CHAPTER-6

Numerical Solution of Fractional Order Vibration Equation using Bernstein Polynomial

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6.1 Introduction

The vibration equation for the large membrane represents the free vibration of a large circular membrane. The governing equation for the vibration of a circular elastic membrane which stretched over a large circular frame is represented by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^\beta u}{\partial t^\beta}, \quad r \geq 0,\ t \geq 0 \text{ and } 1 < \beta \leq 2, \tag{6.1}$$

where $c = \sqrt{T/\rho}$, T is tension in the membrane and ρ is its mass density. Das (2009), and Das and Gupta (2011) have studied the above problem using the semi analytic methods viz., Adomian decomposition method, Homotopy Perturbation Method and Homotopy analysis Method. In last few decades there was lot of interest found among the researchers in the applications of fractional calculus. From the literature survey (Miller and Ross, 1993; Podlubny, 1999; Kilbas *et al.*, 2006) on the fractional calculus, it is seen that the

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subject is widely used by the researchers in many field of science and technology. Previously fractional calculus was the domain of researchers in mathematics but during last few decades its vast applications in different branches of physics and engineering make this subject very popular among the researchers. Recently fractional calculus is used to study fractional Brownian motion (Boufoussi and Hajji, 2011). Fractional calculus deals with the generalization of standard order derivative and integration. During the solutions of fractional differential equations many techniques have been proposed viz., variational iteration method(He, 1998; Wu, 2011) , Adomian decomposition method (Duan *et al,*2013 ; Song and Wang,2013), Homotopy analysis method (Hashim *et al.,* 2009; Mishra *et al.*, 2014; Zhang *et al.,* 2011; Jafari and Seifi, 2009) , Operational matrix of Legendre polynomials (Saadatmandi and Dehghan, 2011; Saadatmandi and Dehghan, 2010), Operational matrix of B-spline functions (Lakestani *et al.*,2012), Legendre multiwavelet collocation method (Yousefi *et al.,* 2011), Operational matrix of Chebyshev polynomials (Doha *et al.*, 2011) etc. Bernstein polynomials frame a complete basis on interval [*a*,*b*], where $a,b \in R$. Many researchers have tried and used Bernstein basis polynomial to solve different kinds of differential and integral equations. Maleknejad *et al.* (2012) have used Bernstein polynomial operational matrix to solve nonlinear Volterra–Fredholm–Hammerstein integral equations. Rad *et al.* (2014) have employed Bernstein polynomial with Tau method to solve fractional order differential equation. Bernstein polynomials had been used to solve variable order linear cable equation by Chen *et al.* (2014), Singular integro-differential equation by Bhattacharya and Mandal (2008), Numerically nonlinear age-structured population models by Yousefi *et al.*(2012). Turnbull and Ghosh (2014) had used Bernstein polynomial basis function to estimate uni-modal density. To the best of my knowledge the fractional order vibration equation using Bernstein polynomial basis function has not yet been solved by any researcher.

In the present chapter, an effort has been given to solve the equation (6.1) for a large membrane using operational matrix of Bernstein polynomial basis. By considering different initial conditions, the nature of the solutions for different particular cases are presented through figures.

6.2 Bernstein polynomials and its properties

The i-th Bernstein polynomial of degree n in the interval $[a, b]$, $a,b \in R$ is defined by the formula

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n} \quad , \qquad (6.2)$$

where as if the above interval is [0,1] then the i-th Bernstein polynomial is represented by

$$B_{i,n}(x)=\binom{n}{i}x^{i}(1-x)^{n-i}\ ,\quad i=0,1,2\ldots n\ , \tag{6.3}$$

using the binomial expansion for $(1-x)^{n-i}$ in the equation (6.3), we get the Bernstein polynomial in the following form as

$$B_{i,n}(x)=\sum_{r=i}^{n}(-1)^{r-i}\binom{n}{i}\binom{n-i}{r-i}x^{r}\ ,\qquad i=0,1,2\ldots,n,. \tag{6.4}$$

