CHAPTER-5

Study of Fractional Order Van der Pol Equation

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5.1 Introduction

Van der Pol oscillator equation was first introduced in 1920 by Balthazar Van der Pol (1920) who introduced the equation to describe the oscillation of triode in the electrical circuit. The mathematical model for this system is a second order differential equation with third degree of nonlinearity as

$$\ddot{u}(t) - \varepsilon(1-u^2(t))\,\dot{u}(t) + u(t) = 0, \tag{5.1}$$

where $\varepsilon > 0$ is a control parameter and $\ddot{u}, \dot{u}$ are the second and first order derivative of u with respect to time. If $\varepsilon = 0$, equation (5.1) represents the simple linear oscillator and for $\varepsilon >> 1$, it represents relaxation oscillation. The equivalent state space formulation of the equation (5.1) is

$$\frac{du_1}{dt} = u_2,$$
$$\frac{du_2}{dt} = -u_1 - \varepsilon(u_1^2 - 1)u_2.$$

In the equation (5.1) for the small value of $u(t)$, the damping force is negative i.e., $-\varepsilon\,\dot{u}(t)$. Again if $u(t)$ is bigger, it becomes dominant and the damping is

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positive. Van der Pol oscillator is an example of self oscillatory system which is now considered as a very useful mathematical model. Equation (5.1) is also known as unforced Van der Pol equation. Van der Pol proposed another version of the above equation by including a periodic forcing term as

$$\ddot{u}(t) - \varepsilon(1 - u^2(t))\dot{u}(t) + u(t) = a\sin wt \quad . \tag{5.2}$$

In 1945, Cartwright and Littlewood (1945) analyzed the Van der Pol equation with large nonlinearity parameter. In 1949, Lavision (1949) studied the Van der Pol equation and had shown that the equation has singular solution. The equation is considered as basic model for oscillatory process for Physics, Biology, Electronics, and Neurology. Van der Pol himself built a number of electronic circuits to model human heart using the equation.

Many researchers have tried to solve and study the Van der Pol equation in various forms. Mickens (2001) proposed the analytical and numerical study of a non-standard finite difference scheme for the unplugged Van der Pol equation. In 2002, Mickens (2002a) studied numerically the Van der Pol equation using a non-standard finite-difference scheme. In the same year, Mickens (2002b) proposed a step-size dependence of the period for a forward-Euler scheme of the Van der Pol equation. In 2003, Mickens (2003) proposed different forms of Fractional Van der Pol oscillators. Researchers have tried many methods to solve the Van der Pol differential equation using Energy balance method (Mehdipour *et al*.,2010; Younesian *et al*.,2010), Parameter expanding method (He *et al*., 2010; Herisanu *et al*.,2010) etc.

Finding the solutions of the Fractional order differential equations have become very popular among the researchers (Atangana and Secer,2013) due to the non local behavior as well as memory effect. Leung *et al.* (2012) have used residue harmonic balance method for fractional order Van der Pol like oscillators. V. Gafiychuk *et al.* (2008) have done the analysis of fractional order Bonhoeffer Van der Pol oscillator. Leung and Guo (2011) have used forward residue harmonic balance for autonomous and non autonomous systems with fractional derivative damping. Guo *et al.* (2011) have given the asymptotic solution of fractional Van der Pol oscillator using the same method. Leung *et al.* (2010) have used the method for discontinuous nonlinear oscillator for fractional power restoring force. Sardar *et al.* (2009) have found the approximate analytical solution of multy term fractionally damped Van der Pol equation. Konuralp (2009) studied numerical solution of Van der Pol equation with fractional damping term. Pereira *et al.*

(2004) have proposed a fractional order Van der Pol equation as

$$\frac{d^{\lambda}u(t)}{dt^{\lambda}}-\varepsilon(1-u^{2}(t))\frac{du(t)}{dt}+u(t)=0, \quad \text{where } 1<\lambda<2, \tag{5.3}$$

with the state space formulation as

$$\frac{du_1}{dt} = u_2,$$
$$\frac{d^{\lambda}u_2}{dt^{\lambda}} = -u_1 - \varepsilon(u_1^2 - 1)u_2 \ .$$

which is obtained by introducing a capacitance by a fractance in the nonlinear RLC circuit. Barbosa *et al.* (2004) proposed fractional order Van der Pol equation by introducing a fractional order time derivative in the state space equation of the classical Van der Pol equation as

$$\frac{d^{\lambda}u_1}{dt^{\lambda}} = u_2,$$
$$\frac{du_2}{dt} = -u_1 - \varepsilon(u_1^2 - 1)u_2.$$

which gives us the Van der Pol equation as

$$\frac{d^{1+\lambda}u(t)}{dt^{1+\lambda}} - \varepsilon(1-u^2(t))\frac{d^{\lambda}u(t)}{dt^{\lambda}} + u(t) = 0\ , \quad \text{where } 0 < \lambda < 1\ . \tag{5.4}$$

