CHAPTER-3

A New Approach in Fractional Variational Iteration Method to Solve the Fractional Order Differential Equations

Chapter 3

A new approach in fractional variational iteration method to solve the fractional order differential equations

3.1 Introduction

Lagrange multiplier method was used by Inokuti *et al.* (1978), with the help of which the variational iteration method (VIM) was proposed by the Chinese scientist, He (1998; 1999). VIM has been successfully used by various authors to solve linear and nonlinear differential equations (He, 1998; 1999; 2000; 2004; Momani and Abuasad, 2006; Wazwaz, 2007; Molliq *et al.* 2009; Porshokouhi *et al.*, 2010; Wu, 2011a) etc. The reliability and simplicity of the method and the reduction in the size of the computation gave the method a wide applicability. He (1998) first applied VIM to solve the fractional order differential equation. Wu and Lee (2010) proposed fractional variational iteration method which is very much useful for solving fractional order differential equations. Wu (2011b) also used fractional variational iteration method to solve fractional order nonlinear differential equations. Khan *et al.* (2011) used fractional variational iteration method for fractional initial boundary value problems arising in the applications of nonlinear science. VIM needs the establishment

The contents of this chapter have been Accepted for publication in ***Journal of Nonlinear Systems and Applications****, 2014*

of correction functional, evaluation of Lagrange's multiplier and the initial approximation. The most challenging step is the calculation of Lagrange's multiplier. Jafari and Abbas (2010) proposed a new method for calculating general Lagrange multiplier in the VIM. Jumarie (2006) proposed a new modified Riemann–Liouville left derivative for fractional order derivative, which is not required to satisfy higher integer-order derivative than α. It is also proposed that α-th derivative of a constant is zero, where α is any arbitrary real number.

In the past century notable work is done in the area related to diffusion equations. An interesting behavior of these diffusion equations is that it generates the Brownian motion (BM). Many authors viz., Wazwaz (2007), Sadighi and Ganji (2007) etc. have used VIM to solve diffusion equations. Fractional diffusion equation is obtained from the classical diffusion equation in mathematical physics by replacing the first order time derivative by a fractional derivative of order α ($0 < \alpha < 1$). It is already described in the previous chapter that the anomalous diffusion is characterized by a diffusion constant whose mean square displacement is calculated as $< x^2(t) > \sim t^\alpha$. Wu (2012a; 2012b) has solved time fractional diffusion equation in porous medium and also fractional order diffusion equation in the case of local verses nonlocal. Das (2009a; 2009b) used VIM to find the solutions of fractional diffusion equation of order α, where $0 < \alpha < 1$ for various initial conditions involving power of x. Later, Vishal and Das (2012) have solved the nonlinear fractional order diffusion equation with absorbent term and external force using optimal homotopy analysis method.

Anomalous diffusion and relaxation behaviors are often described in terms of fractional order equations, and generalized kinetic and stochastic equations. For example, fractional Brownian motion (FBM) is a very useful approach to anomalous diffusion. It represents a random process driven by so-called fractional Gaussian noise. FBM is closely related with the generalized Langevin equation (GLE) for a particle driven by fractional Gaussian noise. An alternative approach to anomalous diffusion is the continuous time random walk (CTRW) which is a random walk subordinated to a renewal process in which each random particle jump is preceded by a random waiting time. The CTRW theory generalizes the results of the standard random walk concept.

Ordinary linear and nonlinear differential equations of fractional order in which unknown functions are operated by derivative of fractional order have already been studied by many researchers (Al-Bassam, 1982; 1986; Campos, 1990; El-Sayed, 1992; 1993; Podlubny, 1999; Elizarraraz and Verde-Star, 2000; Kilbas and Trujillo, 2001; Grin'ko, 1991; Sabatier *et al.*, 2007; Goodrich, 2010; Ahmad and Nieto, 2009; Odibat and Momani, 2006; Kosmatov, 2009; Wang, 2011) for last few decades for the sake of mathematical modeling and useful applications in different branches of science and engineering. The accuracy of the fractional order system is much more than the standard order system since the fractional order system allows great flexibilities in the model than the standard order.