These Bernstein polynomials construct a basis over the interval [0, 1] whereas from (6.4) it is clear that the Bernstein polynomials are also a combination of power basis function $\{1,x,x^2,x^3\ldots x^n\}$. Juttler (1998) introduced the dual basis of the Bernstein basis represented by $\{D_0^n(x),D_1^n(x),D_2^n(x),\ldots D_n^n(x)\}$ for the Bernstein polynomials of degree n for $x\in[0,1]$, which has the property

$$\int_0^1 B_{i,n}(x)\,D_j^n(x)\,dx=\delta_{i,j}\ =\begin{cases}1 & for\ \ i=j\\ 0 & for\ \ i\neq j\end{cases}\qquad \text{(i, j=0,1,2,3, …,n).} \tag{6.5}$$

If $f(x)$ is a square integrable function in an interval [0,1] and if $\psi=\{B_{0,n}(x),B_{1,n}(x),B_{2,n}(x),\ldots,B_{n,n}(x)\}$ is a finite dimensional vector space, then $f(x)$ can have a unique best approximation from ψ expressed in terms of Bernstein basis with first (n+1) terms of the basis as $\Phi(x)=\{B_{0,n}(x),B_{1,n}(x),B_{2,n}(x),\ldots,B_{n,n}(x)\}$, which can be expressed as

$$f(x)\cong\sum_{i=0}^{n}c_iB_{i,n}(x)=c^T\Phi(x)\ , \tag{6.6}$$

where $c^T = [c_0, c_1, c_2 ... c_n]$.

Then $c = Q^{-1}(f, \Phi(x))$, (6.7)

where Q is an $(n+1)\times(n+1)$ matrix which is called as Dual matrix of Φ, and can be obtained as

$$Q = (\Phi(x), \Phi(x)) = \int_0^1 \Phi(x)\ \Phi^T(x)\, dx. \tag{6.8}$$

In the present chapter we approximated the two dimensional function $u(x,t) \in L^2([0,1]\times[0,1])$ as

$$u(x,t) \cong \sum_{i=0}^{n}\sum_{j=0}^{n} u_{i,j} B_{i,n}(x) B_{j,n}(t) = \Phi^T(x) U \Phi(t), \tag{6.9}$$

where

$$U = \begin{bmatrix} u_{00} & u_{01} & . & . & . & u_{0n} \\ u_{10} & u_{11} & . & . & . & u_{1n} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ u_{n0} & u_{n1} & & & & u_{nn} \end{bmatrix}$$

and U can be obtained as

$$U = Q^{-1}(\Phi(x), (\Phi(t), u(x,t)))Q^{-1}. \tag{6.10}$$

6.3 Operational matrix of the derivative using Bernstein polynomial basis

The derivative of the vector $\Phi(x)$ of Bernstein basis is represented as

$$\frac{d\Phi(x)}{dx} = D^{(1)}\Phi(x), \tag{6.11}$$

where $D^{(1)}$ is the $(n+1)\times(n+1)$ operational matrix of derivative calculated as

$$D^{(1)} = BVP^{*}. \tag{6.12}$$

Here

$$B = \begin{bmatrix} (-1)^0\binom{n}{0} & (-1)^1\binom{n}{0}\binom{n-0}{1} & . & . & . & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0} \\ 0 & (-1)^0\binom{n}{1}\binom{n-1}{0} & . & . & . & (-1)^{n-1}\binom{n}{1}\binom{n-1}{n-1} \\ . & . & & & & . \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ . & & & & & . \\ 0 & 0 & . & . & . & (-1)^0\binom{n}{n} \end{bmatrix}_{(n+1)\times(n+1)},$$

$$V = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 \\ 1 & 0 & 0 & . & . & . & 0 \\ 0 & 2 & 0 & . & . & . & 0 \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ 0 & 0 & 0 & . & . & . & n \end{bmatrix}_{(n+1)\times(n)}, \qquad P^{*} = \begin{bmatrix} B^{-1}_{[1]} \\ B^{-1}_{[2]} \\ B^{-1}_{[3]} \\ . \\ . \\ . \\ B^{-1}_{[n]} \end{bmatrix}_{(n)\times(n+1)},$$

where $B^{-1}_{[k]}$ is the *k-th* row of B^{-1} for *k=1,2,…n*. proceeding in this way we can calculate the operational matrix for higher order derivative as

$$\frac{d^n\Phi(x)}{dx^n} = (D^{(1)})^n\,\Phi(x)\,, \qquad (6.13)$$

where, $n \in N$ and superscript of $D^{(1)}$ represents the matrix power.