In the present chapter, the two fractional order time derivatives in the state space equation are considered as

$$\frac{d^{\alpha}u_1}{dt^{\alpha}} = u_2,$$
$$\frac{d^{\alpha}u_2}{dt^{\alpha}} = -u_1 - \varepsilon(u_1^2 - 1)u_2, \quad 0 < \alpha < 1,$$

which generate the fractional order Van der Pol equation as

$$\frac{d^{2\alpha}u(t)}{dt^{2\alpha}} - \varepsilon(1-u^2(t))\frac{d^{\alpha}u(t)}{dt^{\alpha}} + u(t) = 0, \quad 0 < \alpha < 1\ , \tag{5.5}$$

with $u(0) = a$, $\dot{u}(0) = 0$.

Equation (5.5) represents the classical Van der Pol equation for $\alpha = 1$.

In the method HAM proposed by Liao (1992) some parameter terms are used viz., auxiliary linear operator, embedding parameter or Homotopy parameter, initial guess, convergence control parameter, auxiliary parameter etc. In this method there are flexibilities to choose the auxiliary linear parameter, initial guess, auxiliary function and the convergence control parameter. Liao showed the advantages of the method are it is independence of any small or large physical parameters and also provides a convenient way to guarantee the convergence for approximation of series solution. Due to these advantages it can overcome the restrictions and limitations of various existing traditional perturbation and non-perturbation methods. The biggest advantage of the method is the smooth construction of so called zero-th order deformation equation, which is a base of HAM to connect a given non-linear problem and a relatively much simpler linear ones. Keeping in mind these advantages and flexibilities of HAM, an endeavor has been made in this chapter to solve the fractional order Van der Pol equation. The convergence of the series solution (Liao, 2012; Atangana, 2014) with the proper choice of optimal values of convergence control parameter and also the stability analysis of the Van der Pol equation for different fractional order time derivatives through numerical and graphical presentations for different particular cases is the striking feature of this scientific contribution. The remarkable contribution of this study is the presentation of oscillations of the system, which are depicted through phase portraits for various values of control parameters and fractional order derivatives.

5.2 Solution of the problem by HAM

The equation (5.5) can be rewritten as

$$D_t^{2\alpha} u(t) + \varepsilon u^2(t) D_t^{\alpha} u(t) - \varepsilon D_t^{\alpha} u(t) + u(t) = 0 \quad , \ 0 < \alpha \le 1 \ , \tag{5.6}$$

with $u(0) = a,\ \dot{u}(0) = 0$.

The linear auxiliary operator is

$$L[\phi(t,q)] = \frac{\partial^{2\alpha} \phi(t,q)}{\partial t^{2\alpha}} \ , \ t > 0 \ , 0 < \alpha \le 1, \tag{5.7}$$

with the property that

$$L[c] = 0 \ , \tag{5.8}$$

where c is the integrating constant, $\phi(t,q)$ is an unknown function.

The nonlinear operator is defined as

$$N[\phi(t,q)] = D_t^{2\alpha} \phi(t,q) + \varepsilon \phi^2(t,q) D_t^{\alpha} \phi(t,q) - \varepsilon D_t^{\alpha} \phi(t,q) + \phi(t,q) \ . \tag{5.9}$$

Hence the zero-th order deformation equation is

$$(1-q) L[\phi(t,q) - u_0(t)] = q \ \hbar \ N[\phi(t,q)] \ , \tag{5.10}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \ne 0$ is the convergence control parameter, $u_0(t)$ is the initial guess of $u(t)$.

The m-th order deformation equation is

$$L[u_m(t) - \chi_m u_{m-1}(t)] = \hbar \ R_m[\vec{u}_{m-1}(t)] \ , \tag{5.11}$$

with initial condition

$$u_m(0) = 0, \tag{5.12}$$

where χ_m is defined as

$$\chi_m = \begin{cases} 0, m \le 1 \\ 1, m > 1. \end{cases}$$

Therefore solution of the deformation equation is

$$u_m(t) = \chi_m u_{m-1}(t) + \hbar J_t^{2\alpha} R_m[\vec{u}_{m-1}(t)] + c, \tag{5.13}$$

where $J_t^{2\alpha}[f(t)] = \dfrac{1}{\Gamma(2\alpha)}\int_0^t (t-\xi)^{2\alpha-1} f(\xi)d\xi$, c is the integration constant determined from equation. (5.12).

