of penetration depth of elastic mode wave with frequency (low frequency): Thick dot dashed line: $\tau_0 = 0$, Thin line: $\alpha = 0.25$, Thick line: $\alpha = 0.5$, Thin dashed line: $\alpha = 0.75$, Thick dashed line: $\alpha = 0.9$, Thick dotted line: $\alpha = 1$

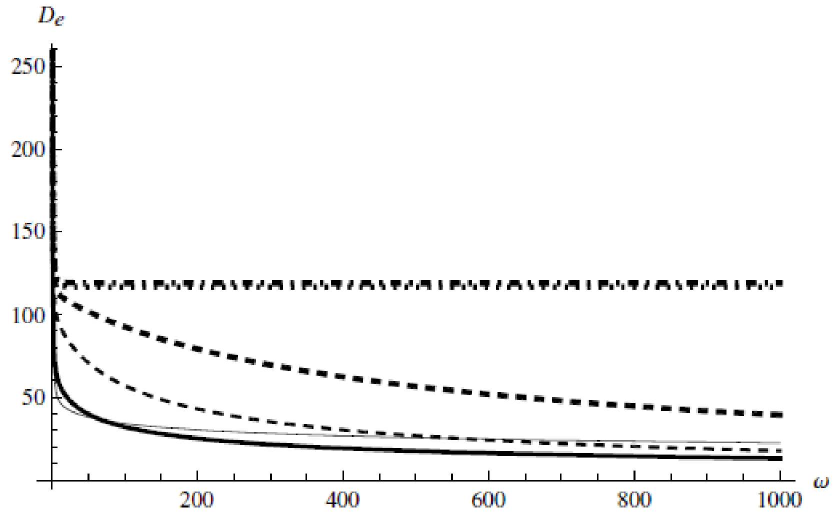

Figure 2.2.12 Variation of penetration depth of elastic mode wave with frequency (high frequency): Thick dot dashed line: $\tau_0 = 0$, Thin line: $\alpha = 0.25$, Thick line: $\alpha = 0.5$, Thin dashed line: $\alpha = 0.75$, Thick dashed line: $\alpha = 0.9$, Thick dotted line: $\alpha = 1$

2.2.9 Conclusions

In the present work, dispersion relation solutions for the plane wave propagating in a thermoelastic media have been determined by employing fractional order thermoelasticity theory. From the derived dispersion relation solution, the components of longitudinal plane wave have been derived and the effects of the fractional order parameter α on the wave components have been analyzed in order to understand the differences in predictions of the present theory as compared to the classical theory and generalized theory of thermoelasticity. Asymptotic expressions of the wave components in two limiting cases of frequency values and also their numerical values for intermediate values of frequency have been obtained. Subsequently, a comparative study of analytical and numerical results has been made to analyze the effect of fractional order parameter in details. While doing analysis of analytical and numerical results, we have found following highlighted results:

1. There are two different modes of longitudinal waves: one is predominantly elastic and the other one is predominantly thermal in nature.

2. We have considered two special cases: $\alpha = 1$, $\tau_0 = 0$ which correspond to generalized theory of thermoelasticity and classical theory of thermoelasticity, respectively and we have attempted to compare the predictions of fractional order theory with these two theories. Our numerical results for the cases when $\tau_0 = 0$ and $\alpha = 1$ match with the corresponding results as reported by Puri and Jordan (2004) and Kumar *et al.* (2015), respectively.

3. It is observed that when we employ the fractional order heat conduction theory, significant changes occurs on the behavior of the both the modes of longitudinal wave (elastic and thermal) when α goes beyond 0.5.

4. The effect of α is more prominent on thermal mode wave.

When we compare the fractional order heat conduction model with the classical model, we achieve that the fractional order heat conduction model is the better one and it predicts better results in comparison to the classical theory of thermoelasticity only when α goes beyond 0.5. Moreover, the fractional order theory indicates almost similar realistic results like generalized theory of thermoelasticity proposed by Lord and Shulman, specially in the case when α is very close to 1. For all other cases when α goes nearer to 0.5 or less than that, the theory predicts significantly different results and suffers from the similar drawback like classical theory of thermoelasticity. This fact is believed to be very useful for further research in this direction.