CHAPTER 5

ERROR ESTIMATION

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Numerical methods provide approximated solution to the problem. The numerical solutions are affected due to errors coming from different sources such as discretization error, modeling error, etc. The minimization of the computational cost and obtaining high accuracy of global solution are achieved by adaptive meshing. In the previous chapter 3, guided wave simulations are computed by proposed wavelet based multiscale method. Subsequently, in chapter 4, nonstandard wavelet based finite element scheme is developed to provide computationally efficient tool for the analysis of linear and nonlinear wave propagation problems. We used wavelets as an error indicator on the basis of available literature and used it in chapter 3 and 4 for adaptive meshing. To further investigate the capability of wavelets as a natural error indicator and as an error estimator for some complex problems, we tested wavelet based a posteriori error estimation technique in the chapter.

This chapter presents a method to estimate local as well as global errors by using wavelet based error estimation technique which is motivated by hierarchical error estimation scheme. The finite element solution of engineering problems is transformed into the multilevel decomposition of wavelet space. The error estimations in the present setting do not have any problem due to complex domain of engineering structures. The proposed method is an efficient technique which can be applied in some small region as well as complete domain.

5.1 Introduction

There is no analytic theory for mesh refinement for most of the boundary value problems (BVP). Therefore, the adaptive mesh cannot be generated to improve the solution on the basis of any prior information. *A posteriori* error estimate is a key ingredient for improving the accuracy of the solution in the successive iteration. The adaptive mesh generation process depends on the successful application of *a posteriori* error estimation techniques which is not only a tool for grid optimization but also provide the estimate of the error in the current numerical solution. There exist a vast literature on various error estimation techniques (Babuska and Rheinboldt (1981), Verfürth (1994), Bornemann *et al.* (1996), Ainsworth and Oden (1997), Mohite and Upadhyay (2002), Erath *et al.* (2009), Gratsch and Bathe (2005), Kim (2012), Lina and Ye (2013), Moore (1999), Rabizadeh *et al.* (2016), Guo and Zhong (2017)) which are recently developed. Most of the estimators use local quantities such as jump of derivatives across the interface between two adjacent elements. The residuals due to local quantities are used for estimating upper and lower bounds of the global error in terms of energy norm. The quality of estimated error by using canonical error estimation techniques depends on the initial mesh which is often overestimated.

The hierarchical approach, which is a multiscale (Yserentant (1986), Bank *et al.* (1988)) expansion of corresponding FEM, is often used for *a posteriori* error estimate due to its simplicity and effectiveness (Bank and Smith (1993), Bank (2004)). The hierarchical basis functions span the space of traditional basis functions of finite element method, and it has desirable properties for fast iterative solvers. *A posteriori* error in the hierarchical approach is represented as a convergent sequence of errors of the Hilbert space. In other words, unlike the other classical error estimators which use total error within an element as an indicator, the higher level component is used as an indicator in this scheme. It is noted that such *a posteriori* error estimators are useful in identifying local as well as global error.

Sweldens (1998) defined hierarchical basis functions as "lazy wavelets". The hierarchical scheme has been applied to various types of partial differential equations but wavelet based techniques are not used frequently. Classical wavelets have many desirable properties such as orthogonality and vanishing moments which do not exist in hierarchical basis. Bertoluzza (1995) used wavelet based error estimator for wavelet Galerkin method. A good mathematical theory for error estimation and adaptive solution of PDEs using wavelet basis are developed (Cohen *et al.* (2001), Dahlke *et al.* (1997)). Implementation of these wavelet based methods are difficult on the wide range of complex boundary value engineering problems. Sudarshan *et al.* (2006) have used wavelet based *a posteriori* error estimation for engineering structures but this method requires customized wavelets. It is very difficult to develop customized wavelets, which should be as efficient as FEM for various types of partial differential equations.

The present work is based on the hierarchical error estimation technique proposed by Bank (1993). It combines the strength of wavelet transform and hierarchical error estimation technique and finds the error in finite element solution. In this method, the solution of finite element method is transformed into the multiresolution wavelet space using linear transformation. For a single variable system, the detail coefficients can be used as a local error indicator but for multiple variable problems, the coupled detail coefficients of multiresolution representation will not be a good error indicator; therefore, some functional of detail coefficients is to be used. The technique can also be applied to finite volume, finite difference and boundary element methods.

5.2 Wavelet Based Error Estimation

Consider a partial differential equation $Lu = f$ in Ω and boundary conditions $Bu = g$. The problem is to find exact solution $u \in \mathcal{H}$. The variational form of the equation can be expressed as

$$a(u,v) = f(v) \tag{5.1}$$

for all $v \in \mathcal{H}$, where $\mathcal{H}$ is an appropriate Sobolev space. $a(.,.)$ is a positive definite bilinear form, and $f(.)$ is a linear functional. The energy norm associated with $a(.,.)$ can be defined as:

$$\|u\|^2 = a(u,u) \tag{5.2}$$

FEM is the best available option to solve Eq. 5.1 and the current solution is assumed as the highest resolution for the wavelet space $u_j \in V_j \subset \mathcal{H}$. This approximate solution process introduces numerical error

$$e = u - u_j \tag{5.3}$$

It will be a highly inefficient process to improve the accuracy of the solution of large and complex BVP containing some singularities using a uniform refinement of the grid in the whole domain. Therefore, *a posteriori* error technique should be used to estimate the local and global error in the current solution. This is essential to form a strategy of adaptive mesh for improvement of solution in the successive iteration. Obtaining exact error e is as difficult as finding exact solution. Thus, various indicators for *a posteriori* error estimations are developed. The problems with the most of the indicators are the incorrect prediction of error on coarse meshes and high computing cost. The wavelet based error indicator uses some functional of wavelet coefficients.