Thus, $R_m[\vec{u}_{m-1}(t)] = D_t^{2\alpha} u_{m-1}(t) + \varepsilon \sum_{i=0}^{m-1}(\sum_{j=0}^{i}(u_j u_{i-j}) D_t^{\alpha} u_{m-1-i} - \varepsilon D_t^{\alpha} u_{m-1}(t) + u_{m-1}(t).$

Taking $u_0 = a$, we get

$$u_1(t) = \frac{\hbar a}{\Gamma(1+2\alpha)} t^{2\alpha} , \tag{5.14}$$

$$u_2(t) = \frac{\hbar(\hbar+1)a}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{\varepsilon a(a^2-1)\hbar^2}{\Gamma(1+3\alpha)} t^{3\alpha} + \frac{\hbar^2 a}{\Gamma(1+4\alpha)} t^{4\alpha} , \tag{5.15}$$

$$u_3(t) = \frac{\hbar(\hbar+1)^2 a}{\Gamma(1+2\alpha)} t^{2\alpha} + [2\varepsilon a(a^2-1)\hbar^2(\hbar+1)]\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + [\varepsilon^2 a(a^2-1)^2 \hbar^3]\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}$$

$$+ [\hbar^3 a + \hbar^2(\hbar+1)a]\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + [2\varepsilon a\hbar^3(a^2-1) + \frac{2\varepsilon a^3\hbar^3\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\,\Gamma(1+\alpha)}]\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}$$

$$+ \frac{\hbar^3 a\, t^{6\alpha}}{\Gamma(1+6\alpha)} , \tag{5.16}$$

$$u_4(t)=\frac{\hbar(\hbar+1)^3\,a}{\Gamma(1+2\alpha)}t^{2\alpha}+[2\varepsilon a\,(a^2-1)\hbar^2(\hbar+1)^2+\hbar^2(\hbar+1)^2\,\varepsilon a(a^2-1)]\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+$$

$$[\hbar^3(\hbar+1)a+\varepsilon^2a(a^2-1)^2\hbar^2(h+1)+2\varepsilon^2a(a^2-1)^2\hbar^3(h+1)+\hbar^2(\hbar+1)^2\,a]\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}$$

$$+[2\varepsilon\hbar^3(\hbar+1)a(a^2-1)+\varepsilon a\hbar^4(a^2-1)+\varepsilon^3a(a^2-1)^3\hbar^3+2\varepsilon a(a^2-1)\hbar^3(\hbar+1)$$

$$+\hbar^3(\hbar+1)\varepsilon a(a^2-1)+\frac{2\varepsilon a^3\hbar^3(\hbar+1)\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\,\Gamma(1+\alpha)}+\frac{2a^3\hbar^3(\hbar+1)\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}+$$

$$+\frac{2\hbar^3(\hbar+1)a^3\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}]\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}$$

$$+[\,\hbar^3(h+1)a+2\varepsilon^2a\,\hbar^4(a^2-1)^2+\hbar^4\,a+a\hbar^3+\varepsilon^2a\hbar^3(a^2-1)^2+\hbar^3(\hbar+1)a$$

$$+\frac{2\varepsilon^2\hbar^4a^3(a^2-1)\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\,\Gamma(1+\alpha)}+\frac{2\varepsilon a^3(a^2-1)\hbar^4\Gamma(1+4\alpha)}{\Gamma^2(1+2\alpha)}$$

$$+\frac{2\varepsilon a^3\hbar^4(a^2-1)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)}]\,\frac{t^{6\alpha}}{\Gamma(1+6\alpha)}$$

$$+[\,(a^2-1)\hbar^4\varepsilon\,a+2\,\varepsilon\,h^4\,a(a^2-1)+\frac{2\varepsilon a^3\hbar^4\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)}+\frac{2\hbar^4a^3\varepsilon\,\Gamma(1+5\alpha)}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)}$$

$$+\frac{2\varepsilon\hbar^4a^3\Gamma(1+5\alpha)}{\Gamma(1+4\alpha)\,\Gamma(1+\alpha)}+\frac{\varepsilon\,\hbar^3a^3\Gamma(1+5\alpha)}{\Gamma^2(1+2\alpha)\Gamma(1+\alpha)}]\,\frac{t^{7\alpha}}{\Gamma(1+7\alpha)}+\frac{\hbar^4a}{\Gamma(1+8\alpha)}\,t^{8\alpha}. \quad (5.17)$$

Proceeding in the similar manner, we can calculate the other components u_n, $n>4$ and hence we get the series solution of the considered problem as

$$u(t)=\lim_{N\to\infty}\phi_N(t)\,, \quad (5.18)$$

where $\phi_N(t)=\sum_{n=0}^{N-1}u_n(t)$, $N\geq 1$.