6.5 Considered problem and its solution

To solve governing equation (6.1), let us consider the following initial conditions

$$u(r,0) = f(r)\,,\quad \frac{\partial}{\partial t}u(r,0) = cg(r)\,. \qquad (6.14)$$

and

$$u(0,t) = 0\ . \qquad (6.15)$$

For solving the above problem, let us consider the following approximation of the solution in the form of Bernstein polynomial basis as

$$u(r,t) \cong \sum_{i=0}^{n}\sum_{j=0}^{n} u_{i,j}B_{i,n}(r)B_{j,n}(t) = \Phi^T(r)U\Phi(t)\,, \qquad (6.16)$$

where U is unknown and

$$\frac{\partial u(r,t)}{\partial r} \cong \frac{\partial(\Phi^T(r)U\Phi(t))}{\partial r} = \Phi^T(r)\big((BVP^*)^T\big)U\,\Phi(t), \qquad (6.17)$$

$$\frac{\partial^2 u(r,t)}{\partial r^2} \cong \frac{\partial^2(\Phi^T(r)U\Phi(t))}{\partial r^2} = \Phi^T(r)\big((BVP^*)^T\big)^2\,U\,\Phi(t). \qquad (6.18)$$

For the fractional order derivative

$$\frac{\partial^{\beta} u(r,t)}{\partial t^{\beta}} \cong \frac{\partial^{\beta} (\Phi^T(r) U \Phi(t))}{\partial t^{\beta}} = \Phi^T(r) U\, D \Phi(t), \qquad (6.19)$$

where D is the operational matrix which is obtained by the algorithm given by Saadatmandi (2014). Substituting the operational matrix structure of all the derivatives from the equations (6.17)-(6.19) in the governing equation, the following algebraic equation is obtained as

$$\Phi^T(r) U D \Phi(t) = c^2 \left(\Phi^T(r) \left((BVP^*)^T \right)^2 U \Phi(t) + \frac{1}{r} \Phi^T(r) \left((BVP^*)^T \right) U \Phi(t) \right). \qquad (6.20)$$

Dispersing the equation (6.20) at (r_i, t_i) for *i*=*1, 2, 3. . .n* and with the help of MATHEMATICA software, the values of U can be calculated.

6.6 Numerical results and discussion

During solutions of our considered problem, we have taken *n*=3 in the equation (6.16), which gives rise to

$$\Phi(r) = \{B_{0,3}(r), B_{1,3}(r), B_{2,3}(r), B_{3,3}(r)\} \;, \Phi(t) = \{B_{0,3}(t), B_{1,3}(t), B_{2,3}(t), B_{3,3}(t)\}.$$

Three cases of initial conditions are considered as

Case (i): $f(r) = r,\ g(r) = 1,$

Case (ii): $f(r) = \sqrt{r},\ g(r) = \frac{1}{\sqrt{r}},$

Case (iii): $f(r) = r^2$, $g(r) = r$.

At $c = 6$ for the order of the derivative $\beta = \frac{3}{2}$. Taking the domain for the considered problem as $\Omega = \{(r,t),\ 0 \le r \le 1\ and\ 0 \le t \le 1\}$ and the points for dispersions are taken as $r_i = \frac{i}{5}$ and $\mathrm{t_i} = \frac{i}{6}$, $\mathrm{i} = 1,2,3,4$, the operational matrix is obtained as

$$D = \frac{1}{1155\sqrt{\pi}}\begin{bmatrix} 1824 & 9664 & 5824 & 4864 \\ -3712 & -18432 & -6912 & -512 \\ 1952 & 7872 & -3648 & -13568 \\ -64 & 896 & 4736 & 9216 \end{bmatrix}.$$