Due to the immense applications of fractional order diffusion equations and differential equations in various areas in science and engineering, I have been motivated to solve those types of equations using a new technique to achieve the solutions in approximate analytical form. Soltani and Shirzadi (2010,) proposed a modification to the VIM for the evaluation of Lagrange multiplier through addition and subtraction of a linear term in the correction functional. In the present chapter the author has proposed correction functional using G. Jumarie's (2006) Modified fractional order derivative and introduced a new idea for the calculations of Lagrangian multiplier using Jumarie's fractional order Taylor's series. In this chapter a sincere attempt has been taken to solve fractional order differential equations using the modified fractional variational iteration method to show the validity of the method. The salient feature of the chapter is finding the solutions of two considered problems in terms of Mittag-leffler function.

3.2 Basic definitions

Definition 3.2.1 Jumarie (2006) defined the fractional derivative in the following limit form

$$f^{\alpha} = \lim_{h \longrightarrow 0} \frac{\Delta^{\alpha}[f(x) - f(0)]}{h^{\alpha}}, 0 < \alpha \leq 1 \ . \tag{3.1}$$

This definition is closed to the standard definition of derivatives, and as a direct result, the α -th derivative of a constant is zero.

Definition 3.2.2 If $f : [0,1] \to R$ is a continuous function and α is a real number in the interval $(0,1)$, then Jumarie's modified Reimann- Liouville fractional derivative (2006) is defined by

$$f^{\alpha}(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} (f(\tau) - f(0)) d\tau \ . \tag{3.2}$$

Definition 3.2.3 The formula for Integration by parts can be used in the fractional calculus as (Jumarie, 2009)

$${}_aI_b^{\alpha} u^{\alpha} v = uv\Big|_a^b - {}_aI_b^{\alpha} uv^{\alpha} \ . \tag{3.3}$$

Definition 3.2.4 Assume that the continuous function $\mathrm{f : R \to R, x \to f\,(x)}$ has a fractional derivative of order $k\alpha$ for any positive integer k and for any α, $0 < \alpha \leq 1$; then(Jumarie ,2006)

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\Gamma(1+\alpha k)} f^{\alpha k}(x), \qquad 0 < \alpha \leq 1, \tag{3.4}$$

which provides the local fractional Mc-Laurin's series as

$$f(x)=\sum_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+\alpha k)}f^{\alpha k}(0),\qquad 0<\alpha\le 1. \tag{3.5}$$

3.3 Algorithm of the proposed method

Consider the fractional order differential equation

$$D_t^{\alpha}u(x,t)+R[u(x,t)]+N[u(x,t)]=k(t), \tag{3.6}$$

where $R[u(x,t)]$ represents the linear terms, $N[u(x,t)]$ represents nonlinear terms in the differential equation and $k(t)$ is a continuous function.

Consider the correction functional as

$$u_{n+1}=u_n+{}_0I_t^{\alpha}\left\{\lambda(D_t^{\alpha}u_n(x,t)+R[u_n(x,t)]+N[u_n(x,t)]-k(t))\right\}. \tag{3.7}$$

Introducing a linear term u_n in equation (3.7), we obtain

$$u_{n+1}=u_n+{}_0I_t^{\alpha}\left\{\lambda(D_t^{\alpha}u_n(x,t)-u_n+\tilde{u}_n+\tilde{R}[u_n(x,t)]+\tilde{N}[u_n(x,t)]-\tilde{k}(t))\right\}. \tag{3.8}$$

It is obvious that the successive approximation $u_n, n\ge 0$ can be established by determining Lagrange multiplier λ. The function $\tilde{u}_n$ is a restricted variation, which means $\delta\tilde{u}_n=0$. The successive approximation $u_{n+1}(x,t),\ n\ge 0$, of the solution $u(x,t)$ will be readily obtained upon using the Lagrange's multiplier and by using any selective function u_0. The initial value $u(x,0)$ is usually used for selecting the zero-th order approximation u_0.