As given by Liao (1992), at the m-th order of approximation, one can define the exact square residual error as

$$E_m = \int_{\Omega} \left(N\left[\sum_{i=0}^{m} u_i(t) \right] \right)^2 dt \tag{5.19}$$

During numerical computation the limits of t in the equation (5.19) is taken from 0 to 1.

The optimal value of $\hbar$ can be obtained by means of minimizing the so called exact residual error defined by equation (5.19), corresponding to the nonlinear algebraic equation $\frac{\partial E_m}{\partial \hbar} = 0$.

Convergence Theorem: If the series solution defined by the equation (5.18) is convergent then it converges to an exact solution of the nonlinear problem (5.6) as mentioned in Liao (1992) as theorem (4.21) and theorem (4.22)

5.3 Numerical results and discussion

In this section, the numerical results of $u(t)$ for the considered non-linear fractional Van der Pol oscillator equation have been obtained. The optimal values of $\hbar$, for comparison of minimum exact residual errors for $a = 1$, $\varepsilon = 1$ and various values of α are provided through Tables 5.1-5.3 and are displayed through Figs. 5.1-5.3. It is observed from Tables 5.1-5.3 that with increase in the order of approximations, the residual error is decreasing and optimal value of $\hbar$ goes away from $\hbar=-1$. Tables 5.1-5.3 also depict that with decrease in the value of α, residual error is decreasing for $\hbar=-1$.

Order of Approximation	$\hbar$	E_m	E_m at $\hbar=-1$
1	- 1.02178	7.23448×10^{-3}	7.56674×10^{-3}
2	- 0.729311	7.59265×10^{-2}	1.09266×10^{-1}
3	- 0.76059	1.07886×10^{-4}	5.11962×10^{-2}

Table 5.1. Comparison of exact residual error for different values of $\hbar$ at $\alpha=1$.

Order of Approximation	$\hbar$	E_m	E_m at $\hbar=-1$
1	- 1.04575	1.39174×10^{-2}	1.50387×10^{-2}
2	- 0.624875	1.41559×10^{-1}	2.35311×10^{-1}
3	- 0.758726	2.03645×10^{-4}	9.34828×10^{-2}

Table 5.2. Comparison of exact residual error for different values of $\hbar$ at $\alpha=0.75$.

Order of Approximation	$\hbar$	E_m	E_m at $\hbar=-1$
1	- 1.09418	8.14262×10^{-3}	1.21135×10^{-2}
2	- 0.550819	1.87993×10^{-1}	3.65587×10^{-1}
3	- 0.725017	2.56436×10^{-3}	1.22567

Table 5.3. Comparison of exact residual error for different values of $\hbar$ at $\alpha=0.5$.

The phase portraits between u_1 and u_2 are presented through Figs 5.4(a)-(c). It is observed that for $\varepsilon = 1$ and $\alpha = 0.5$, system approaches towards an equilibrium point whereas for $\varepsilon = 1$ and $\alpha = 0.75$ the system gives us a stable limit cycle and with the increase of the values of α from 0.75 to 1 it is seen from Figs 5.4 (b) - (c) that amplitude of the limit cycle is increasing. In Fig. 5.5 drawn for $\alpha = 0.75$ and $\varepsilon = 1\ (1)\ 4$, the same nature is found in the amplitude of the limit cycle. When $\varepsilon = 0.5$, the system approaches towards the equilibrium point at $\alpha = 0.75$ (Fig.5.6 (a)). An interesting phenomenon is observed at $\varepsilon = 0.5, \alpha = 0.85$ and $\varepsilon = 0.5, \alpha = 0.95$. In both occasions limit cycles obtained are displayed through Fig. 5.6(b) and Fig.5. 6 (c). In the first case the path of the orbit approaches towards the limit cycle from outside whereas in the later one the nature is opposite. Again for $\varepsilon \geq 7$, some strange natures are found in the limit cycles at $\alpha = 0.75$ depicted through Fig.5.7 which may be described as bad bands (Guckenheimer, 1980).