A brief review of wavelets will be useful for this error estimation technique. The concept of multiscale analysis is to interpolate an unknown field at a coarse level with the help of so called scaling functions. Any improvement to the initial approximation is achieved by adding 'details' provided by new functions known as wavelets. A multiscale analysis uses multiresolution properties of wavelets (Mallet (1998)). Each subspace V_j is spanned by a set of scaling function $\left\{\phi_{j,k}(x), \forall k \in Z\right\}$. The complement of V_j in V_{j+1} is defined as subspace W_j such that $V_{j+1} = V_j \oplus W_j$ for all $j \in Z$. The space V_{j+1} can be decomposed in a consecutive manner as:

$$V_{j+1} = V_0 \oplus W_0 \oplus W_1 \oplus W_2 ... \oplus W_j \tag{5.4}$$

The basis functions in W_j are called wavelet functions and are denoted by $\psi_{j,k}$. Let $V_j \subset L^2(\Omega)$ be a finite dimensional subspaces, and consider the approximate solution $u_j \in V_j$ such that $a(u_j, v) = f(v)$ for all $v \in V_j$. Its solution holds the approximation property $\left\| u - u_j \right\| = \inf_{v \in V_j} \left\| u - v \right\|$. Similarly, for $V_j \subset V_{j+1} \subset L^2(\Omega)$ Galerkin solution $u_{j+1} \in V_{j+1}$ will satisfy $a(u_{j+1}, v) = f(v)$ for all $v \in V_{j+1}$. Here $\left\| u - u_j \right\| \to 0$, as $j \to \infty$.

Let us assume that the approximate solutions u_{j+1} is closer to u than u_j. This is defined in terms of the saturation assumption

$$\|u - u_{j+1}\| \le \xi \|u - u_j\| \tag{5.5}$$

where $\xi < 1$. The characteristic length of the element $h \to 0$ as $j \to \infty$. For higher degree of approximation in the space V_{j+1}, Bank and Smith (1993) estimated $\xi = O(h^r)$ for some $r > 0$.

For $v_{j+1} \in V_{j+1}$ the unique decomposition $v_{j+1} = v_j + w_j$, where $v_j \in V_j$ and $w_j \in W_j$, we assume a strengthened Cauchy inequality for the decomposition,

$$|a(v_j, w_j)| \le \tilde{\lambda} \|v_j\| \|w_j\| \tag{5.6}$$

where $\tilde{\lambda} < 1$ independent of j.

Instead of Eq. 5.3, Bank and Smith (1993) used two-scale error in the multiresolution approach

$$r_j = u_{j+1} - u_j \tag{5.7}$$

where $r_j \in W_j$. It means that two level error is the projection of true error onto the wavelet space. It is proved (Bank and Smith (1993)) that

$$c_1 \|u - u_j\| \le \|r_j\| \le c_2 \|u - u_j\| \tag{5.8}$$

where $c_1, c_2 > 0$ are constants and independent of multiresolution level j.

By using orthogonality relation

$$a(u - u_j, v) = 0 \;\; \forall \;\; v \in V_j \tag{5.9a}$$

$$a\left(u-u_{j+1},v\right)=0 \; \forall \;\; v\in V_{j+1} \tag{5.9b}$$

From Eq. (5.9a -5.9b), we get

$$a\left(u-u_{j},v\right)-a\left(u-u_{j+1},v\right)=0 \qquad \forall \;\; v\in V_{j}\subset V_{j+1}$$

$$a\left(u_{j+1}-u_{j},v\right)=0 \; \forall \;\; v\in V_{j} \tag{5.9c}$$

and

$$a\left(u-u_{j}-w_{j},v\right)-a\left(u-u_{j+1},v\right)=0 \quad \forall \;\; v\in W_{j}$$

$$a\left(u_{j+1}-u_{j}-w_{j},v\right)=0 \qquad \forall \;\; v\in W_{j} \tag{5.9d}$$

The orthogonality property (Figure 5.1)

$$\left\|u-u_{j}\right\|^{2}=\left\|u-u_{j+1}\right\|^{2}+\left\|u_{j+1}-u_{j}\right\|^{2} \tag{5.10}$$

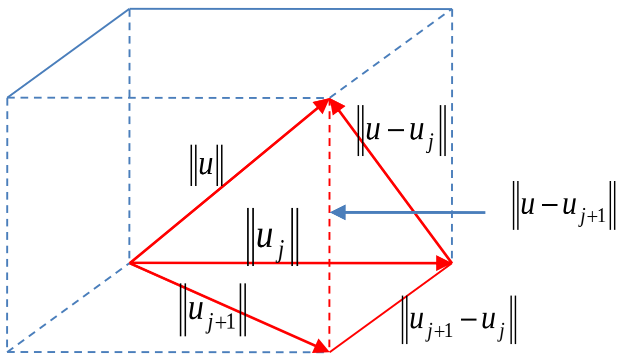


Figure 5.1. Orthogonality at two scales

and saturation property are used to get the lower bounds $c_1=\left(1-\xi^2\right)$. The two scale error will always be less than the total error, i.e. $\|r_j\|\le\|u-u_j\|$. Therefore, two level error is a good estimate of the actual error. Since w_j is the projection of r_j hence

$$\|w_{j+1}\| \le \|r_{j+1}\| \le \|w_j\| \le \|r_j\| \le \|u-u_j\| \tag{5.11}$$

The two consecutive levels of FEM refinements can be used to calculate two scale error $\|r_{j+1}\|$ or $\|r_j\|$. This method will be expensive to estimate error. Therefore, FE solution is transformed using discrete wavelets. The norm of some functional calculated using wavelet coefficients will show sudden jump in some areas which will indicate the need of mesh refinement in the zone. The cost of computing $\|w_{j+1}\|$ or $\|w_j\|$ is considerably less than the two scale error.

