CHAPTER-2

On the Solution of the Nonlinear Fractional Diffusion-Wave Equation with Absorption: A Homotopy Approach

Chapter 2

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2.1 Introduction

In this chapter, a sincere attempt has been taken to solve the nonlinear fractional diffusion equation with reaction term as

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial}{\partial x}\left(u^{n}(x,t)\frac{\partial u(x,t)}{\partial x}\right) - \int_{0}^{t} a(t-\xi)u(x,\xi)d\xi\,, \tag{2.1}$$

where $u(x, t)$ is a field variable which is assumed to vanish for $t < 0$. The absorbent team related to the reaction diffusion process is described as

$$a(t) = a_0 \frac{t^{-\beta}}{\Gamma(1-\beta)},\ 0<\beta<1, \tag{2.2}$$

which possesses a long time correlation with the exponent β, which may be determined by dynamical mechanism of the physical process. The equation will represent a death process for the sink term as $a_0 > 0$ and a birth process for a source term as $a_0 < 0$ mentioned by Angstmann. *et al.* (2013). The equation (2.1) is said to be fractional diffusion equation for $0 < \alpha < 1$ and fractional wave equation for $1 < \alpha < 2$. The difference between these two cases

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can be seen in the formula for the Laplace transform of the Caputo fractional derivative of order $\alpha \quad (m-1<\alpha \le m, m\in N)$ as

$$L\left[\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}\right]=s^{\alpha}L[u(x,t)]-\sum_{k=0}^{m-1}s^{\alpha-k-1}\left.\frac{\partial^{k}u(x,t)}{\partial t^{k}}\right|_{t=0^{+}} . \tag{2.3}$$

For the case $0<\alpha\le 1$, we have initial condition

$$u(x,0)=x^{k} . \tag{2.4}$$

For the case $1<\alpha\le 2$, we have initial conditions

$$u(x,0)=x^{k}, \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=0 . \tag{2.5}$$

Due to the presence of the reaction term, the equation (2.1) may be useful to investigate, several situations by choosing an appropriate $a(t)$, for example, catalytic processes in regular, heterogeneous, or disordered systems (Havlin Ben-Avraham, 1987; Lee, 1994; Alemany *et al*. 1994). Another example is an irreversible first-order reaction of transported substance so that the rate of removal is $\alpha\rho$ as given by Crank (1956). The above type of anomalous diffusion is a ubiquitous phenomenon in nature and appears in different branches of science and engineering. Equation (2.1) for $\alpha=1$ and without absorption represents a model of plasma diffusion for $n=-1/2$, thermal limit approximation of Carlemans model of the Boltzmann equation for $n=-1$, diffusive in higher polymer systems for $n=-2$, isothermal percolations of perfect gas through a micro-porous medium for $n=1$ and process of melting and evaporation of metals for $n=2$ (Wazwaz,2001). Equation (2.1) for

$n=1$, $a_0=0$ i.e., the nonlinear time-fractional diffusion equation in absence of absorption has exact solution $u(x,t)=x+t$. Similarly for $n=2$ $a_0=0$ the exact solution is $u(x,t)=\frac{x}{\sqrt{(1-4t)}}$ (Das *et al.*, 2011).

For $\alpha=1$, equation (2.1) represents a Fickian or normal diffusion process. When $0<\alpha<1$, $1<\alpha<2$, equation (2.1) describes a diffusion process which is temporally non-Fickian but specially Gaussian. For $\alpha=2$, equation (2.1) represents a wave equation, which is also known as Ballistic diffusion.

Einstein's theory of Brownian motion reveals that the mean square displacement of a particle moving randomly is proportional to time. But after the advancement of fractional calculus, it is seen that the mean square displacement for an anomalous diffusion equation having time fractional derivative grows slowly with time i.e., $\left\langle X^2(t)\right\rangle \sim t^{\alpha}$, where $0<\alpha<1$ is the anomalous diffusion exponent. When $n=0$, equation (2.1) reduces to the linear fractional order diffusion equation as for $0<\alpha<1$. In this case after a lengthy mathematical calculation it is seen that $M_{2k}(t)\sim t^{\alpha k}E_{\alpha-\beta+1,k\alpha+1}(-rat^{\alpha-\beta+1})$, where $r^n=\begin{bmatrix}k\\n\end{bmatrix}$ and $M_{2k+1}(t)=0$ .Thus the mean square displacement $\left\langle X^2(t)\right\rangle \sim t^{\alpha}E_{\alpha-\beta+1,\alpha+1}(-rt^{\alpha-\beta+1})$, where, $E_{\alpha,\beta}(t)$ is the Mittag-Leffler function.