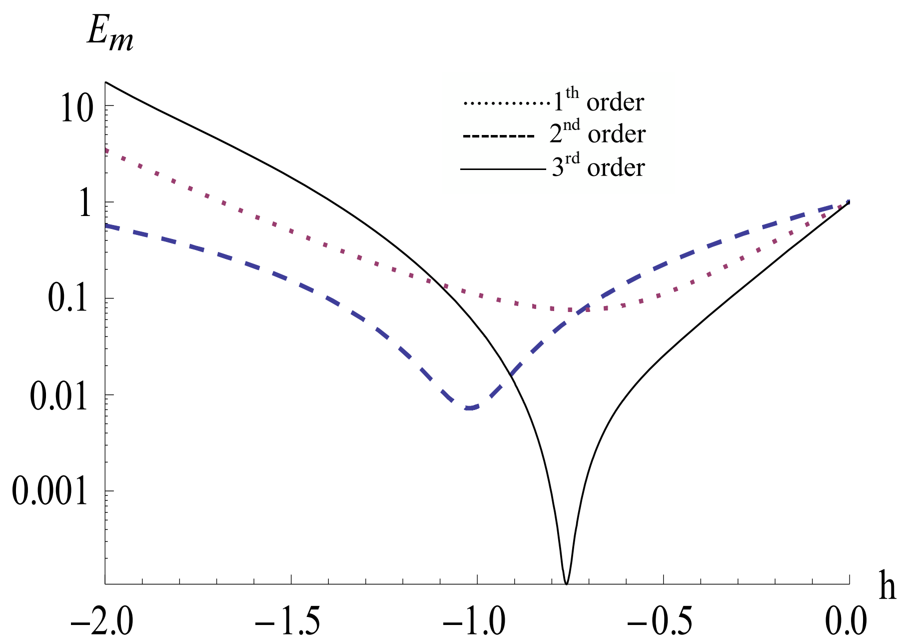


Fig.5.1. Plots of exact residual error E_m vs. $\hbar$ for $a = 1, \varepsilon = 1$ and $\alpha = 1$.

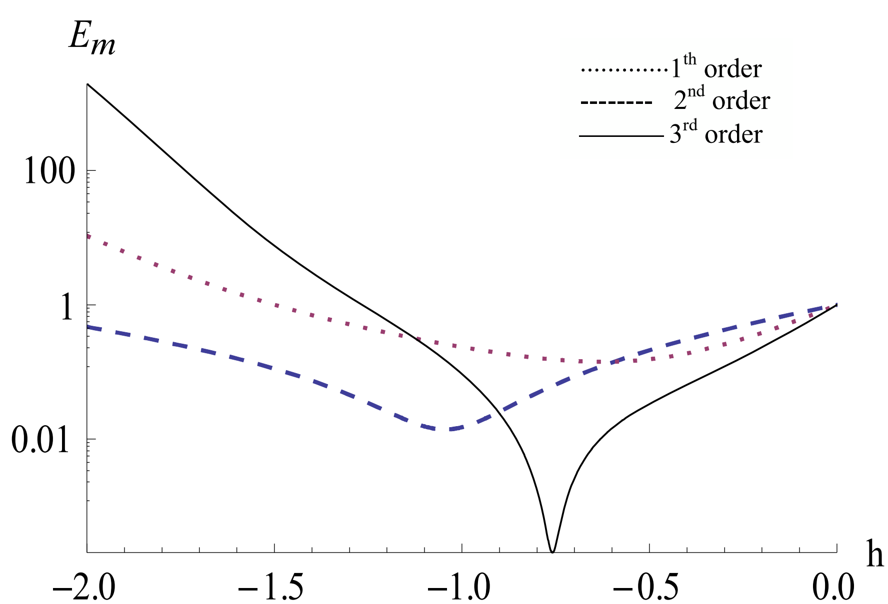


Fig.5.2. Plots of exact residual error E_m vs. $\hbar$ for $a = 1, \varepsilon = 1$ and $\alpha = 0.75$.

