Chapter-1

Introduction and Literature review

1.1 Thermoelasticity

A deformable solid body, when it is subjected to the action of external loads through heat sources and/or non-uniform heating and body forces, it gets deformed. This deformation produces strain and stress in the body. Even in absence of any external mechanical forces, the deformation and consequent stresses can develop in the body. An extensive research work has been carried out in this area due to its importance in various applications related to engineering and technology. In the initial period, the investigations in this area have been devoted to the "uncoupled theory of thermoelasticity" with the simplifying assumption that the effect of strain on the temperature field may be neglected. However, the experimental evidence indicates that deformation of a body is associated with a change of its heat content. This implies that the time varying external loading of a body causes in it not only displacements but also temperature distribution changing with time. Conversely, the heating of a body leads to deformation as well as a change in temperature. In such circumstances, the motion of a body is controlled by mutual interactions between deformation and temperature fields. Hence, the classical "uncoupled theory of thermoelasticity" is influenced with the drawback that the temperature field is independent from the elastic changes and vice-versa. The domain of science that deals with the mutual interaction of these two different fields is called coupled thermoelasticity, consequent of two combined theories, namely 'theory of heat conduction' and 'theory of elasticity'. In this case, the internal energy of the body is a function of the deformation and temperature. As a result of the coupling between these two fields, the temperature term appears in the displacement equations of motion, and the deformation is included in the equation of heat conduction. It must be mentioned that the coupling between deformation and temperature fields was first postulated by Duhamel (1837,1838) who advocated first the theory of thermal stresses and introduced the dilatation term in the equation of thermal conductivity. Neumann (1841) also formulated stress-strain-temperature relations, similar to the relations given by Duhamel. Hence, these approaches are now known as 'Duhamel-Neumann' relations. An extensive research has made a considerable progress in this area during last few decades.

1.2 Classical coupled theory of thermoelasticity

In nineteenth century, Biot (1956) worked on the field of thermoelasticity based on irreversible thermodynamics and successfully derived the constitutive relations and basic governing equations of thermoelasticity by taking into account the coupling between thermal and strain fields on the basis of Duhamel-Neumann relations. He also presented the fundamental methods for solving the thermoelasticity equations and established variational theorem. The following fundamental equations represent the system of linear equations of the theory of coupled dynamical thermoelasticity for anisotropic materials as given by Biot (1956):

Strain-displacement relation:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.1}$$

where, i, j varies from 1 to 3.

Equations of compatibility:

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \tag{1.2}$$

where, $i, j, k, l = 1, 2, 3$

Equation of motion:

$$\sigma_{ij,j} + F_i = \rho \ddot{u}_i \tag{1.3}$$

Equation of heat conduction:

$$q_{i,i} = \rho(R - \dot{S}\theta_0) \tag{1.4}$$

Constitutive relations:

$$\sigma_{ij} = c_{ijkl}e_{kl} - \gamma_{ij}\theta \tag{1.5}$$

$$\rho S = \frac{\rho c_e}{\theta_0}\theta + \gamma_{ij}e_{ij} \tag{1.6}$$

$$q_i = -K_{ij}\theta_{,j} \tag{1.7}$$

where R is the strength of the internal heat source, S denotes the entropy, c_{ijkl} is the elasticity tensor, γ_{ij} is the thermoelasticity tensor, K_{ij} is the thermal conductivity tensor and c_e is the specific heat per unit mass, in the isothermal state.

From equations (1.3) and (1.5), we obtain

$$c_{ijkl}u_{k,lj} - \gamma_{ij}\theta_{,j} + \rho F_i = \rho \ddot{u}_i \tag{1.8}$$

Further, from equations (1.6) and (1.7), we obtain heat conduction equation in the following form:

$$K_{ij}\theta_{,ij} + \rho R = \rho c\dot{\theta} + \theta_0\gamma_{ij}\dot{u}_{i,j} \tag{1.9}$$

When the material is considered to be isotropic, then equations (1.5)-(1.7) reduce to

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e_{kk} - \gamma\theta)\delta_{ij} \tag{1.10}$$

$$\rho S = \frac{\rho c}{\theta_0}\theta + \gamma e_{kk} \tag{1.11}$$

$$q_i = -K\theta_{,i} \tag{1.12}$$

We observe that equation (1.12) is identical to the Fourier law of heat conduction. In this case, equations (1.8) and (1.9) take the forms as

$$\mu\nabla^2 u_i + (\lambda + \mu)u_{k,ki} - \gamma\theta_{,i} + \rho F_i = \rho\ddot{u}_i \tag{1.13}$$

$$K\nabla^2\theta + \rho R = \rho c\dot{\theta} + \theta_0\gamma\dot{u}_{k,k} \tag{1.14}$$

Biot's thermoelasticity theory as represented by above system of equations is the first coupled dynamical thermoelasticity theory that describes a broad range of phenomena. It is the generalization of the classical theory of elasticity and of the theory of thermal conductivity. Biot's theory has been considered as an elegant model of thermoelasticity. Several eminent researchers including Boley and Weiner (1960), Chadwick (1960), Nowacki (1962, 1975), Parkus (1976), Nowinski (1978), Dhaliwal and Singh (1980), Chandrasekharaiah (1986) have contributed significantly providing the wide and detailed discussions along with interesting applications and theorems based on it. However, subsequently it has been realized through theoretical as well as experimental research work that although the theory proposed by Biot (1956), removes the drawback of uncoupled theory of thermoelasticity but it suffers from the deficiency of admitting thermal signals propagating with infinite speed. This is considered as a paradox inherent in this theory. In addition to this paradox, this theory also exhibits unsatisfactory description of a solid's response to fast transient heating, like short laser pulses. Due to the shortcomings of this theory in several cases, researchers have put their efforts in recent years to modify the concept of this theory. Basically, this shortcoming arose from the inherent limitation in Fourier law of heat conduction which has been discussed in the next section.