Replacing integer order with fractional order time derivative changes the fundamental concept of time and with it the concept of evolution in the foundations of physics. The fractional order α can be identified and has a physical meaning related to the statistics of waiting times in the Montroll-Weiss theory. The relation was established in two steps. First, it was shown by Hilfer and Anton (1995) that Montroll-Weiss continuous time random walks with a Mittag-Leffler waiting time density are rigorously equivalent to a fractional master equation. Then Hilfer (1995) explained that this underlying random walk model was connected to the fractional time diffusion equation in the usual asymptotic sense of long times and large distances.

A simple model for simulating diffusive phenomena is the random walk approach. A random walker can be regarded as a diffusing particle, performing a random motion, similar to the Brownian motion, on an appropriate discrete lattice in discrete time steps. However, diffusion then is a stochastic process of many moving particles. So we have to simulate not only one diffusing particle, but a large number of particles. Both, the diffusive process and its simulation, can be characterized by the time development of their mean square displacement. It is already mentioned that the anomalous diffusion is characterized by a diffusion constant and the mean square displacement of diffusing species in the form $\left\langle X^2(t)\right\rangle \sim t^{\alpha}$ and the phenomena of anomalous diffusion is usually divided into anomalous sub-diffusion for $0 < \alpha < 1$ and anomalous super-diffusion for $1 < \alpha < 2$. The strictly time fractional diffusion of order α, $0 < \alpha < 1$ generates a class of symmetric densities whose moments of order *2m* are proportional to the $m\alpha$ power of time (Gorenflo *et al.*, 2002). We thus obtain a class of Non-Markovian stochastic processes (they possess a memory!) which exhibit a variance consistent with slow anomalous diffusion. In 1999, applying a fractional order Fokker-Plank equation approach, Metzler *et al.* (1999) have shown that anomalous diffusion is based upon Boltzman statistics. Many researchers have used fractional equations to describe Levy flights or diverging diffusion.

The integer-order model can be viewed as a special case from the more general fractional order model since it can be retrieved by putting all fractional orders of the derivatives equal to unity. In other words, the ultimate behavior of the fractional order system response must converge to the response of the integer-order version of the model. This shows that fractional calculus is the extension of classical mathematics. In the last two decades, fractional differential equations have been widely used by the researchers not only in science and engineering but also in economics and finance. It is also a powerful tool in modeling multi scale problems, characterized by wide time or length scale. The attribute of fractional order systems for which they have gained popularity in the investigation of dynamical systems is that they allow greater flexibility in the model. An integer order differential operator is a local operator whereas the fractional order differential operator is non-local in the sense that it takes into account the fact that the future state not only depends upon the present state but also upon all of the history of its previous states. An important characteristic of these evolution equations is that they generate the fractional Brownian motion which is a generalization of Brownian motion. For physical systems, one should have to keep in mind two things for application of fractional order in the system for making a decisive step for the penetration of mathematics of fractional calculus into a body of natural sciences. Firstly to analyze the importance and physical influence of the memory effects on time or space or both and secondly to give proper interpretation of general meaning of non integer operator. The main advantage of the fractional calculus is that fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes.