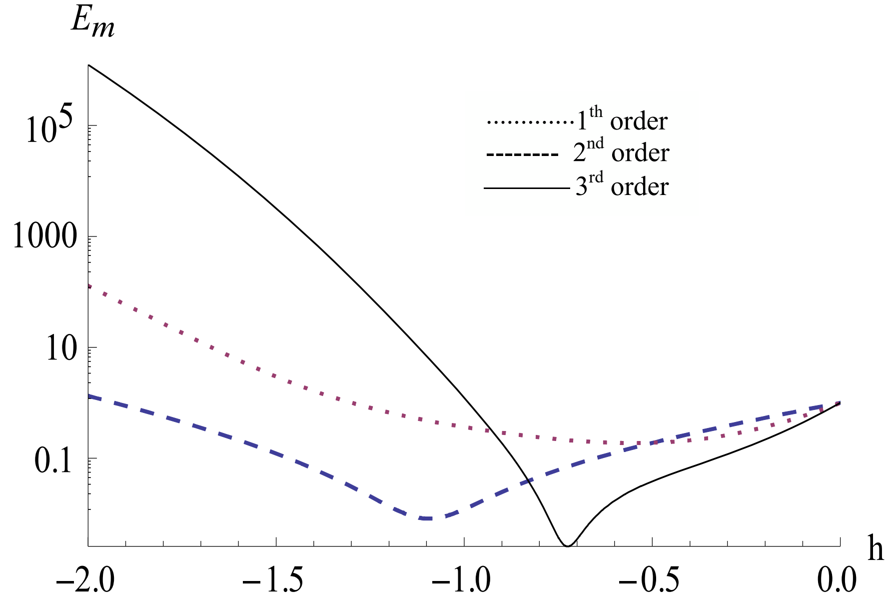
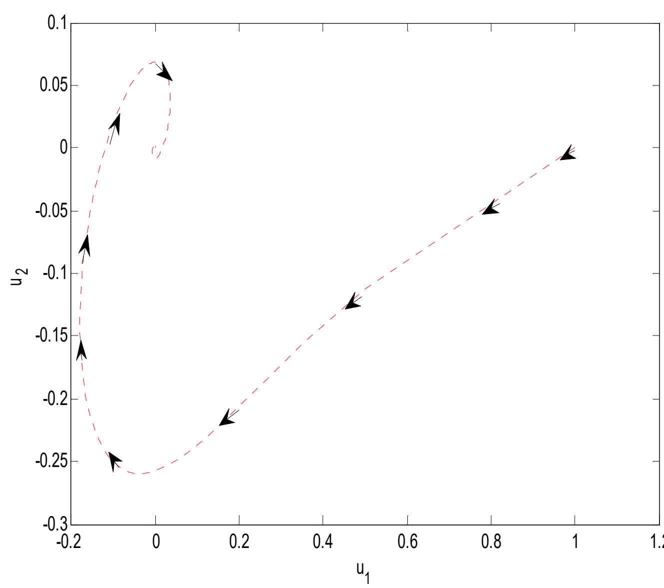
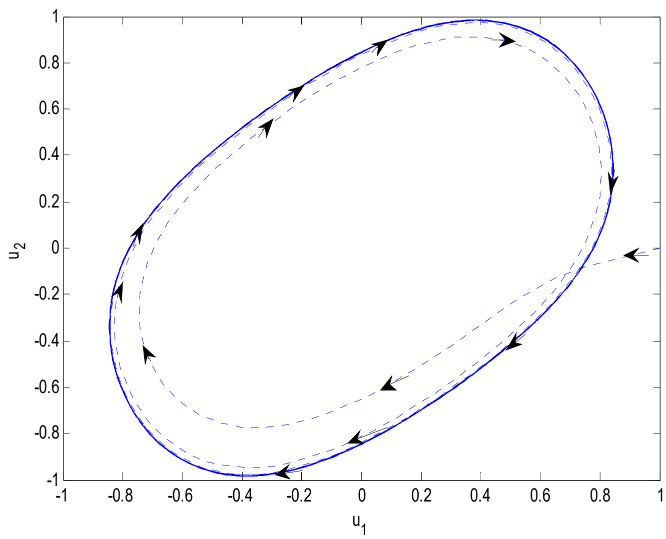