1.3 Drawbacks of Fourier law and its generalization

Fourier law implies that heat flux is the instantaneous result of a temperature gradient established at a point of a body. Mathematically, for isotropic material it is of the form

$$\vec{q}(\vec{r},t) = -K\vec{\nabla}\theta(\vec{r},t) \tag{1.15}$$

where $\vec{\nabla}\theta$ is the temperature gradient vector, $\vec{r}$ is the position vector, t is the time. Correspondingly, the heat conduction equation is given by

$$K\nabla^2\theta = \rho C_e\dot{\theta} - Q \tag{1.16}$$

It has been realized that this law is successfully applicable to the problems that involve large spatial dimension and long time response. However, it is physically unrealistic for the transient behavior of heat conduction, specially at extremely short time, e.g., on the order of a fraction of second (10^{-12}s to 10^{-15}s). In fact, it yields unacceptable results in the situations that involve extreme thermal gradients, high heat flux condition and short time behavior (such as laser-material interactions). Moreover, heat conduction of many nano-scale devices demonstrates several distinct phenomena, which are not captured by the conventional Fourier law. It must be mentioned here that in 1867, Maxwell postulated the occurrence of a wave-type heat flow and suggested that the thermal disturbance is a wave like phenomenon rather than diffusion phenomenon. Accordingly, he suggested the modification of Fourier law for the first time. The wave-type heat flow is now called as 'second sound effect' (see Chandrasekharaiah (1986)). Possibility of 'second sound effect' was also speculated by Nernst (1917), Landau (1941), Tisza (1947). Landau described 'second sound' as the propagation of phonon density disturbance for super-fluid helium and predicted that its speed should be equal to $\frac{v_p}{\sqrt{3}}$ at 0 K temperature. The second sound was first detected in liquid helium by Peshkov (1944) and experimentally, its speed was found to be equal to 19 m/s at 1.4 K. Later on, Tisza's and Landau's predictions were verified experimentally by Maurer and Herlin (1949), Pellam and Scott (1949), and Atkins and Osborne (1950). Lifshitz (1958) observed that in fluid helium second sound occurs at low temperatures. Subsequently, second sound had also been detected by several workers including Ackerman *et al.* (1966), Ackerman and Overton (1969) and Bertman and Standiford (1970), McNelly *et al.* (1970), Jackson *et al.* (1970), Jackson and Walker (1971), Rogers (1971). We refer the review article by Chandrasekharaiah (1986) for details in this respect. Parallel to experimental research work to account for the inadequacy of Fourier law, several theoretical work have also been carried out. Several non-Fourier heat conduction theories have accordingly been established. A brief discussion for some of the well established models is given below.

1.4 Non-Fourier heat conduction models

1.4.1 Cattaneo-Vernotte model

Cattaneo (1958) and Vernotte (1958, 1961) recommended independently a model (CV model) of heat conduction for the first time by including the flux rate term into Fourier law. They proposed the heat conduction law in the following manner:

$$\overrightarrow{q}(\overrightarrow{r},t) + \tau \frac{\partial \overrightarrow{q}(\overrightarrow{r},t)}{\partial t} = -K\overrightarrow{\nabla}\theta(\overrightarrow{r},t) \tag{1.17}$$

Here $\tau \geq 0$ is referred to as thermal relaxation time which is defined as the finite built-up time (phase-lag), for the onset of heat flow at $\overrightarrow{r}$ after a temperature gradient is imposed there. Equation (1.17) yields the following hyperbolic type heat conduction equation:

$$K\nabla^2\theta = (1 + \tau\frac{\partial}{\partial t})(\rho C_e\dot{\theta} - Q) \tag{1.18}$$

Above equation represents the combined diffusion and wave-like behavior of heat transport and predicts a wave-like thermal signal propagating with the finite speed, $\sqrt{\frac{K}{\rho C_e \tau}}$ when $\tau > 0$. This modified heat conduction law is also called as Maxwell-Cattaneo law. It yields successful results in the cases that involve a localized moving heat source with high intensity, a rapidly propagating crack tip, shock wave propagation, thermal resonance, interfacial effects between dissimilar materials, laser material processing, laser surgery which involve short time intervals and high heat fluxes. Mengi and Turhan (1978) carried out experiment where they determined the actual value of τ for a given material and observed that the values of τ for gases range from 10^{-10}s , for metals to 10^{-14}s for liquids and insulators falling within this range. Francis (1972) provided a table of values of τ for some materials. Laser penetration and welding, explosive bonding, melting and nuclear bonding possess the transient process of heat conduction at extremely short time of very high heat flux and this modified law is applicable in such circumstances. The review articles by Chandrasekharaiah (1986, 1998a), Hetnarski and Ignaczak (1999) and the books by Straughan (2011), Ignaczak and Starzewski (2010), Wang *et al.* (2008) etc. report this aspect in details.