Calculating the operational matrix for the first and second order derivatives are

$$D^{(1)} = \begin{bmatrix} -3 & -1 & 0 & 0 \\ 3 & -1 & -2 & 0 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \qquad (D^{(1)})^2 = \begin{bmatrix} 6 & 4 & 2 & 0 \\ -12 & -6 & 0 & 6 \\ 6 & 0 & -6 & -12 \\ 0 & 2 & 4 & 6 \end{bmatrix}$$

and substituting in the equation (6.20), a linear system of equations with '16' unknowns is obtained. Solving this system of equations, the matrix U can be obtained for the considered three cases as

$$\text{(i) } U = \begin{bmatrix} 0.011991 & 2.039855 & -14.434862 & 62.034989 \\ 0.308689 & 2.251410 & -27.684852 & 76.285440 \\ 0.705303 & 2.795088 & -29.904249 & 83.629464 \\ 0.952037 & 2.840581 & -34.926396 & 95.318201 \end{bmatrix},$$

$$\text{(ii)}\quad U=\begin{bmatrix} 0.212847 & 7.14811 & -52.8038 & 229.343 \\ 0.647742 & 2.28134 & -42.7944 & 209.507 \\ 0.852851 & 3.75551 & -47.4568 & 224.884 \\ 0.953673 & 2.86898 & -45.436 & 232.734 \end{bmatrix},$$

$$\text{(iii)}\quad U=\begin{bmatrix} 0.056203 & 0.086813 & -1.33832 & 6.53857 \\ -0.115528 & 0.488218 & -17.5144 & 20.7175 \\ 0.514431 & 1.9464 & -22.8141 & 33.2038 \\ 0.775189 & 2.65275 & -31.2323 & 47.5582 \end{bmatrix}$$

and respective solutions are displayed through Figs. 6.1-6.3.

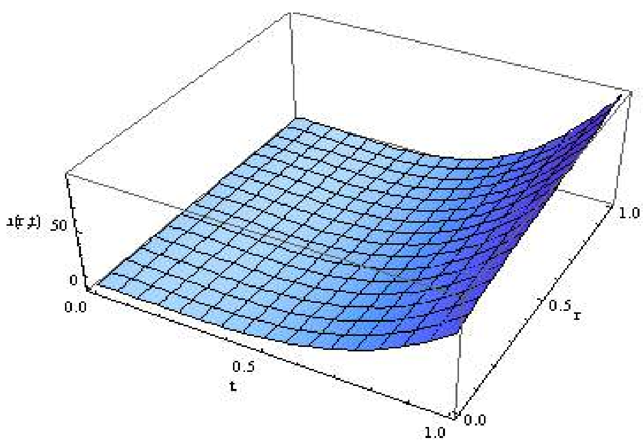


Fig.6.1. Plot of *u (r, t)* vs. *r,t* for the Case (i)

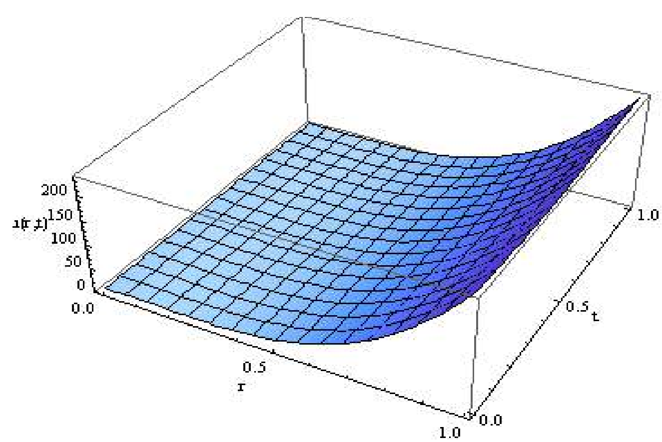


Fig.6.2. Plot of *u (r, t)* vs. *r,t* for the Case (ii)