To find the optimal value of λ, we have

$$\delta u_{n+1}=\delta u_n+\delta\,{}_0I_t^{\alpha}\left\{\lambda(D_t^{\alpha}u_n(x,t)\right\}-{}_0I_t^{\alpha}\left\{\lambda\delta u_n\right\}=0.$$

i.e., $\delta u_n + \left\{ \lambda \delta u_n \big|_{\tau=t} - {}_0I_t^{\alpha} \lambda^{\alpha} \delta u_n \right\} - {}_0I_t^{\alpha} \left\{ \lambda \delta u_n \right\} = 0 .$ (3.9)

This yields the stationary conditions

$$1 + \lambda(\tau,t)\big|_{\tau=t} = 0 \ . \tag{3.10}$$

and $\lambda^{\alpha} + \lambda = 0 ,$ (3.11)

which give rise to $\lambda(\tau,t)\big|_{\tau=t} = -1$

and $\lambda^{\alpha}(\tau,t)\big|_{\tau=t} = -\lambda(\tau,t)\big|_{\tau=t} = 1,$

$$\lambda^{2\alpha}(\tau,t)\big|_{\tau=t} = -\lambda^{\alpha}(\tau,t)\big|_{\tau=t} = -1,$$

$$\lambda^{3\alpha}(\tau,t)\big|_{\tau=t} = -\lambda^{2\alpha}(\tau,t)\big|_{\tau=t} = 1 .$$

Putting all these values in the fractional order Taylors series expansion, we obtain

$$\lambda(\tau,t) = \lambda(\tau,t)\big|_{\tau=t} + \lambda^{\alpha}(\tau,t)\big|_{\tau=t} \frac{(\tau-t)^{\alpha}}{\Gamma(1+\alpha)} + \lambda^{2\alpha}(\tau,t)\big|_{\tau=t} \frac{(\tau-t)^{2\alpha}}{\Gamma(1+2\alpha)} + \lambda^{3\alpha}(\tau,t)\big|_{\tau=t} \frac{(\tau-t)^{3\alpha}}{\Gamma(1+3\alpha)} + \ldots$$

$$= -1 + \frac{(\tau-t)^{\alpha}}{\Gamma(1+\alpha)} - \frac{(\tau-t)^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{(\tau-t)^{3\alpha}}{\Gamma(1+3\alpha)} + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(\tau-t)^{k\alpha}}{\Gamma(1+k\alpha)} = (-1)E_{\alpha}[-(\tau-t)^{\alpha}] \ . \tag{3.12}$$

Substituting this value of Lagrangian multiplier in equation (3.7), we get the following iteration formula

$$u_{n+1} = u_n - {}_0I_t^{\alpha} \left\{ E_{\alpha}[-(\tau-t)^{\alpha}](D_t^{\alpha} u_n(x,t) + R[u_n(x,t)] + N[u_n(x,t)] - k(t)) \right\}. \tag{3.13}$$

Starting with the initial approximation $u_0(x,t)=u(x,0)$, we get the successive terms $u_1, u_2, u_3 \ldots$ using this procedure for sufficiently large value of *n*, we get $u_n(x,t)$ as an approximation of the exact solution for large n and finally the exact solution is obtained as

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) \quad . \tag{3.14}$$

Example 3.3.1 Consider the diffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\beta u}{\partial t^\beta} \quad \text{with } 0<\alpha\le 1,\ 1<\beta\le 2, \tag{3.15}$$

where $u(x,0)=f(x)$. (3.16)

Here the correctional functional is taken as

$$u_{n+1} = u_n + {}_0I_t^{\,\alpha}\left\{\lambda\left(\frac{\partial^\alpha u_n}{\partial t^\alpha} - \frac{\partial^\beta u_n}{\partial x^\beta}\right)\right\}. \tag{3.17}$$

Introducing a linear term u_n, we get

$$u_{n+1} = u_n + {}_0I_t^{\,\alpha}\left\{\lambda\left(\frac{\partial^\alpha u_n}{\partial t^\alpha} - u_n + \tilde{u}_n + \frac{\partial^\beta \tilde{u}_n}{\partial x^\beta}\right)\right\}. \tag{3.18}$$