1.4.2 Dual phase-lag heat conduction model

It has now been obvious that the advancement of short-pulse laser technology and their huge applications to modern micro-fabrication technology are attracting attention of the researchers towards the issues of high rate heating on thin films (Tzou (1995a)). It has been noticed that laser pulses can be made shorter to the range of femtoseconds (10^{-15}s). As the response time is shorter, the non-equilibrium thermodynamic transition and the microscopic effects in the energy exchange during heat transport procedure become prominent. In view of recent experiments, the heat conduction theory of Cattaneo and Vernotte also fails in some cases, specially during heating of thin films. Hence, to surmount the drawbacks of the classical heat conduction model as well as of the Cattaneo-Vernotte model, Tzou ((1995 a,b) proposed the dual phase-lag (DPL) theory of heat conduction. This model establishes that either the temperature gradient may dominate the heat flux or the heat flux may dominate the temperature gradient. It must be mentioned that the dual phase-lag model of Tzou was motivated by some prior established models. It may be pointed out that in order to capture the microscopic effects in heat transport mechanism, phonon-scattering model was put forward by Joseph and Preziosi (1989, 1990) and Guyer and Krumhansl (1966). The phonon-electron interaction model developed by Brorson *et al.* (1987), Anisimov *et al.* (1974) and Fujimoto *et al.* (1984), microscopic two-step model by Qiu and Tien (1992, 1993). In 1995, Tzou (1995a) has pronounced the effect of micro-structural interactions in the fast transient process of heat transport phenomenon and developed a more generalized and accurate law of heat conduction, known as dual-phase lag model, in the form,

$$\overrightarrow{q}(\overrightarrow{r}, t+\tau_q) = -K\overrightarrow{\nabla}T(\overrightarrow{r}, t+\tau_t) \tag{1.19}$$

Here τ_q, τ_t are two delay times, where τ_q represents the phase-lag of the heat flux vector and it captures the thermal wave behavior, a small-scale response in time for heat flux and τ_t is the phase lag of the temperature gradient and it captures the effect of phonon-electron interactions, a micro scale response in space. Thus, the dual phase lag concept is capable of predicting the small-scale response in both space and time. The phase-lags τ_q and τ_t are assumed to be positive and they are the intrinsic properties of the medium (Tzou (1997)).

Equation (1.19) represents a universal model which is a good explanation of all fundamental behaviors in diffusion, thermal wave, phonon-electron scattering associated with the shortening of the response time. Many workers such as Tzou (1997), Al-Nimr and Al-Huniti (2000), Chen *et al.* (2002), Lee and Tsai (2008) and Abdallah (2009), Kothari (2013), Quintanilla (2002) carried out various applications of this model. Some of its silent features are described below:

(a) Using Taylor series expansion of (1.19) by retaining only the terms up to the first order for τ_q and τ_t, we obtain the Jeffery-type heat flux equation (Joseph and Preziosi (1989, 1990))

$$\overrightarrow{q} + \tau_q\frac{\partial\overrightarrow{q}}{\partial t} = -K[\overrightarrow{\nabla}\theta + \tau_t\frac{\partial\overrightarrow{\nabla}\theta}{\partial t}] \tag{1.20}$$

The corresponding heat conduction equation is

$$(1+\tau_t\frac{\partial}{\partial t})\nabla^2\theta = \frac{\rho C_e}{K}(1+\tau_q\frac{\partial}{\partial t})\frac{\partial\theta}{\partial t} - \frac{1}{K}(1+\tau_q\frac{\partial}{\partial t})\frac{\partial Q}{\partial t} \quad (1.21)$$

(b) The heat conduction model (1.19) reduces to

(i) Classical Fourier law of heat conduction equation when $\tau_q = \tau_t = 0$.

(ii) Hyperbolic heat conduction when $\tau_t = 0$ and $\tau_q > 0$.

(iii) the energy equation in phonon-scattering model (Joseph and Preziosi (1989), Guyer and Krumhansl (1966)) when $\frac{K}{\rho C_e} = \frac{\tau_R c^2}{3}$, $\tau_t = \frac{9\tau_N}{5}$ and $\tau_q = \tau_R$,

where τ_R is the relaxation time for Umklapp process in which momentum is lost from the phonon system and τ_N denotes the relaxation time for normal processes in which momentum is conserved for the phonon system (Tzou (1995a)).

(c) Now applying Taylor series expansion of (1.19) by retaining terms up to the second order in τ_q but only up to the first order in τ_t (Tzou (1995b)), we obtain

$$\overrightarrow{q} + \tau_q\frac{\partial\overrightarrow{q}}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2\overrightarrow{q}}{\partial t^2} = -K[\overrightarrow{\nabla}\theta + \tau_t\frac{\partial\overrightarrow{\nabla}\theta}{\partial t}] \quad (1.22)$$

Corresponding heat conduction equation is

$$(1+\tau_t\frac{\partial}{\partial t})\nabla^2\theta = \frac{\rho C_e}{K}(1+\tau_q\frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2})\frac{\partial\theta}{\partial t} + \frac{1}{K}(1+\tau_q\frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2})\frac{\partial Q}{\partial t} \quad (1.23)$$

Equation (1.23) is hyperbolic type heat conduction equation admitting the thermal wave to propagate with a finite speed, $V_T = \frac{1}{\tau_q}\sqrt{\frac{2K\tau_t}{\rho C_e}}$ (Tzou (1995b)).