Theorem: Let $V_{j+1} = V_j \oplus W_j$, strengthen Cauchy inequality and saturation property holds then

$$\left(1-\xi^2\right)\left(1-\lambda^2\right)\|u-u_j\|^2 \le \|w_j\|^2 \le \|u-u_j\|^2$$

Proof

The proposed posteriori error estimate $w_j \in W_j$ is the error of the solution $u_j \in V_j$ such that $a\left(w_j, v\right) = f\left(v\right) - a\left(u_j, v\right) \ \forall \ \ v \in W_j$

or $$a\left(u-u_j-w_j, v\right) = 0 \qquad \forall \ \ v \in W_j \tag{5.12}$$

Substitution of $v = w_j$ in the preceding equation gives right inequality of the theorem. Let $u_{j+1} = \hat{u}_j + \hat{w}_j$ where $\hat{u}_j \in V_j$ and $\hat{w}_j \in W_j$ then

$$\|u_{j+1} - u_j\|^2 = \|\hat{u}_j + \hat{w}_j - u_j\|^2$$

$$= \|\hat{u}_j - u_j\|^2 + \|\hat{w}_j\|^2 - 2\lambda\|\hat{u}_j - u_j\|\|\hat{w}_j\|$$

$$= \left\| \hat{u}_j - u_j \right\| \left(\left\| \hat{u}_j - u_j \right\| - 2\tilde{\lambda} \left\| \hat{w}_j \right\| \right) + \left\| \hat{w}_j \right\|^2$$

$$\geq \left\| \hat{u}_j - u_j \right\| \left(\tilde{\lambda} \left\| \hat{w}_j \right\| - 2\tilde{\lambda} \left\| \hat{w}_j \right\| \right) + \left\| \hat{w}_j \right\|^2$$

$$\geq \tilde{\lambda} \left\| \hat{w}_j \right\| \left(-\tilde{\lambda} \left\| \hat{w}_j \right\| \right) + \left\| \hat{w}_j \right\|^2$$

$$\geq \left(1 - \tilde{\lambda}^2 \right) \left\| \hat{w}_j \right\|^2 \qquad (5.13)$$

By using orthogonality relations $a\left(u - u_j, v\right) = 0 \ \forall \ v \in V_j$ and $a\left(u - u_{j+1}, v\right) = 0$ $\forall \ v \in V_{j+1}$, we get $a\left(u_{j+1} - u_j, v\right) = 0 \ \forall \ v \in V_j$. Substitution of $v = \hat{u}_j - u_j$, gives

$$\left\| u_{j+1} - u_j \right\|^2 = a\left(u_{j+1} - u_j, \hat{w}_j \right) \qquad (5.14)$$

On subtracting Eq. 5.12 from the $a\left(u - u_{j+1}, v\right) = 0 \ \forall \ v \in V_{j+1}$, we get $a\left(u_{j+1} - u_j - w_j, v\right) = 0 \quad \forall \ v \in W_j$.

With $v = \hat{w}_j$, we get

$$a\left(u_{j+1} - u_j, \hat{w}_j \right) = a\left(w_j, \hat{w}_j \right) \qquad (5.15)$$

Using Eq. 5.14 and Eq. 5.15, we have

$$\left\| u_{j+1} - u_j \right\|^2 = a\left(w_j, \hat{w}_j \right)$$

Using Eq. 5.6 in the preceding equation, it can be shown that

$$\left\|u_{j+1}-u_j\right\|^2 \le \left\|w_j\right\|\left\|\hat{w}_j\right\| \tag{5.16}$$

From Eq. 5.13 and Eq. 5.16, we obtain

$$\left\|w_j\right\| \ge \left(1-\bar{\lambda}^2\right)\left\|\hat{w}_j\right\| \tag{5.17}$$

On substituting Eq.5.16 in Eq. 5.10, we get

$$\left\|u-u_j\right\|^2 = \left\|u-u_{j+1}\right\|^2 + \left\|w_j\right\|\left\|\hat{w}_j\right\|$$

Using saturation assumption, we obtain

$$\left\|u-u_j\right\|^2 \le \xi^2\left\|u-u_j\right\|^2 + \left\|w_j\right\|\left\|\hat{w}_j\right\|$$

Combining this with Eq.5.17, we have

$$\left(1-\xi^2\right)\left\|u-u_j\right\|^2 \le \frac{1}{1-\bar{\lambda}^2}\left\|w_j\right\|^2$$

□

Now we have to determine error in a computationally efficient way. Bank and Smith (1993) suggested easily computable operator instead of original operator. In this paper, we used functional of wavelet coefficients. A function $u \in L^2(R)$ is approximated by its projection $P^j u$ onto the space V_j and the projection of u on W_j as $Q^j u$, we have

$$P^j u = P^{j-1}u + Q^{j-1}u \tag{5.18}$$

If the coefficient vector of $P^j u$ in terms of some scaling function basis is $u_j = \{u_{j,0},\ldots\ldots,u_{j,\upsilon(j)}\}^T$ and coefficient vector of $Q^j u$ in terms of some wavelet basis is

$w_j = \{w_{j,0}, \ldots\ldots, w_{j,\omega(j)}\}^T$ then we can write wavelet transform as

$$u_j = [\mathbf{P}_j \mid \mathbf{Q}_j] \left[\frac{u_{j-1}}{w_{j-1}} \right] \tag{5.19}$$

where $\mathbf{P}_j$ is a $\upsilon(j) \times \upsilon(j-1)$ matrix and $\mathbf{Q}_j$ is a $\omega(j) \times \omega(j-1)$ matrix. $\upsilon(j)$ and $\omega(j)$ are dimensions of V_j and W_j spaces, respectively.

5.3. Numerical Experiments

We developed the wavelet based cost effective error indicator. To test the performance of proposed error estimators, we compare it with exact error and two scale error indicator for standard problems.