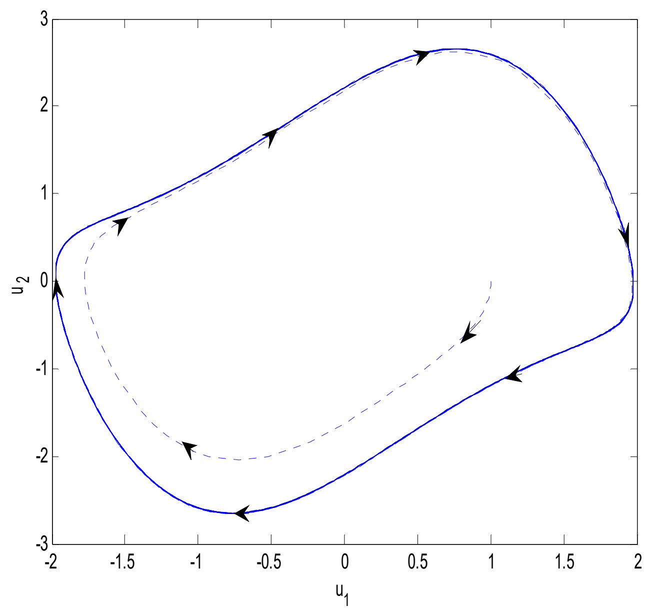
Fig.5.3. Plots of exact residual error E_m vs. $\hbar$ for $a=1, \varepsilon=1$ and $\alpha=0.5$.



(a)



(b)



(c)

Fig.5.4. Phase Portrait between u_1 and u_2 (a) for $a=1, \varepsilon=1$ and $\alpha=0.5$ (b) for $a=1, \varepsilon=1$ and $\alpha=0.75$ (c) for $a=1, \varepsilon=1$ and $\alpha=1$.

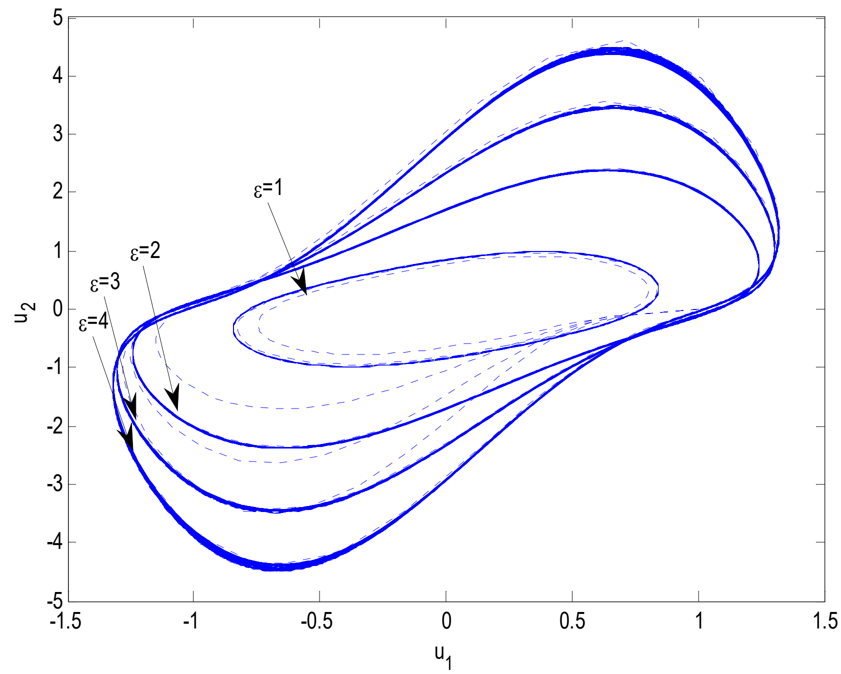
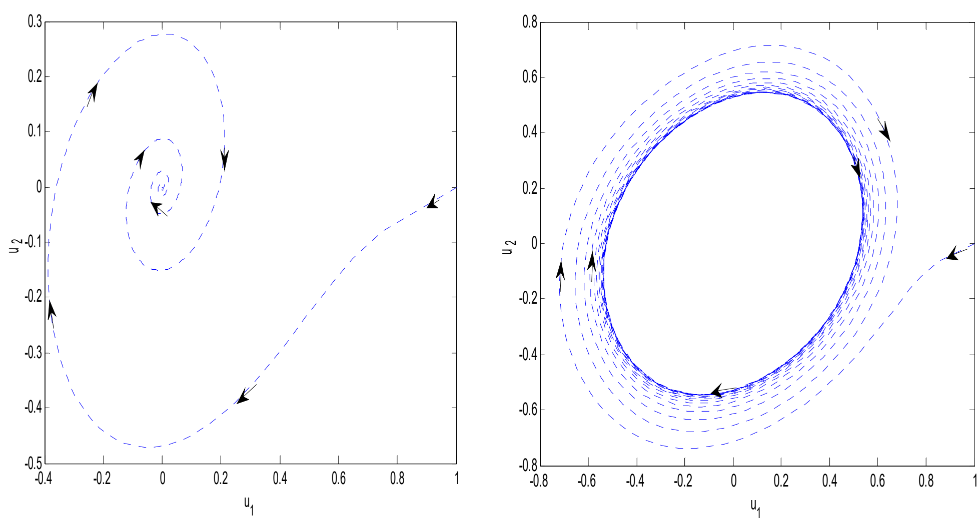
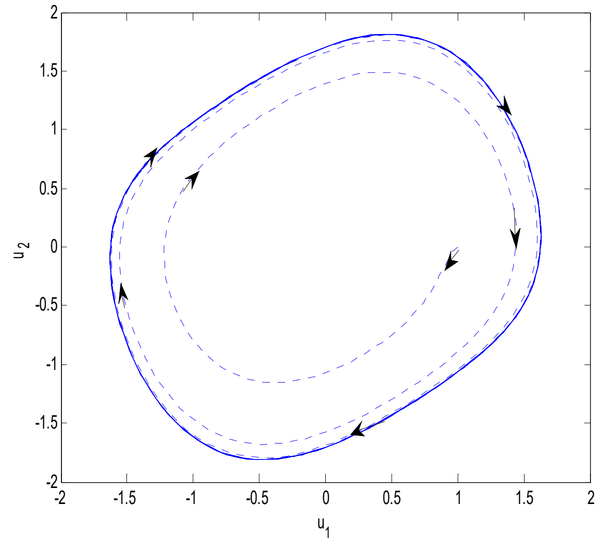