1.5 Generalized thermoelasticity theories

Parallel research activities have been carried out in the field of thermoelasticity for providing the major growth of the area of thermoelasticity accounting for non-Fourier heat conduction in elastic materials. Accordingly, several models have been proposed which are capable of removing the drawbacks of Fourier law of heat conduction. Such theories are called as generalized theory of thermoelasticity or hyperbolic thermoelasticity. A brief description of such models are given below:

1.5.1 Extended thermoelasticity theory (ETE)

Lord and Shulman (1967) proposed one generalized thermoelastic model which includes one thermal relaxation parameter for isotropic and thermoelastic body. In this model of thermoelasticity, the flux rate term was incorporated into the Fourier law of heat conduction. Basically this theory is based on Cattaneo-Vernotte law (1.17) and as a result the heat conduction equation in this theory exhibits wave-type heat phenomenon i.e. the propagation speed for both elastic and thermal waves is finite. This thermoelasticity theory is renamed as 'Extended thermoelasticity theory (ETE)' or 'Lord and Shulman theory (LS theory)'. The present theory is considered as the first generalization to the coupled thermoelasticity theory. This theory was subsequently extended by Dhaliwal and Sherief (1980) to general anisotropic media in the presence of heat sources.

1.5.2 Temperature-rate-dependent thermoelasticity theory (TRDTE)

The second generalization to the coupled thermoelasticity is known as the theory of thermoelasticity with two relaxation times or the theory of temperature-rate dependent thermoelasticity that admits the "second sound effect". This theory is advocated by Green and Lindsay (1972). It must be recalled that prior to this, Muller (1971), in a review of thermodynamics of thermoelastic solids, proposed an entropy production inequality with some restrictions on a class of constitutive equations. A generalization to this inequality was proposed by Green and Laws (1972). Subsequently, Green and Lindsay (1972) obtained an explicit version of the constitutive equations. These equations were also obtained independently by Suhubi (1975). This theory contains two non negative constants that act as thermal relaxation times or temperature-rate and modify the constitutive relations of the coupled thermoelasticity theory. However, the classical Fourier law of heat conduction is not violated in this theory if the medium under consideration has a center of symmetry. Temperature rate dependent thermoelasticity theory (TRDTE) has also been investigated by researchers like ETE. The review/survey articles by Chandrasekharaiah (1986, 1998) and Hetnarski and Ignaczak (1999) provide detailed discussion about ETE and TRDTE generalized thermoelasticity theories.

1.5.3 Theory of thermoelasticity of type I, II and III (Green and Naghdi)

Next generalization to the coupled thermoelasticity has been made by Green and Naghdi (1991, 1992, 1993, 1995) who have introduced their theory as an alternative one. The propagation of heat has been modeled in a very elegant way to produce a fully consistent theory of thermoelasticity. This theory is capable of organizing the thermal wave transmission in a reasonable manner and is based on the firm ground of thermodynamic principles. Moreover, to account for the finite speed for thermal wave, Green and Naghdi (1993) speculated a new concept in generalized thermoelasticity which is known as the thermoelasticity with no energy dissipation. The paramount characteristic of this theory is that it is completely in contrast to the classical thermoelasticity associated with Fourier law of heat conduction. Furthermore, the potential function which is used to derive the stress tensor is used to determine the constitutive equation for the entropy flux vector. Basically, Green-Naghdi (GN) theory depends on entropy balance law rather than the usual entropy inequality. The theory proposed by Green and Naghdi (1991, 1992, 1993, 1995) has been categorized into three parts which have been labeled as thermoelasticity of type-I, II and III. Linearized version of type-I theory is similar to the classical theory of thermoelasticity predicting an infinite speed of thermal wave propagation. Type-II model describes the finite speed of heat propagation wave as a special case of type III, i.e. in the heat equation of type III, the heat flux is the combination of type-I and type-II theories. For isotropic medium, the heat conduction equation in the theories proposed by Green and Naghdi can be expressed in the following way:

Green and Naghdi type-I (GN-I) theory of thermoelasticity:

$$K\nabla^2\dot{\theta} = \rho C_v\ddot{\theta} + \gamma T_0\ddot{u}_{i,i} \tag{1.24}$$

Green and Naghdi type-II (GN-II) theory of thermoelasticity:

$$K^\star\nabla^2\theta = \rho C_v\ddot{\theta} + \gamma T_0\ddot{u}_{i,i} \tag{1.25}$$

Green and Naghdi type-III (GN-III) theory of thermoelasticity:

$$K^\star\nabla^2\theta + K\nabla^2\dot{\theta} = \rho C_v\ddot{\theta} + \gamma T_0\ddot{u}_{i,i} \tag{1.26}$$

1.5.4 Thermoelasticity with dual phase-lags (DPLTE)

This theory of thermoelasticity has been developed in the frame of ETE theory by introducing the dual phase-lag heat conduction law in place of Fourier law. This theory has been introduced by Tzou (1997) and subsequently, it has been formulated and discussed by Chandrasekharaiah (1998a).

1.5.5 Thermoelasticity with three phase lags (TPLTE)

Roychoudhuri (2007a) proposed a model of thermoelasticity in which Fourier law of heat conduction is modified by introducing three different phase-lags for the heat flux vector, the temperature gradient and the thermal displacement gradient vectors. Hence, this model is known as the three-phase lag thermoelasticity theory (TPLTE). In the present model, the generalized heat conduction model is proposed as

$$\vec{q}(\vec{r}, t+\tau_q) = -[K\vec{\nabla}T(\vec{r}, t+\tau_t) + K^{\star}\vec{\nabla}\nu(\vec{r}, t+\tau_\nu)] \tag{1.27}$$

Here τ_ν denotes the phase-lag in thermal displacement gradient and $\vec{\nabla}\nu$ is the gradient of thermal displacement such that $\dot{\nu} = T$. Therefore, TPLTE is considered as the generalization of GN-III thermoelasticity theory.