Example 1: In the first example, we consider the linear initial-boundary value hyperbolic problem (Arney and Flaherty (1986)):

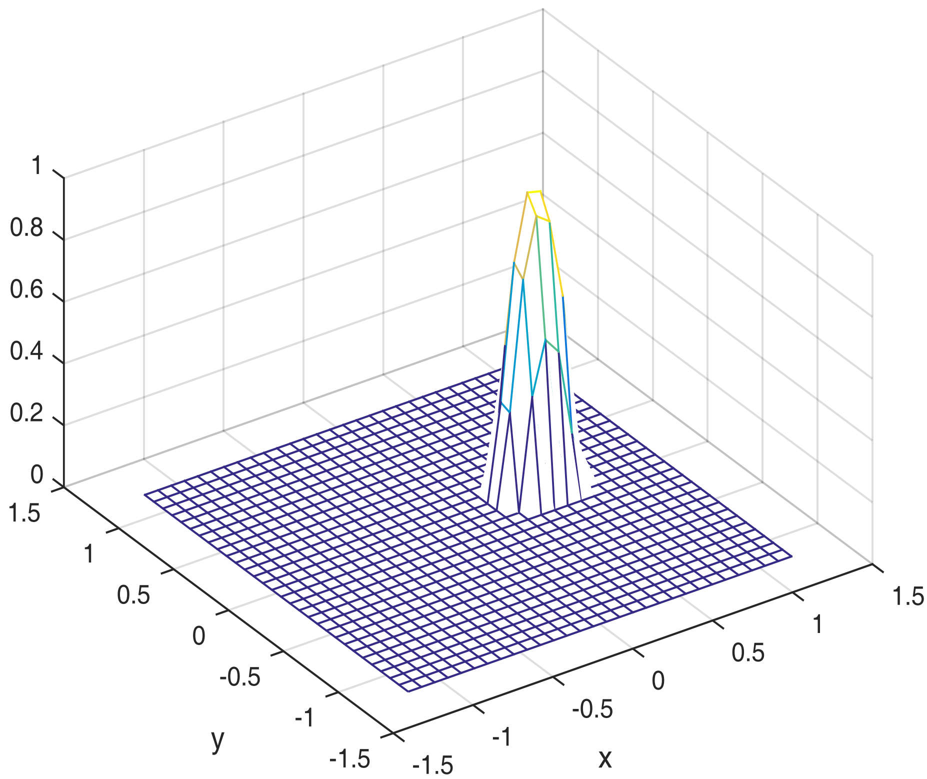
$$\frac{\partial u}{\partial t} - y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0,\ -1.2 \le x,\ y \le 1.2,\ \ t > 0, \quad (5.20)$$

and initial conditions $u(x,y,0) = \begin{cases} 0 & \text{if } (x\text{-}0.5)^2 + 1.5y^2 \ge 0.0625 \\ 1 - 16((x-0.5)^2 + 1.5y^2) & \text{otherwise} \end{cases}$

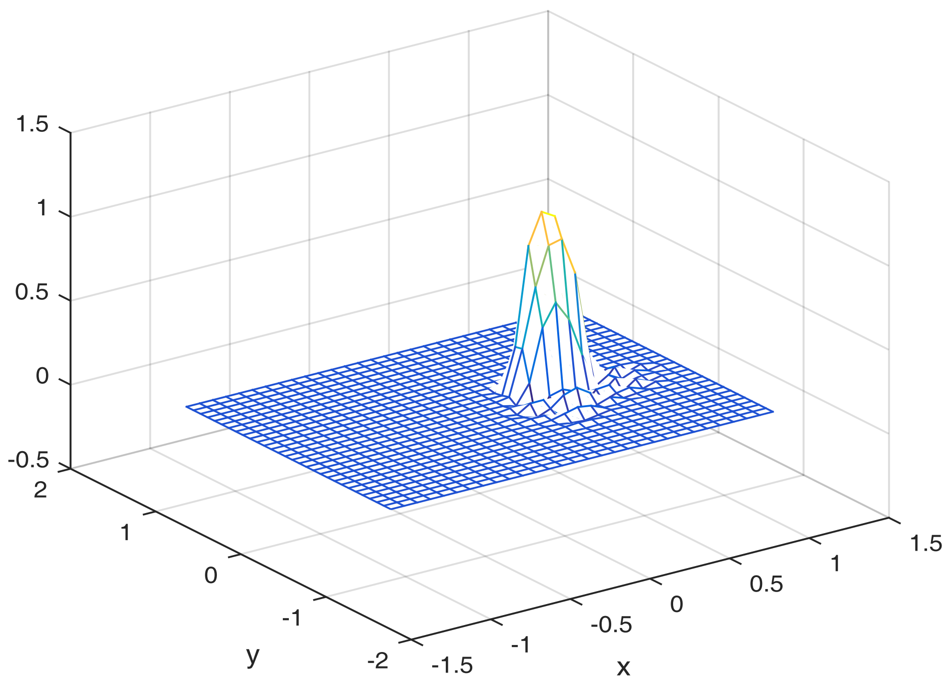
The Dirichlet boundary conditions on all four sides $u(1.2,y,t) = u(-1.2,y,t) = u(x,-1.2,t) = u(x,1.2,t) = 0$ are considered. The solution of this problem is a moving elliptical cone that rotates counter clockwise direction around the origin with period π. It can be written in the form $u(x,y,t) = \begin{cases} 0 & \text{if } C < 0 \\ C & \text{if } C \ge 0, \end{cases}$

where $C = 1 - 16[(x\cos t + y\sin t - 0.5)^2 + 1.5(y\cos t + x\sin t)^2]$.