Applying restricted variations to the linear term and space fractional term in equation (3.18), we get

$$\delta u_{n+1} = \delta u_n + \delta\, {}_0I_t^{\,\alpha}\left\{\lambda\left(\frac{\partial^\alpha u_n}{\partial t^\alpha} - u_n + \tilde{u}_n + \frac{\partial^\beta \tilde{u}_n}{\partial x^\beta}\right)\right\}$$

$$= \delta u_n + \lambda\delta u_n - {}_0I_t^{\,\alpha}\lambda^\alpha \delta u_n - {}_0I_t^{\,\alpha}\lambda\delta u_n .$$

The stationary condition yields

$$1+\lambda(\tau,t)\big|_{\tau=t}=0,$$

$$\lambda^{\alpha}+\lambda=0,$$

which finally give the Lagrangian multiplier as

$$\lambda(\tau,t)=(-1)E_{\alpha}[-(\tau-t)^{\alpha}].$$

Thus the correction functional becomes

$$u_{n+1}=u_n-{}_0I_t^{\alpha}\left\{E_{\alpha}(-(\tau-t)^{\alpha})\left(\frac{\partial^{\alpha}u_n}{\partial t^{\alpha}}-\frac{\partial^{\beta}u_n}{\partial x^{\beta}}\right)\right\}. \tag{3.19}$$

Taking $u_0=f(x)$, the next approximation is obtained as

$$u_1=u_0-{}_0I_t^{\alpha}\left\{\sum_{k=0}^{\infty}\frac{(-(\tau-t)^{\alpha})^k}{\Gamma(1+k\alpha)}\left(0-f^{\beta}(x)\right)\right\},$$

$$=u_0-(-t)^{\alpha}f^{\beta}(x)E_{\alpha,\alpha+1}(-(-t)^{\alpha}). \tag{3.20}$$

For the sake of calculation, taking term upto k=2, we get the next iterated value as

$$u_2=u_1-{}_0I_t^{\alpha}\left\{E_{\alpha}(-(\tau-t)^{\alpha})\left(\frac{\partial^{\alpha}u_1}{\partial t^{\alpha}}-\frac{\partial^{\beta}u_1}{\partial x^{\beta}}\right)\right\}$$

$$=f(x)-f^{\beta}(x)(-1)^{\alpha}t^{\alpha}E_{\alpha,\alpha+1}[-(-t)^{\alpha}]-(-1)^{\alpha}f^{\beta}(x)t^{\alpha}E_{\alpha,\alpha+1}\left[-(-t)^{\alpha}\right]-f^{\beta}(x)t^{\alpha}E_{\alpha,\alpha+1}\left[-(-t)^{\alpha}\right]$$

$$-f^{\beta}(x)t^{2\alpha}E_{\alpha,2\alpha+1}\left[t^{\alpha}\right]-(-1)^{\alpha}f^{2\beta}(x)t^{2\alpha}E_{\alpha,2\alpha+1}\left[t^{\alpha}\right]+f^{2\beta}(x)t^{3\alpha}E_{\alpha,3\alpha+1}\left[t^{\alpha}\right] \tag{3.21}$$

Proceeding in the similar way, we get $u_n, n>2$ and thus approximate solution is obtained using equation (3.14).

Example 3.3.2 Wu (2011b): The nonlinear fractional order differential equation

$$y^{2\alpha} = y^2 + 1, \quad 0 < \alpha \le 1, \; 0 \le x \le 1, \tag{3.22}$$

given that y(0)=0 and $y^{\alpha}(0) = 1$.

The Correction functional is taken as

$$y_{n+1} = y_n + {}_0I_x{}^{\alpha}\left\{\lambda\left(y_n{}^{2\alpha} - y_n^2 - 1\right)\right\}. \tag{3.23}$$

Introducing a linear term y_n, we get

$$y_{n+1} = y_n + {}_0I_x{}^{\alpha}\left\{\lambda\left(y_n{}^{2\alpha} - y_n + \tilde{y}_n - \tilde{y}_n^2 - 1\right)\right\}. \tag{3.24}$$