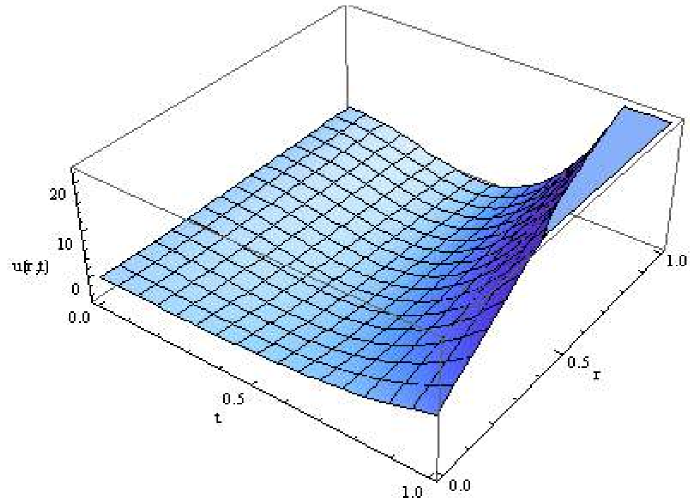


Fig.6.3. Plot of *u (r, t)* vs. *r,t* for the Case (iii)

The two dimensional figures of *u (r, t)* vs. *t* for *r=0.5* and *r = 0.9* for the three considered cases are shown through Fig 6.4 and Fig 6.5 respectively. It is seen from the figures that for both the considered values of *r*, *u(r,t)* increases as the values of *f(r)* and *g(r)* increase. The obtained results are in complete agreement with the results obtained by Das (2009), and Das and Gupta (2011). It is also observed from Fig. 6.6 that in all three cases, at *r=0.5* the increase rate of *u(r ,t)* is faster when the system approaches from standard order to fractional order.