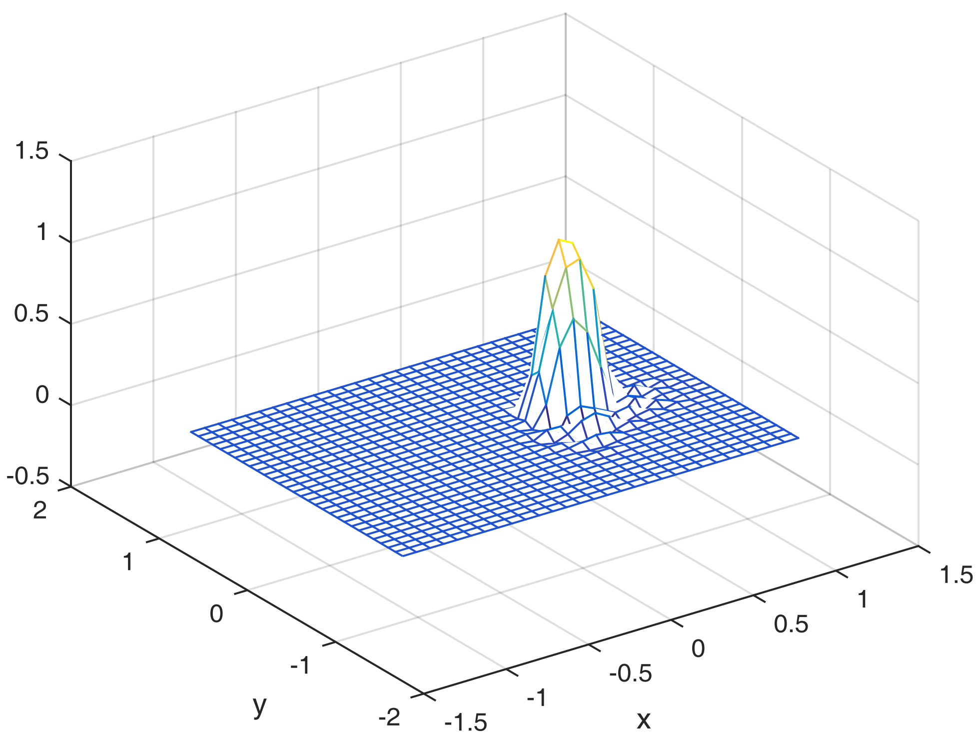
The equation is solved by using finite element method for various level of uniform mesh. At $0 < t \leq 3.2$, the Newmark time integration scheme is used with a time step Δt = 0.00625 s. Figures 5.2 shows the comparison of exact solution with FEM solution and wavelet based solution at $t = 1.6$ s with mesh size 40×40. Figure 5.3 shows the errors $\|u - u_j\|$, $\|r_j\|$ and the proposed estimator $\|w_j\|$ with the increasing degrees of freedom. Initially at $j = 1$, a uniform grid of 20×20 is used. The grid is doubled at every successive level j for multiscale wavelets. The figure shows the convergence of exact error as the degrees of freedom increases. It can be observed that the two level indicator and the proposed indicator show the similar trend of convergence of global error with the level of refinement. This ensures the reliability of the proposed wavelet based estimator.