Fig.5.5. Phase Portrait between u_1 and u_2 for $a=1,\ \alpha=0.75$ and $\varepsilon=1,2,3,4$.



(a) (b)



(c)

Fig. 5.6. Phase Portrait between u_1 and u_2 (a) for $a = 1,\ \varepsilon = 0.5$ and $\alpha = 0.75$ (b) for $a = 1,\ \varepsilon = 0.5$ and $\alpha = 0.85$ (c) for $a = 1,\ \varepsilon = 0.5$ and $\alpha = 0.95$.

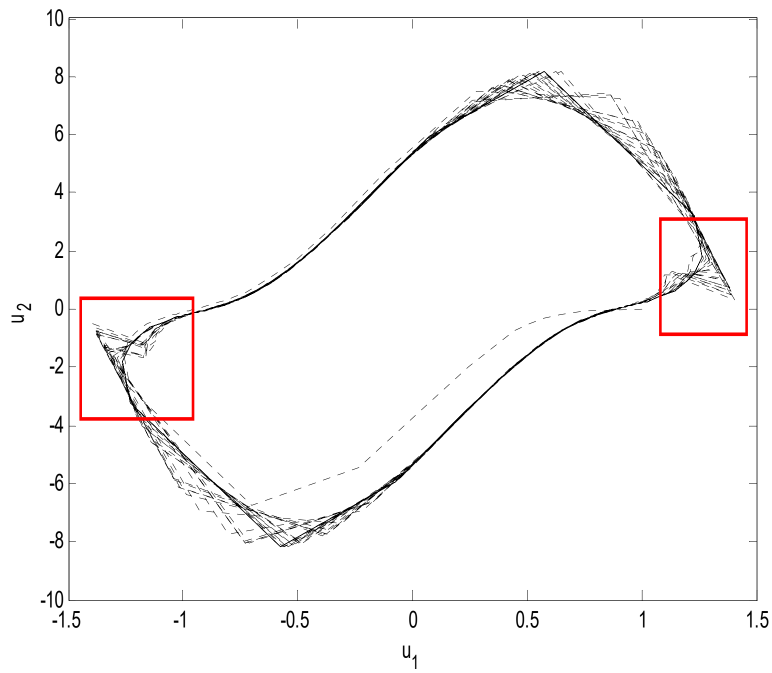


Fig. 5.7. Phase Portrait between u_1 and u_2 (a) for $a = 1,\ \varepsilon = 8$ and $\alpha = 0.75$.

5.4 Conclusion

There are two important goals that have been achieved through the present study. First one is how the convergence of approximate solution can be accelerated using convergence control parameter which demonstrates computationally efficient approximate solutions with low residual errors during the solution of the historical nonlinear equation in fractional order system. This clearly reveals the reliability and potential of the method HAM during the solution of nonlinear partial differential equations even in fractional order system. Second one is the observation of limit cycles for small values of ε when α is close to the standard one, and also the large value of ε when α is close to 0.5, which clearly demonstrate the variations of achieved stable limit cycles of the system with changes in small value of control parameter and higher value of fractional order time derivative to the large value of control parameter and small value of fractional order derivative.