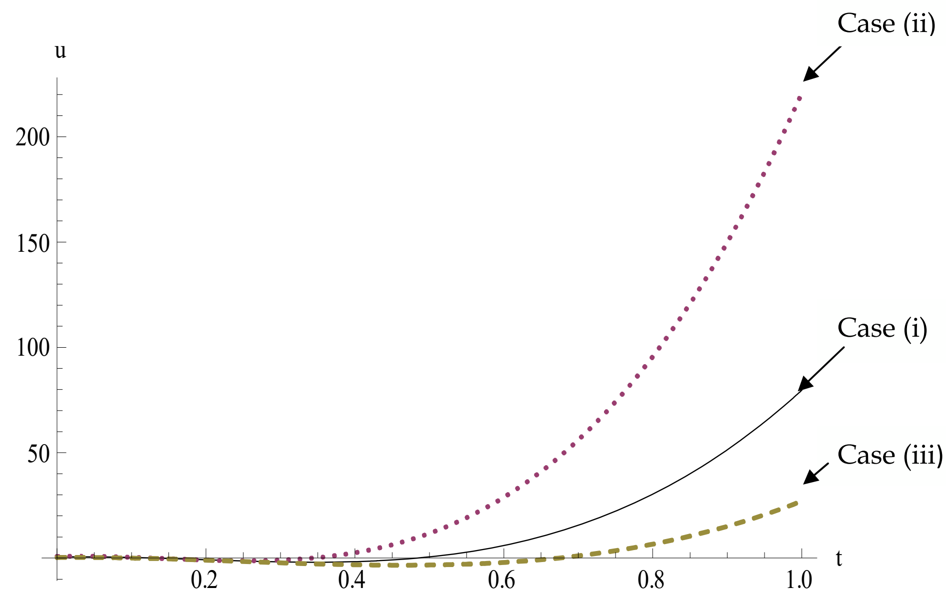


Fig.6.4. Plot of *u(r, t)* vs. *t* in all the three cases for *r=0.5*

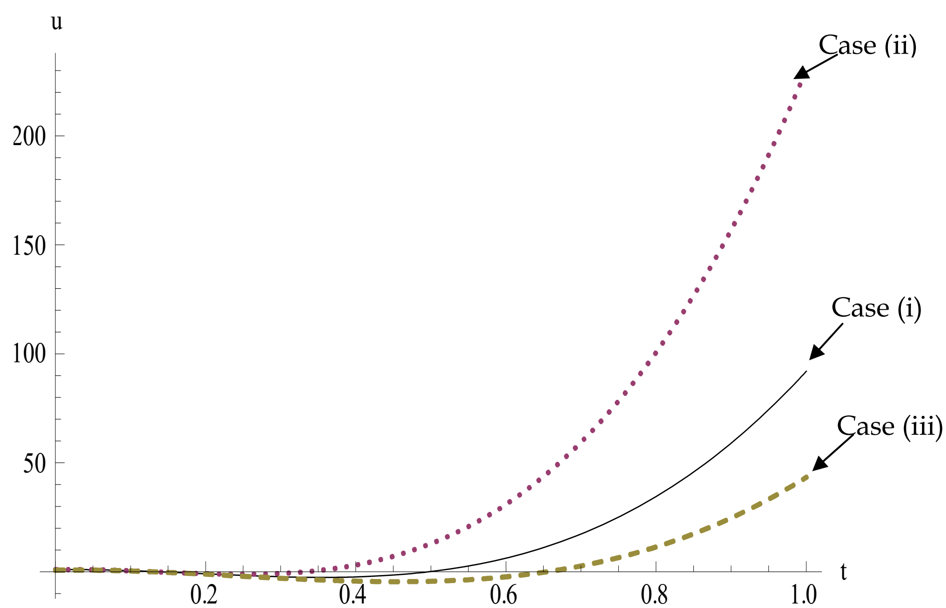
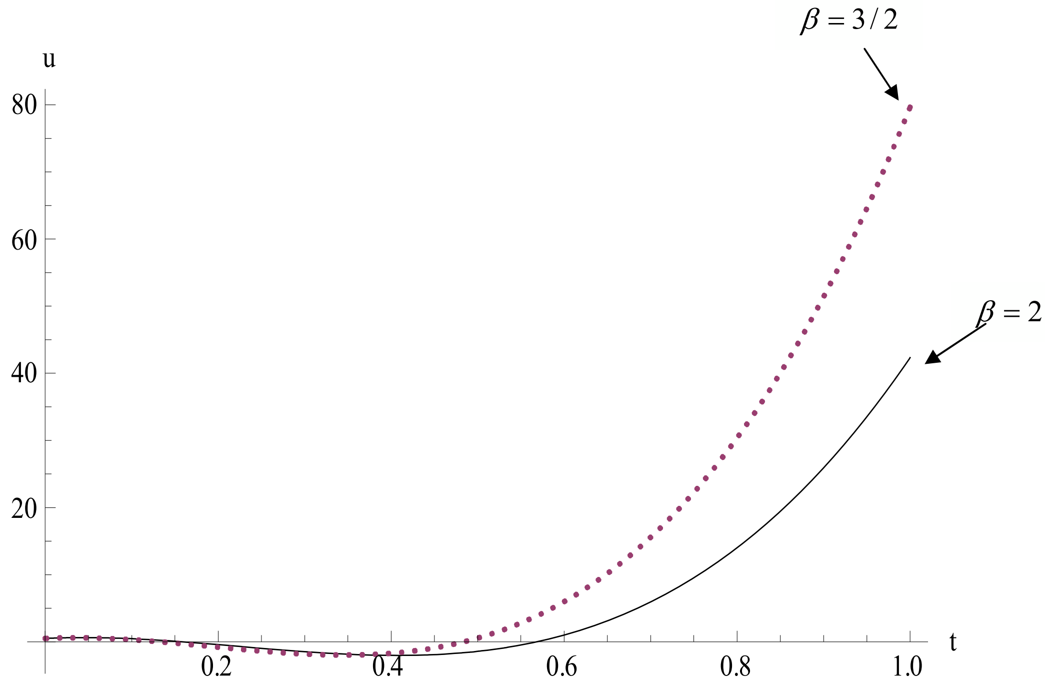
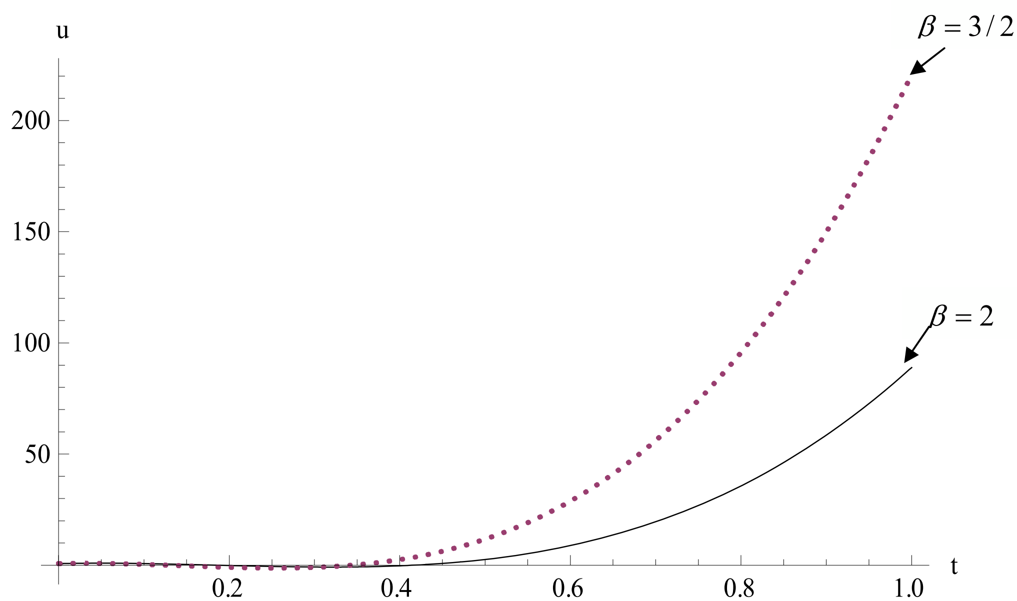