(a) Exact solution



(b) FEM Solution



(c) Wavelet based solution

Figure 5.2. The solution at $t = 1.6$ **with mesh size** 40×40

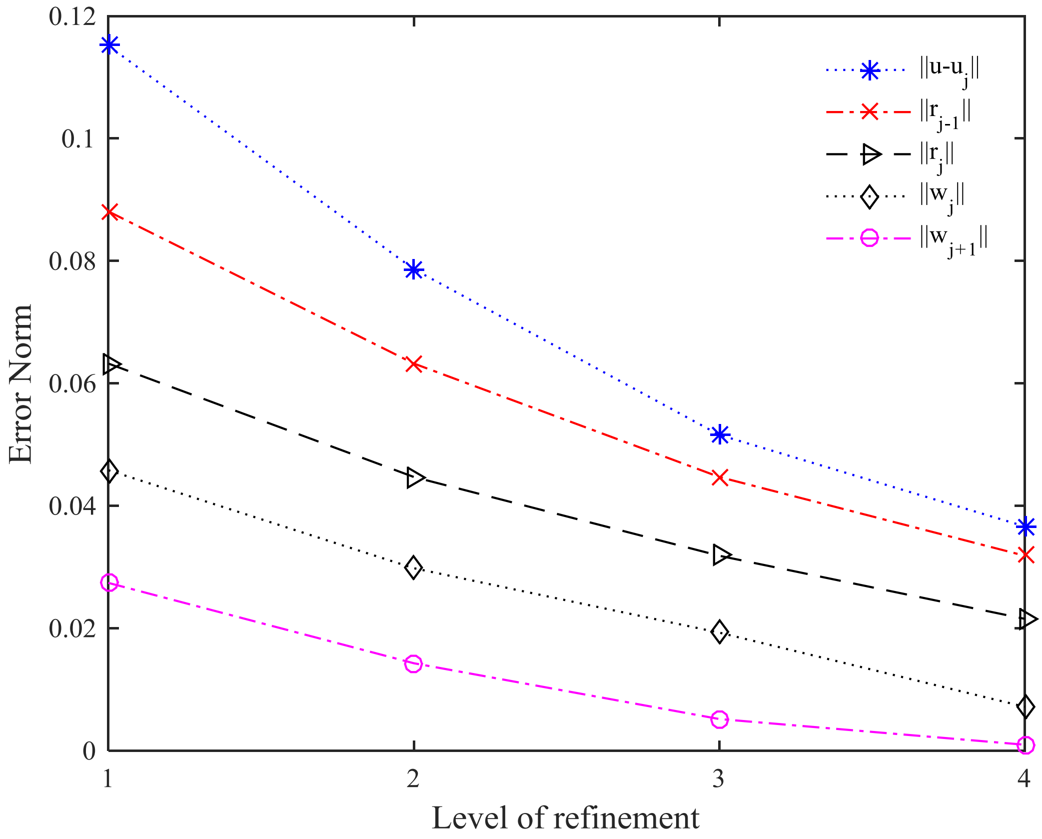


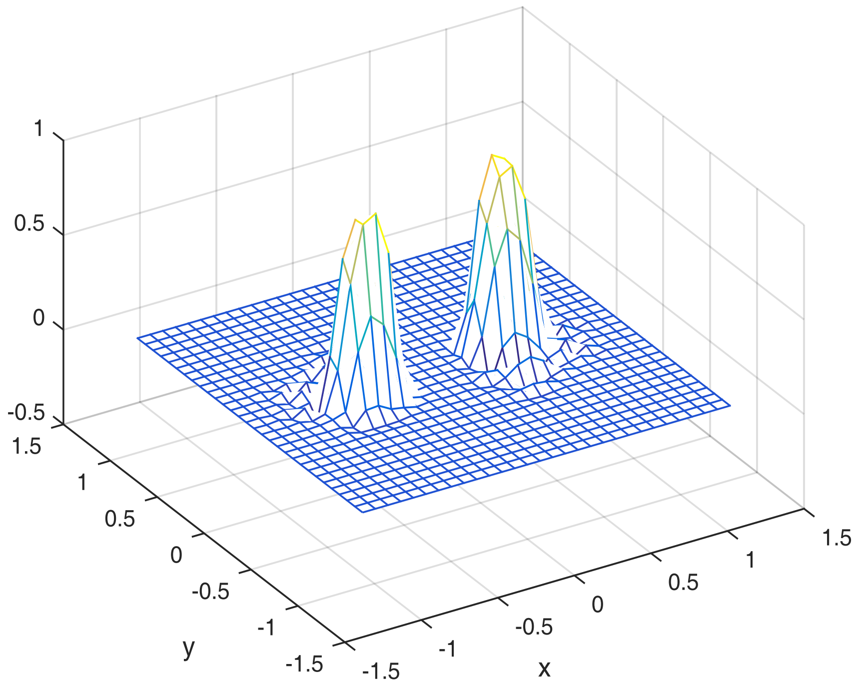
Figure 5.3. Comparison of exact, two scale, and proposed error estimators

Example 2. The problem is similar to Example 1 except new initial conditions (Arney and Flaherty (1986)):

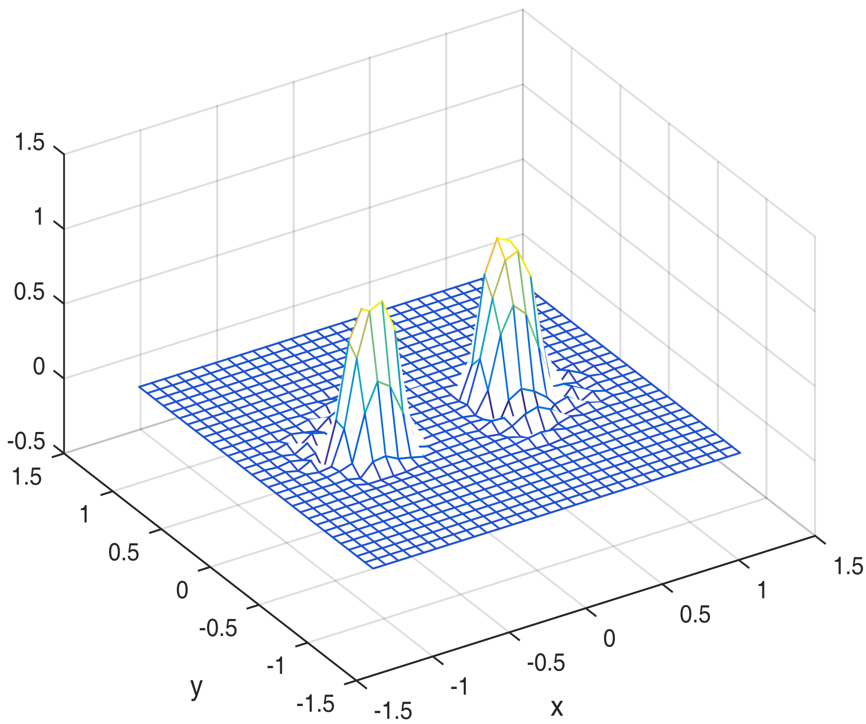
$$u(x,y,0)=\begin{cases}1-16((x-0.5)^2+1.5y^2) & \text{if } (x\text{-}0.5)^2+1.5y^2\le 0.0625\\ 1-16((x+0.5)^2+1.5y^2) & \text{if } (x+0.5)^2+1.5y^2\le 0.0625\\ 0 & \text{otherwise}\end{cases} \quad (5.21)$$

in this case, two symmetric cones rotating anticlockwise direction about the origin.

Figure. 5.4 shows the FEM and wavelet based solution of mesh size 40×40 at $t=1.1$s. Unlike the Example 1, exact solution is not available in this example. Figure 5.5 shows the convergence of the estimators. It can be observed that proposed estimator is a very good error indicator.