1.5.6 Fractional order thermoelasticity

In recent years, fractional calculus is playing a crucial role in developing several models and it has been verified that the use of fractional order derivatives/integrals lead to the formulation of certain physical problems which are more economical and appropriate as compared to the classical one. Fractional calculus is a natural extension of classical mathematics and since the inception of the theory of differential and integral calculus, mathematicians such as Euler and Liouville developed their ideas about the calculation of non-integer order derivatives and integrals. The subject is therefore more aptly called as “integration and differentiation of arbitrary order. Nowadays, it has been realized that fractional calculus can be very useful in the areas of diffusion, heat conduction, viscoelasticity, continuum mechanics, electromagnetism etc. It must be mentioned here that the first use of a fractional operation is due to Abel (1823) and Liouville (1832). Abel applied fractional calculus in the solution of an integral equation that arises in the formulation of tautochrone problem. After him, Liouville made the first major study of fractional calculus. Caputo (1967), Caputo and Mainardi (1971a 1971b) and Caputo (1974) employed the fractional order derivatives for the purpose of analyzing the elastic-energy dissipation based on a memory mechanism in viscoelastic materials, established the connection between fractional derivatives and the theory of linear viscoelasticity and achieved a wider agreement with experimental results (Oldham and Spanier (1974)). One can see various alternative definitions of fractional derivatives in the articles by Oldham and Spanier (1974), Samko *et al.* (1993), Miller and Ross (1993), Gorenflo and Mainardi (1997) and Hilfer (2000) etc. Caputo and Mainardi (1971), Caputo (1974), Bagley and Torvik (1983), Koeller (1984) and Rossikhin and Shitikova (1997). The book by Podlubny (1999) can be reffered for a survey of applications of fractional calculus. At present, fractional calculus has been employed in the area of thermoelasticity theory. Povstenko (2005) has developed a quasi-static uncoupled thermoelastic model based on the heat conduction equation with fractional order time derivatives. He used the Caputo fractional derivative (Caputo (1967)) and obtained the stress components corresponding to the fundamental solution of a Cauchy problem for the fractional order heat conduction equation in both the one-dimensional and two-dimensional cases. In 2010, a new theory of thermoelasticity in the frame of a new consideration of the heat conduction equation with fractional order time derivatives has been proposed by Youssef (2010). The uniqueness of the solution has been established (see also Youssef (2012)). Youssef and Al-Lehaibi (2010) have investigated a problem on an elastic half-space using this theory. The uniqueness of the solution has been proved in the same work. Sherief *et al.* (2010) has independently constructed another model in generalized thermoelasticity by using fractional order time-derivatives in which the fractional order derivative is introduced in Fourier law of heat conduction a

$$\overrightarrow{q} + \tau_0 \frac{\partial^\alpha \overrightarrow{q}}{\partial t^\alpha} = -K\theta_{,i} \tag{1.28}$$

Here τ_0 is the thermal relaxation parameter andα is a real constant such that $0 < \alpha \leq 1$.

It has been considered here

$$\frac{\partial^\alpha}{\partial t^\alpha} f(x,t) = f(x,t) - f(x,0); \quad \alpha \to 0.$$

$$\frac{\partial^\alpha}{\partial t^\alpha} f(x,t) = I^{\alpha-1} \frac{\partial f(x,t)}{\partial t}; \quad 0 < \alpha < 1.$$

$$\frac{\partial^\alpha}{\partial t^\alpha} f(x,t) = \frac{\partial f(x,t)}{\partial t}; \quad \alpha \to 1.$$

Above cases represent the limiting cases of heat conduction equation (1.28). This theory is derived by using the Caputo definition of fractional derivatives of order α that belongs to (0,1] of the absolutely continuous function $f(t)$ and is given by $\frac{d^\alpha}{dt^\alpha} f(x,t) = I^{\alpha-1} f'(t)$, where I^β is the fractional integral of the function $f(t)$ of order β defined by Miller and Ross (1993) as

$I^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds.$

Here, $f(t)$ is a Lebesgue integrable function and $\beta > 0$.

Youssef (2010) constructed his model of thermoelasticity by considering the heat conduction law with fractional order as follows:

$$q_i + \tau_0 \dot{q}_i = -K I^{\alpha-1} \theta_{,i} \tag{1.29}$$

where $\alpha \epsilon (0, 2]$.

Subsequently, Ezzat and El-Karamany (2011), Ezzat (2012), Ezzat *et al.* (2012) derive fractional dual and three-phase lag thermoelasticity model by using fractional Taylor series. Ezzat and Fayik (2011) introduced the theory of fractional order theory of thermoelastic diffusion. Later on Yu *et al.* (2013a, 2013b) investigated the unified fractional order generalized thermoelasticity with electro-magnetic effect and micro-modeling by introducing fractional calculus into LS, GL, GN and DPL models. Abbas (2014) studied fractional thermoelasticity theory based on Green-Naghdi model.

1.5.7 Thermoelasticity using memory dependent derivatives

This is a new concept and it has come after some generalization of fractional order derivative theory. Diethelm (2010) analyzed fractional differential equations by applying the concept of Caputo fractional derivative (1967) defined as

$$D_a^{\alpha} f(t) = \int_a^t K_{\alpha}(t-\xi) f^m(\xi) d\xi \tag{1.30}$$

where $K_{\alpha}(t-\xi)$ is denoted as kernel of function and it is defined as $K_{\alpha}(t-\xi) = \frac{(t-\xi)^{m-\alpha-1}}{\Gamma(m-\alpha)}$.

In above equation, the kernel $K_{\alpha}(t-\xi)$ is fixed, where a is a fixed integer and m is an integer such that $m-1 < \alpha < m$.