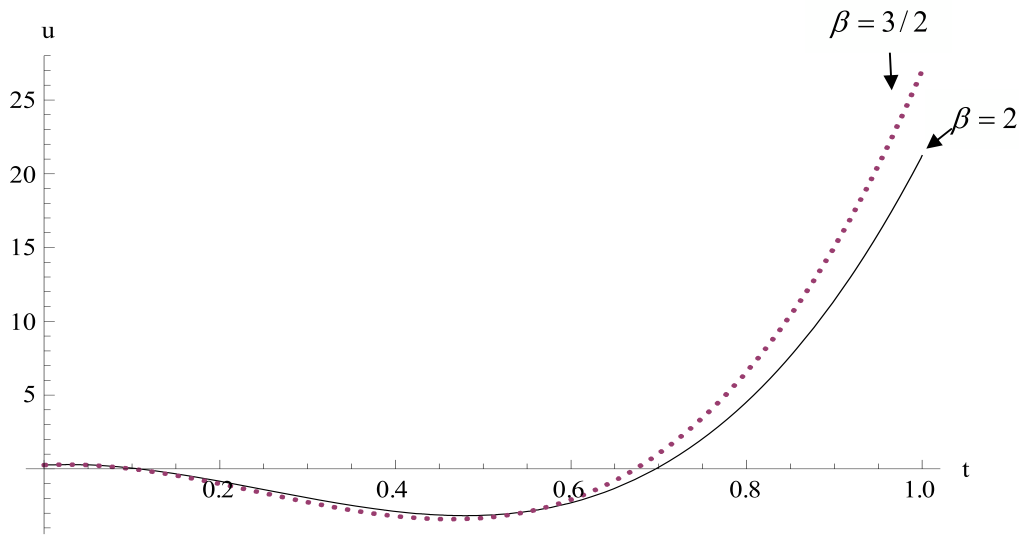
Fig.6.5. Plot of *u(r, t)* vs. *t* in all the three cases for *r=0.9*



(a) Case (i)



(b) Case (ii)



(c) Case (iii)

Fig.6.6. Plots of *u(r, t)* vs. *t* in all the three cases for $\beta = 2$ and $\beta = 3/2$ at *r=0.5*.

6.7 Conclusion

The present chapter has achieved the solutions of the fractional order vibration equation using the operational matrix of Bernstein polynomial basis for the different initial conditions. The attribute of the present study is fractional order vibration equation is successfully converted into a linear system of equations which are dispersed in the domain $([0,1]\times[0,1])$ and solved numerically. As a check on accuracy of the proposed method solution, the results obtained are found to be in complete agreement with the results obtained by Das (2009), and Das and Gupta (2011).