(a) FEM solution



(b) wavelet based solution

Figure 5.4. The solution at $t = 1.1\,\text{s}$ **with mesh size** 40×40

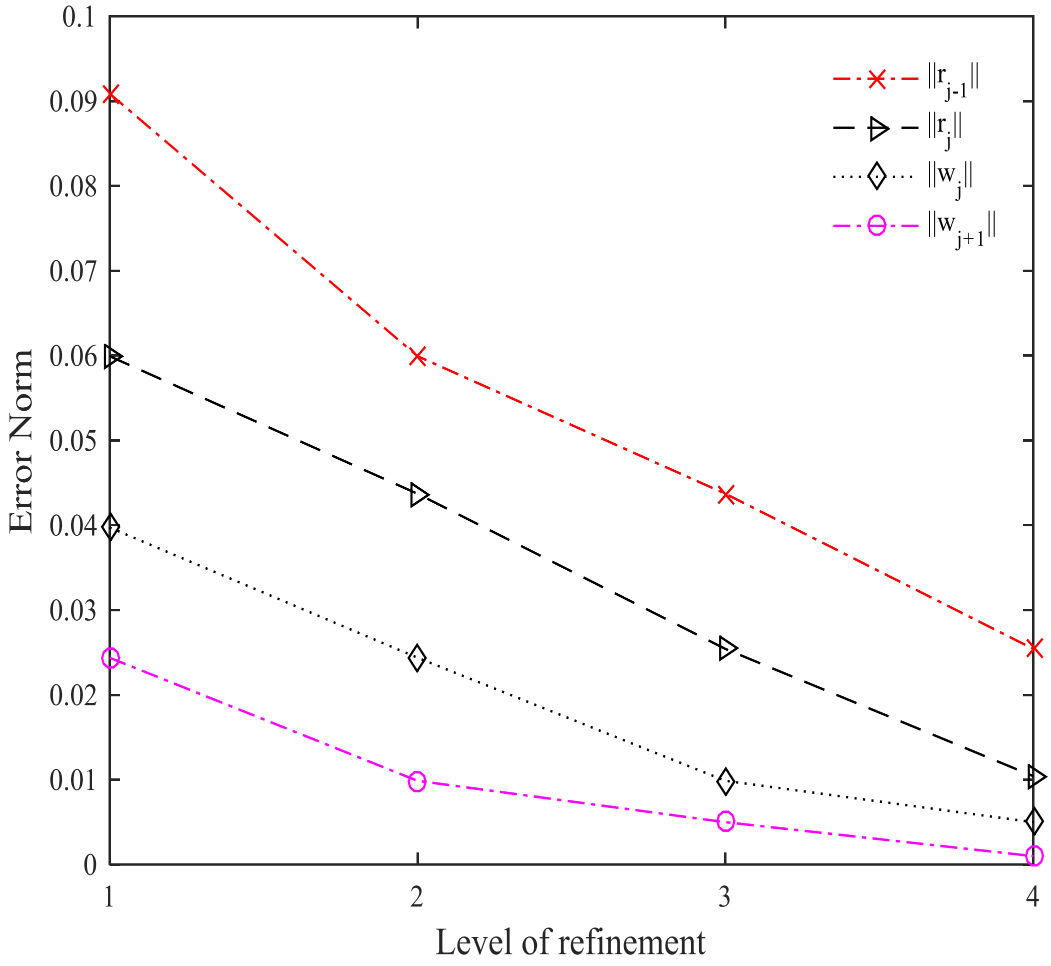


Figure 5.5. Comparison of two scale, and proposed error estimators

Example 3. Plate with hole

To demonstrate the effectiveness of two-level error indicator and proposed error indicator for local error estimation, both the methods are tested on a standard 2D linear elasticity problem. The zone of mesh refinement for a plate with a hole under longitudinal tensile loading is very well known. Both the error indicators are tested by considering one-fourth of the plate as shown in Figure 5.6. The region of interest in this problems are the elements in the neighborhood of the circular hole where stress concentration is very high. The functional used for two level error indicator is the difference of strain energy at two consecutive levels of finite element solutions. The peaks in the Figure 5.7. indicates the elements with high local error, i.e. these elements are to be refined. For the proposed error indicator, the strain energy is calculated by using wavelet coefficients and shown in Figure 5.8. As in the previous example, we see the similar peaks at high stress concentration elements but the proposed error indicator is much more economical than the two level error indicator. Convergence of stress is used to stop the refinement process.