From the above definition it is clear that $\alpha-$order fractional derivative at time t is not defined locally at time t, but it depends on the total effects of $m-$order integer derivative on the interval $[a, t]$. Hence, this concept of fractional derivative can be used to describe the variation of a system in which the instantaneous change rate depends on the past state which is known as 'memory effect' (Diethelm (2010)).

However, we know that the memory effect of real process basically arises in a segment of time $[t-\tau, t]$, where τ denotes the time delay and it is always positive. In spite of several applications of fractional calculus, it has some demerits. Due to this, the concept of fractional order derivative has been modified and a new concept of derivative has been established by Wang and Li (2011) which has been named as 'memory dependent derivative' and can be written mathematically as

$$D^{m}_{\tau,K}f(t) = \frac{1}{\tau}\int\limits_{t-\tau}^{t} K(t-s)f^{m}(s)ds \tag{1.31}$$

This concept has some specific properties which is not applicable in fractional order derivative. Here the kernel $K(t-s)$ and time delay parameter, τ can be chosen freely as per the necessity of problem. For physical point of view, generally we take $0 < K(t-s) \leq 1$ and τ should be smaller than an upper bound determined by the kernel function to ensure the uniqueness and existence of the solution (Wang and Li (2011)). Several examples like, weather forecast, population model etc. need the data of recent past and this is possible in memory dependent derivative because it will fail if the lower terminal has a very less value in fractional order derivative. In recent years, the concept of memory dependent derivatives has drawn attention of the researchers. Yu *et al*. (2014) introduced memory-dependent derivative into the generalized theory of thermoelasticity provided by Lord and Shulman (1967). Furthermore, Ezzat *et al.*, (2015) developed a new generalized thermo-viscoelasticity theory using memory-dependent derivatives.

1.6 Magneto-thermoelasticity

An elastic body, when it is placed in a primary uniform magnetic field, is subjected to non-uniform temperature distribution, will give rise to the interactions among the elastic, magnetic and thermal fields. The theory of coupled magneto-thermoelasticity deals with the mutual interactions among strain, temperature and electromagnetic fields. Electric current gives rise to magnetic field and vice-versa. The combined effect is therefore sometimes called as thermo-magneto-electro-elasticity. The governing equations in this theory includes Maxwell equations with modified

Ohm's law which contains temperature gradient term and also modified Hooke-Duhamel-Newman law of stress-strain-temperature relations. The stress equation of motion also includes an additional body force term, called as Lorentz ponderomotive force and the total stress in the combined field consists of the Hooke's mechanical stress due to thermoelastic deformation and the Maxwell's electro-magnetic stress. Efforts are being devoted to understand the interaction between magnetic field in a thermoelastic solid due to its useful applications in the field of geophysics, optics, acoustics, damping of acoustic wave in magnetic fields and other related topics involving sensing and actuation. At the early stage, the field of magneto-elasticity was addressed notably by Knopoff (1955), Kaliski (1963), Nowacki (1975) etc. Kaliski (1965) provided a symmetric treatment of the magneto-thermo-elastic field equations and those of wave equations of thermo-electro-magnetoelasticity. The dynamic problems concerning such interactions were studied by several workers including Paria (1962), Wilson (1963), Purushotama (1965), Massalas and Dalamangas (1983a,b), Chatterjee and Roychoudhuri (1985), Roychoudhuri and Chatterjee (1990b), Saxena, Dhaliwal and Rokne (1991). Purushotama (1965) have used the classical theory of thermoelasticity (Biot (1956)) along with the electromagnetic theory to characterize harmonically time dependent plane waves of assigned frequency in a homogeneous, isotropic and unbounded solid. Furthermore, Nayfeh and Nemat-Nesser (1972) and later on, Agarwal (1979) reported a detailed study on electro-magneto-thermoelastic plane waves in solids in the context of generalized thermoelasticity theory with the effects of thermal relaxation parameters. Roychoudhuri (1984) and Roychoudhuri and Debnath (1983) studied propagation of magneto-thermoelastic plane waves in rotating thermoelastic media permeated by a primary uniform magnetic field using generalized heat conduction equation of Lord and Shulman (1967). A systematic presentation of the subject of magneto-thermoelasticity is available in the books by Parkus (1972, 1979), Moon (1984), Dhaliwal and Singh (1980), Eringen and Maugin (1990).

1.7 Literature review

Theory of thermoelasticity has aroused much interest among researchers in last few decades due to its wide applications in science, engineering and technology. This theory has been applied by several researchers to solve various problems relevant to the thermoelastic interactions in different types of media. Chandrasekharaiah (1986, 1998a) mentioned extensive research work that has been carried out on the various theories of thermoelasticity till 1998 in his two review articles. The recent books by Hetnarski and Eslami (2010) and Ignaczak and Ostoja-Starzewski (2010) as well as the Ph.D. thesis of Roushan Kumar (2010), Rajesh Prasad (2012) and Sweta Kothari (2014) may be referred in this regard. We state below some of the important works in the relevance of the present study:

Danilovskaya (1950) was the first one who studied a half space problem under the theory of classical thermoelasticity by neglecting the coupling term. This problem has been named as Danilovskaya's problem. Several authors like, Paria (1959), Hetnarski (1961, 1964a) and Boley and Tolins (1962) investigated half space problems under different conditions. After that Hetnarski (1964b) achieved the fundamental solutions under classical thermoelasticity (CTE) theory for small values of time.