To find the optimal value, we have

$$\delta y_{n+1} = \delta y_n + \delta\, {}_0I_x{}^{\alpha}\left\{\lambda\left(y_n{}^{2\alpha} - y_n + \tilde{y}_n - \tilde{y}_n^2 - 1\right)\right\} = 0, \tag{3.25}$$

which gives

$$1 - \lambda^{\alpha}(\xi, x)\Big|_{\xi = x} = 0, \tag{3.26}$$

$$\lambda\,(\xi, x)\Big|_{\xi = x} = 0, \tag{3.27}$$

$$\lambda^{2\alpha} - \lambda = 0. \tag{3.28}$$

Using fractional order Taylor's series for $\lambda\,(\xi, x)$, we get

$$\lambda\,(\xi,x) = \lambda\,(\xi,x)\Big|_{\xi=x} + \lambda^{\alpha}(\xi,x)\Big|_{\xi=x}\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)} + \lambda^{2\alpha}(\xi,x)\Big|_{\xi=x}\frac{(\xi-x)^{2\alpha}}{\Gamma(1+2\alpha)} +$$

$$\lambda^{3\alpha}(\xi,x)\Big|_{\xi=x}\frac{(\xi-x)^{3\alpha}}{\Gamma(1+3\alpha)} + \lambda^{4\alpha}(\xi,x)\Big|_{\xi=x}\frac{(\xi-x)^{4\alpha}}{\Gamma(1+4\alpha)} + \lambda^{5\alpha}(\xi,x)\Big|_{\xi=x}\frac{(\xi-x)^{5\alpha}}{\Gamma(1+5\alpha)} + \ldots$$

$$= \frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)} + \frac{(\xi-x)^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{(\xi-x)^{5\alpha}}{\Gamma(1+5\alpha)} + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{(\xi - x)^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}$$

$$= (\xi - x)^{\alpha} E_{2\alpha,\alpha+1}[(\xi - x)^{2\alpha}]. \tag{3.29}$$

Hence correction functional (3.23) reduces to

$$y_{n+1} = y_n + {}_0I_x^{\alpha}\{(\xi - x)^{\alpha} E_{2\alpha,\alpha+1}[(\xi - x)^{2\alpha}](y_n^{2\alpha} - y_n^2 - 1)\}. \tag{3.30}$$

Using Mc-laurian series for fractional derivative as

$$y = y(0) + y^{\alpha}(0)\frac{x^{\alpha}}{\Gamma(1+\alpha)},$$

and putting the values of $y(0) = 0, y^{\alpha}(0) = 1,$

we get initial approximation as $y_0 = \frac{x^{\alpha}}{\Gamma(1+\alpha)}.$

Putting *n=0* in equation (3.30), we get the next approximation as

$$y_1 = y_0 + {}_0I_x^{\alpha}\{(\xi - x)^{\alpha} E_{2\alpha,\alpha+1}[(\xi - x)^{2\alpha}](0 - \frac{\xi^{2\alpha}}{\Gamma^2(1+\alpha)} - 1)\}$$

$$= \frac{x^{2\alpha}}{\Gamma(1+\alpha)} + \frac{\Gamma(1+2\alpha)x^{4\alpha}}{\Gamma^2(1+\alpha)} E_{2\alpha,4\alpha+1}(x^{2\alpha}) + x^{2\alpha} E_{2\alpha,2\alpha+1}(x^{2\alpha}).$$

For the sake of calculation, taking term upto *k=0*, we get the next iterated value as

$$y_2 = y_1 + {}_0I_x^{\alpha}\{(\xi - x)^{\alpha} E_{2\alpha,\alpha+1}[(\xi - x)^{2\alpha}](y_1^{2\alpha} - y_1^2 - 1)\}$$

$$= y_1 - \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k\alpha + \alpha + 1)\Gamma^2(1+2\alpha)} {}_0I_x^{\alpha}\{(\xi - x)^{2k\alpha+\alpha} \xi^{4\alpha}\}$$

$$-\frac{\Gamma^2(1+2\alpha)}{\Gamma^4(1+\alpha)\Gamma(1+4\alpha)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(2k\alpha+\alpha+1)}\,{}_0I_x^{\alpha}\{(\xi-x)^{2k\alpha+\alpha}\xi^{8\alpha}\}$$