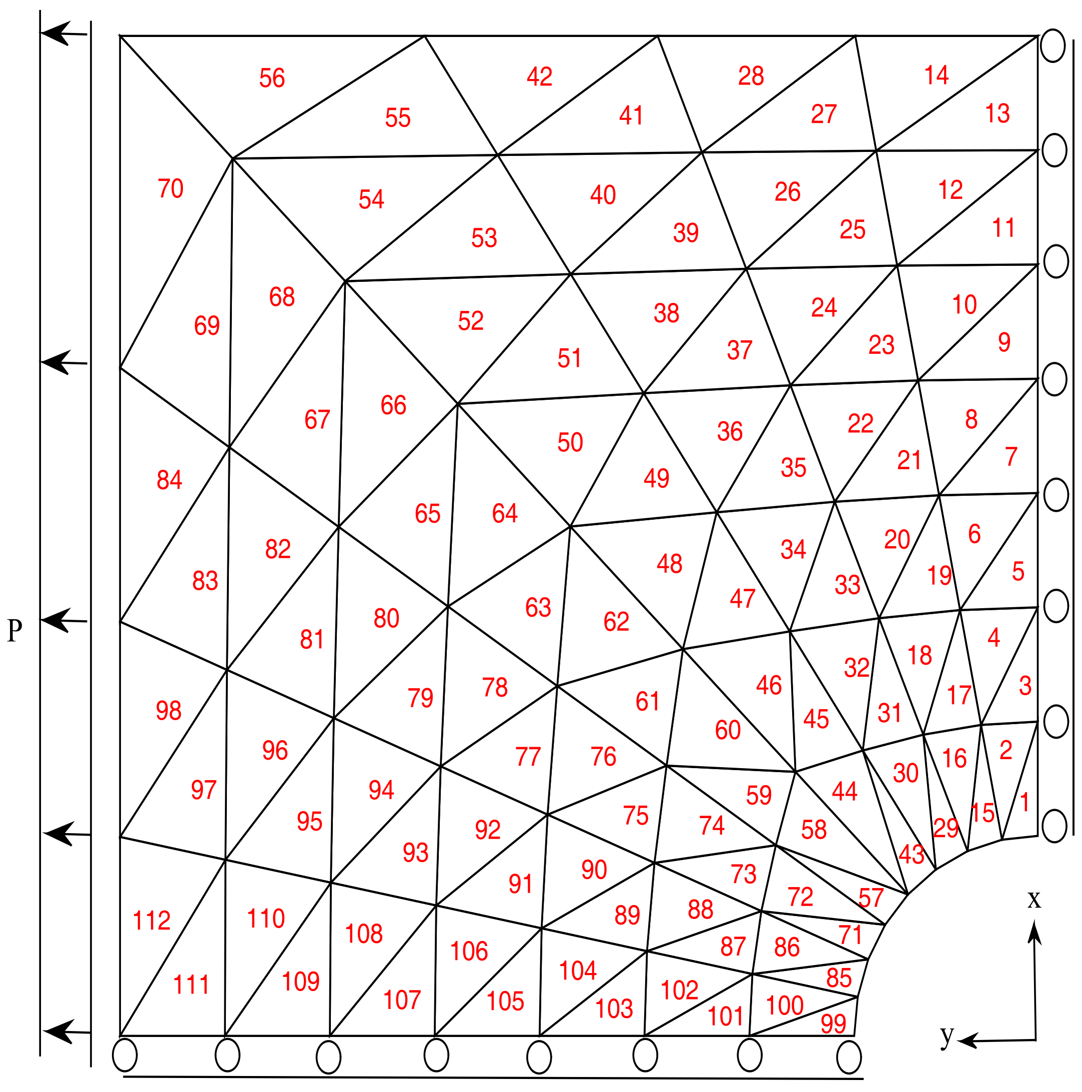


Figure 5.6. Discretized one quarter of the rectangular plate with a hole

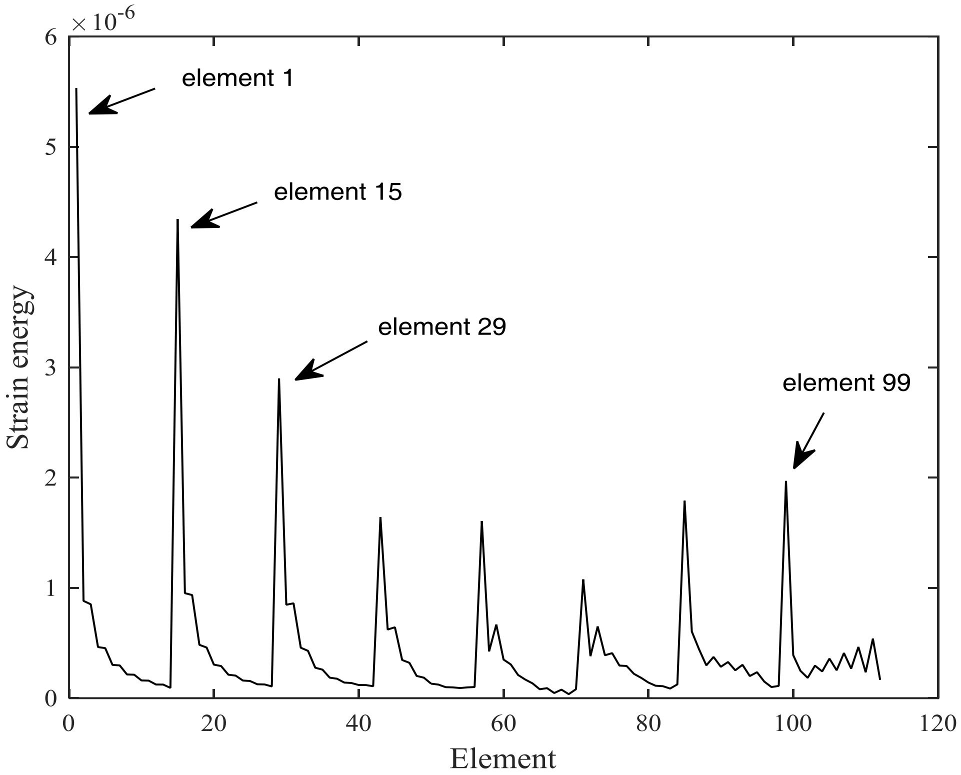


Figure 5.7. Strain energy using two scale difference

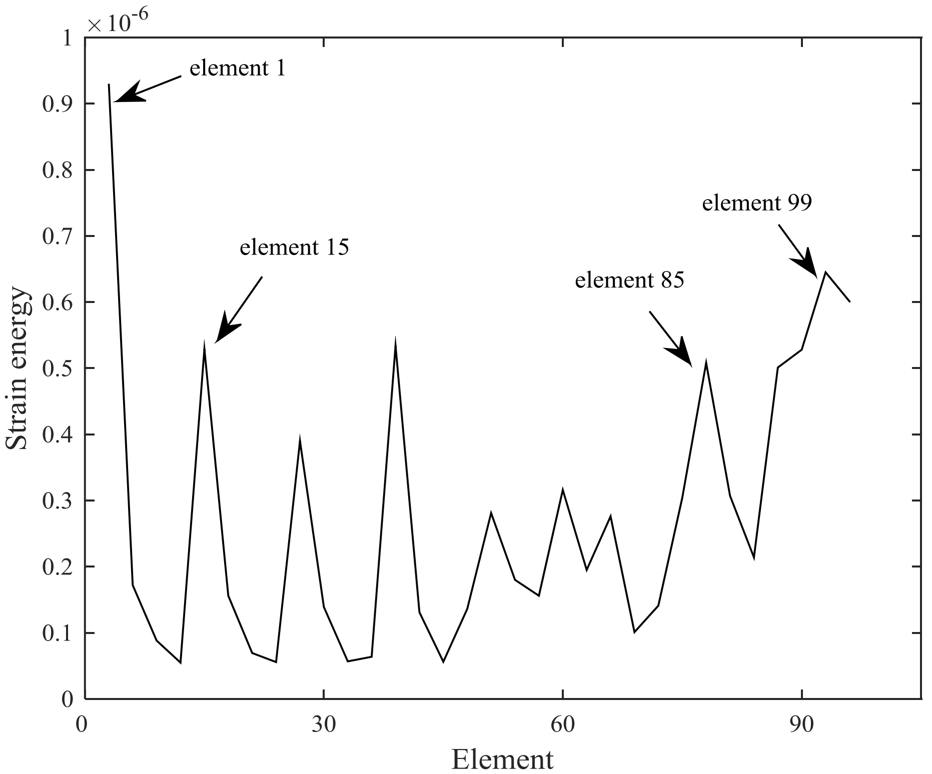


Figure 5.8. Strain energy using wavelet coefficient