It has been already mentioned above that the theory of classical thermoelasticity is not capable of providing accurate results for physical point of view. Several efforts have therefore been made to remove this paradox of heat conduction. Several authors such as Achenbach (1968), Lord and Lopez (1970), Ramamurthy (1978, 1979), Popov (1967), Norwood and Warren (1969), Sherief and Dhaliwal (1981), Chattopadhyay et al. (1982), Sharma (1987a), Dhaliwal and Rokne (1988), Bioko (1986), Bykovtsev and Shatalov (1987), Anwar and Sherief (1988), Anwar (1991), Balla (1991), Ramamurthy and Sharma (1991) investigated half space problems having different types of boundary conditions in the context of Lord and Shulman (LS) theory of thermoelasticity. Wadhawan (1972, 1973), Sharma (1987b), Furukawa *et al.* (1990), Sharma and Chand (1991, 1996), Roychoudhuri and Bhatta (1983) have investigated various other problems concerning thermoelastic interactions in elastic medium under LS theory. Distinct problems on viscoelastic medium based on LS- theory have been investigated by Misra *et al.* (1987), Mukhopadhyay and Bera (1989), Mukhopadhyay *et al.* (1991) and Banerjee and Roychoudhuri (1995). The boundary initiated axisymmetric waves in an annular cylinder under different boundary conditions have been discussed by Sherief and Anwar (1988, 1989). Sherief (1986) has obtained the fundamental solutions for spherically symmetric space. Chattopadhyay *et al.* (1985) studied a problem of an infinite allotropic medium having a cylindrical hole. Problems of thermoelastic interactions due to heat sources in an unbounded elastic medium have been investigated by many researchers including Roychoudhuri and Bhatta (1981), Roychoudhuri and Sain (1982), Sherief and Anwar (1986), Sharma (1986), Mishra *et al.* (1987), Chandrasekharaiah (1988), Sherief and Anwar (1992), Das *et al.* (1997) and Chakravorty and Chakravorty (1998). El-Maghraby (2005), Kulkarni and Deshmukh (2008) and Deshmukh *et al.* (2009) have reported their work concerning thermoelastic interactions in thick plate subjected to different thermoelastic loading.

Thermoelasticity theory proposed by Green and Naghdi is very famous among the researchers due to its efficiency for providing accurate results of problems. This theory has been originated by Green and Naghdi (1991, 1992, 1993, 1995). Li and Dhaliwal (1996) studied Danilovskaya problem under different conditions depending on GN-III theory of thermoelasticity. Chandrasekharaiah (1996a) reported one-dimensional wave propagation in an elastic medium under Green and Naghdi-II (GN-II) theory of thermoelasticity. After that Chandrasekharaiah (1996b) investigated free plane harmonic wave in an unbounded medium in context of GN-II theory of thermoelasticity. Further Chandsekharaiah and Srinath (1997a) extended the same problem for rotating body. After that Chandrasekharaiah and Srinath (1997b, 1998a, 1998b) investigated a problem of cylindrical and spherical cavity in an unbounded medium subjected to some loads on boundary and due to heat source in an unbounded medium under GN-II theory of thermoelasticity. Chandrasekharaiah (1998b) proposed a Biot type variational and reciprocal relation under the linear theory of GN-II. Mishra *et al.* (2000) investigated a half space problem of thermoelastic wave propagation in the context of GN theory. Some qualitative research based on Green and Naghdi theory have been published by Quintanilla (2001a), Quintanilla (2001b), Quintanilla (2003a) and Quintanilla and Straughan (2004). In the context of GN-II model, Roychoudhuri and Datta (2005) investigated thermoelastic interactions in an isotropic, homogeneous thermoelastic solid containing periodically varying time dependent distributed heat source. Mukhopadhyay (2002) studied a thermal shock related problem with spherical cavity in an unbounded medium under GN-II model. Puri and Jordan (2004) presented a detailed study on harmonic plane wave propagation in thermoelastic medium in the context of GN-III theory of thermoelasticity. Several problems under GN-III model have also been investigated by Taheri *et al.* (2005), Malik and Kanoria (2006, 2007a), Roychoudhari and Bandyopadhyay (2007), Banik et al. (2007), Kar and Kanoria (2007b). Quintanilla (2007) studied a problem related to the Green and Naghdi theory of thermoelasticity. Mallik and Kanoria (2008) reported two dimensional problem of transversely isotropic problem based on GN-II and GN-III models. Later on Kovalev and Radayev (2010) investigated plane harmonic waves under GN-III model.

Mukhopadhyay and Kumar (2008a, 2008b) studied problems in context of GN-III model. Further Chirita and Ciarletta (2010) and Mukhopadhyay and Prasad (2011) studied the convolution type variational and reciprocity theorems in the context of linear theory of GN-II and GN-III. Prasad *et al.* (2013) studied a two dimensional crack problem in GN-III thermoelastic medium. Further Swantje and Bargmann (2013) presented some remarks on Green and Naghdi theory of heat conduction. Recently Othman and Sarhan (2015) investigated a two dimensional problem of generalized thermo-microstretch elastic solid under GN theory. Further Ahmed El-Karamany and Ezzat (2016) reported phase lag GN thermoelasticity theory.