$$-\frac{2\Gamma(1+2\alpha)}{\Gamma^3(1+\alpha)\Gamma(1+4\alpha)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(2k\alpha+\alpha+1)}\,{}_0I_x^{\alpha}\{(\xi-x)^{2k\alpha+\alpha}\xi^{5\alpha}\}$$

$$-\frac{2}{\Gamma^2(1+\alpha)\Gamma(1+4\alpha)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(2k\alpha+\alpha+1)}\,{}_0I_x^{\alpha}\{(\xi-x)^{2k\alpha+\alpha}\xi^{6\alpha}\}$$

$$-\frac{2}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(2k\alpha+\alpha+1)}\,{}_0I_x^{\alpha}\{(\xi-x)^{2k\alpha+\alpha}\xi^{3\alpha}\}$$

$$=\frac{x^{2\alpha}}{\Gamma(1+\alpha)}+\frac{\Gamma(1+2\alpha)x^{4\alpha}}{\Gamma^2(1+\alpha)}E_{2\alpha,4\alpha+1}(x^{2\alpha})+x^{2\alpha}E_{2\alpha,2\alpha+1}(x^{2\alpha})+\frac{\Gamma(4\alpha+1)x^{6\alpha}}{\Gamma^2(1+2\alpha)}E_{2\alpha,6\alpha+1}(x^{2\alpha})$$

$$+\frac{\Gamma^2(1+2\alpha)\Gamma(8\alpha+1)\ x^{10\alpha}}{\Gamma^4(1+\alpha)\Gamma(1+4\alpha)}E_{2\alpha,10\alpha+1}(x^{2\alpha})+\frac{2\Gamma(1+2\alpha)\ \Gamma(5\alpha+1)\ x^{7\alpha}}{\Gamma^3(1+\alpha)\Gamma(1+4\alpha)}E_{2\alpha,7\alpha+1}(x^{2\alpha})$$

$$+\frac{2\Gamma(6\alpha+1)\ x^{8\alpha}}{\Gamma^2(1+\alpha)\Gamma(1+4\alpha)}E_{2\alpha,8\alpha+1}(x^{2\alpha})+\frac{2\Gamma(3\alpha+1)\ x^{5\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}E_{2\alpha,5\alpha+1}(x^{2\alpha}).$$

Proceeding in the similar way, we get $y_n, n>2$ and thus the approximate solution is obtained using equation (3.14).

Particular case: Taking *k=0* in y_2, we get

$$y_2=\frac{x^{2\alpha}}{\Gamma(1+\alpha)}+\frac{\Gamma(1+2\alpha)x^{4\alpha}}{\Gamma^2(1+\alpha)\Gamma(1+4\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\Gamma(4\alpha+1)x^{6\alpha}}{\Gamma^2(1+2\alpha)\Gamma(1+6\alpha)}$$

$$+\frac{\Gamma^2(1+2\alpha)\Gamma(8\alpha+1)\ x^{10\alpha}}{\Gamma^4(1+\alpha)\Gamma(1+4\alpha)\Gamma(1+10\alpha)}+\frac{2\Gamma(1+2\alpha)\ \Gamma(5\alpha+1)\ x^{7\alpha}}{\Gamma^3(1+\alpha)\Gamma(1+4\alpha)\Gamma(1+7\alpha)}+\frac{2\Gamma(6\alpha+1)\ x^{8\alpha}}{\Gamma^2(1+\alpha)\Gamma(1+4\alpha)\Gamma(1+8\alpha)}$$
$$+\frac{2\Gamma(3\alpha+1)\ x^{5\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+5\alpha)},$$

which is in complete agreement with Wu (2011b).

3.4 Conclusion

In the present chapter a modified VIM is proposed for solving fractional order nonlinear differential equation and linear partial differential equation. The proposed method clearly exhibits the suitability for a large class of fractional order differential equations. The beauty of the chapter is the finding of Lagrangian multiplier in terms of Mittag-Leffler function which helps to obtain the solutions of the considered problems in approximate analytical form.