The theory of magneto-thermoelasticity is highly useful in understanding the effect of earth's magnetic field on seismic waves, emissions of electromagnetic radiations from nuclear devices, development of highly sensitive superconducting magnetometer and electrical power engineering. The phenomenon of wave reflection and refraction is a elemental topic in various fields e.g., seismology, physics, geophysics, earthquake engineering, non-destructive evaluation, etc. Jeffreys (1930) and Gutenberg (1944) assumed the reflection of elastic plane waves at a solid half space. The reflection of plane waves from a plane stress-free boundary has been investigated by Deresiewicz (1960) in the coupled theory of thermoelasticity. Knott (1899) studied the general equations for reflection and refraction of waves at plane boundaries. Wu *et al.* (1990) studied the reflection and transmission of elastic waves from the boundary of a fluid-saturated porous solid. Lin *et al.* (2005) investigated the reflection of plane waves in a poroelastic half-space saturated with inviscid fluid. Knopoff (1955), Chadwick (1957), Kaliski (1963) and Nowacki (1975) have given a significant contribution in the field of magneto-thermoelasticity. Kaliski (1965c) developed the wave equations of thermo-electro-magneto-elasticity. Purushothama (1965), Massalas and Dalamangas (1983a, 1983b), Chatterjee (Roy) and Roychoudhuri (1985) studied several dynamic problems concening such interactions. It should be indicated here that Parkus (1972, 1979), Eringen and Maugin (1990) and Moon (1984) provided a systematic representation of the subject of magneto thermoelasticity in their books. Propagation of electromagnetic waves in thermoelastic materials is a very famous topic among researchers. A number of articles have been published regarding the propagation of electromagnetic plane waves. Paria (1962) and Wilson (1963) applied the heat conduction equation predicted by Fourier law along with the electromagnetic theory to explain harmonically time dependent plane waves of assigned frequency in a homogeneous, isotropic and unbounded solid. Neyfeh and Nemat-Nessar (1972) and Agrawal (1979) studied electromagnetic plane waves in solids in the context of generalized thermoelasticity theory. Roychoudhuri and Debnath (1983) and Roychoudhuri (1984) investigated propagation of magneto-thermoelastic plane waves in rotating thermoelastic media permeated by a primary uniform magnetic field using generalized thermoelasticity of Lord and Shulman. Roychoudhuri and Banerjee (2005) reported magneto-elastic plane waves in rotating media under thermoelasticity of type-II model. Das and Kanoria (2009) also studied magneto-thermo-elastic waves in a medium with perfect conductivity in the context of Green and Naghdi-III theory. Yu *et al.* (2013) studied fractional order generalized electro-magneto thermoelasticity. Atwa and Sarhan (2011) have made a significant contribution in this area. Reccently, Abd-Alla *et al.* (2016) studied a problem of wave propagation for half space. The theory of fractional order thermoelasticity is also investigated by several researchers. Further, Povstenko (2008-2011) reported some problems under fractional order thermoelasticity in the framework of quasi-static uncoupled theory of thermoelasticity and discussed the effects of fractional order parameter. Youssef (2010) has proved a uniqueness and a reciprocity theorem as well as a variational principle. Furthermore, Youssef and Al-Lehaibi (2010, 2011), Youssef (2012) studied some problems on thermoelastic interactions in the context of Youssef (2010) model. Kothari and Mukhopadhyay (2011), Abouelregal (2011), Sarkar and Lahiri (2012), Ezzat et al. (2013), Deswal and Kalkal (2013), Sheoran et al. (2016) also worked in the field of fractional order thermoelasticity. Povstenko (2012a,b) developed theories of thermal stresses for the generalized telegraph equations considering time and space fractional derivatives. Youssef (2013) studied fractional order two temperature generalized thermoelasticity subjected to moving heat source. Later on, Youssef and Abbas (2014) studied fractional order generalized thermoelasticity with variable thermal conductivity. Recently, several researchers such as Sherief and Abd-El Latief (2013), Ezzat and El-Bary (2016), Abbas (2016) solved dynamic problems using the approach of fractional order thermoelasticity. A new theory of generalized micropolar thermoelasticity with two temperatures using fractional calculus has been derived by Shaw and Mukhopadhyay (2011). Ezzat *et al.* (2012) formulated the field equations of three-phase lag heat conduction model of linear theory of thermoelasticity with time-fractional order derivatives and proved uniqueness and reciprocity theorems. Hamza *et al.* (2014) established the theory of thermoelasticity associated with two relaxation times using the methodology of fractional calculus and derived uniqueness and reciprocity theorems. Wang et al. (2015) constructed fractional order theory of generalized thermoelasticity for elastic media with variable properties.

1.8 Objective of the present thesis

Thermoelasticity theory has become very relevant in various fields of engineering sciences. It comprises of the heat conduction and corresponding stress and strain arising out due to the flow of heat. Various non-Fourier heat conduction models have been developed with the purpose to eliminate the inherent drawback in classical coupled thermoelasticity theory.

The main objective of the present thesis is to discuss some important aspects of thermoelasticity and magneto-thermoelatsicity by employing the recently developed non-Fourier heat conduction models. We concentrate on various problems of real homogeneous continuous medium in which deformation occurs within the range of elasticity. Since the deformation as well as temperature increase is very small, the linearized theories connecting temperature and strain have been considered. In the generalized theory of magneto-thermoelasticity, a large number of unknowns are involved and due to that it becomes very difficult to solve even simple one or two dimensional problems. In order to achieve solutions, certain assumptions are considered to linearize the governing equations. Solutions of some physical problems associated with the thermoelastic and magneto-thermoelastic interactions based on these theories are derived. We pay attention to Green-Naghdi theory, dual phase-lag theory, fractional order theory and memory dependent derivative theory of thermoelasticity and investigate the effects of employing these models in various cases of mutual interactions. Detailed aspects on the propagation of harmonic plane wave are discussed by implementing various models. The results predicted by concerned models are analyzed and are compared with the corresponding results of other models of thermoelasticity. Significant points have been highlighted to display the characteristic features of the models.