Fractional derivatives and integrals are useful to explore the characteristic features of anomalous diffusion, transport and fractal walks through setting up of fractional kinetic equations, master equations etc. Fractional kinetic equations have proved particularly useful in the context of anomalous sub-diffusion (Metzler and Klafter, 2000). The fractional diffusion equation, which demonstrates the prevalence of anomalous sub-diffusion, has led to an intensive effort in recent years to find the solution accurately in straight forward manner (Langlands and Henry,2005). The fractional diffusion equation is valuable for describing reactions in the dispersive transport media (Yuste and Lindenberg, 2001). Anomalous diffusion processes occur in many physical systems for various reasons including disorder in terms of energy or space or both (Weiss, 1994; Hughes, 1995). Fractional reaction-diffusion equations or continuous time random walk models are also introduced for the description of nonlinear reactions, propagating fronts and two species reactions in sub-diffusive transport media (Henry and Wearne, 2000). Chen *et al.* (2007) proposed an implicit difference approximation scheme (ISAS) for solving fractional diffusion equation, where the stability and convergence of the method had been analyzed by Fourier method. Schot *et al.* (2007) have given an approximate solution of the diffusion equation in terms of Fox H-function. Zahran (2009) has offered a closed form solution in Fox H-function of the generalized fractional reaction-diffusion equation subjected to an external linear force field, one that is used to describe the transport processes in disorder systems. It is to be noted that some works on fractional diffusion equations have already been done by Angulo *et al.* (2000), Pezat and Zabczyk (2000), Schneider and Wyss (1989),Yu and Zhang (2006), Mainardi (1996), Mainardi *et al*. (2001), Anh and Leonenko (2003). Recently, Das (2012) has solved the fractional order nonlinear reaction diffusion equation using a mathematical tool variational iteration method and has shown that sub-diffusions occur even for cubic order non-linearity and also cubic order of x in the initial condition.

The theory of fractional time evolutions describes novel three parameters susceptibility functions which contain only a single stretching exponent. Also it shows that two widespread characteristics of relaxation spectra in the glass forming materials, (Hilfer, 2002; Hilfer, 2003a) has shown that a power-law tail in the waiting time density is not sufficient to guarantee the emergence of the propagator of fractional diffusion in the continuum limit. Another work of Hilfer (2003b) underlines that fractional relaxation equations providing a promising mathematical framework for slow and glassy dynamics. In particular fractional susceptibilities are seen to reproduce not only broadening or stretching of the relaxation peaks but also the high frequency wing and shallow minima observed in the experiment.

Homotopy Analysis Method (HAM) is an analytical approach to get the series solutions of linear and nonlinear differential equations. The difference with the other perturbation methods is that this method is independent of small/large physical parameters. Another important advantage of this method as compared to the other existing perturbation and non-perturbation method lies in the flexibility to choose proper base function to get better approximate solution of the problems. This method offers certain advantages over routine numerical methods. Numerical methods use discretization which give rise to rounding off errors causing loss of accuracy, and require large computer memory and time. This computational method is better since it does not involve discretization of the variables and hence is free from rounding off errors and does not require large computer memory or time .This method has been successfully applied by many researchers for solving linear and non-linear partial differential equations,(Liao ,1995; 1997; 2003; 2009) and by Das *et al*. (2011). Reaction-diffusion appears during the propagation of flames and migration of biological species. Tumor growths are the examples of such phenomena. Therefore, in this chapter a drive is taken to see the nature of these types of equations with memory effect due to the presence of fractional order time derivatives after solving the equation using HAM technique. The salient feature of the chapter is the graphical presentations and numerical discussions of the damping behaviors of the field variable $u(x,t)$ in order to obtain sub-diffusion of the time fractional nonlinear equations due to the presence of various parameters of physical interest.

2.2 Solution of the problem by homotopy analysis method

Taking the Laplace transform on both sides of equation (2.1), we get

$$L\left[\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}\right]=L\left[\frac{\partial}{\partial x}\left(u^{n}(x,t)\frac{\partial u(x,t)}{\partial x}\right)\right]-L\left[\int_{0}^{t}a(t-\xi)u(x,\xi)\,d\xi\right]. \tag{2.6}$$

Now for $0<\alpha<1$, we have

$$L\left[u(x,t)\right]=\frac{1}{s}u(x,0)+\frac{1}{s^{\alpha}}L\left[\frac{\partial}{\partial x}\left(u^{n}(x,t)\frac{\partial u(x,t)}{\partial x}\right)\right]-\frac{1}{s^{\alpha}}L\left[\int_{0}^{t}a(t-\xi)u(x,\xi)\,d\xi\right], \quad (2.7)$$

and for $1<\alpha<2$, we have

$$L\left[u(x,t)\right]=\frac{1}{s}u(x,0)+\frac{1}{s^{2}}\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}+\frac{1}{s^{\alpha}}L\left[\frac{\partial}{\partial x}\left(u^{n}(x,t)\frac{\partial u(x,t)}{\partial x}\right)\right]$$

$$-\frac{1}{s^{\alpha}}L\left[\int_{0}^{t}a(t-\xi)u(x,\xi)\,d\xi\right]. \quad (2.8)$$

In the view of equations (2.4) and (2.5), equations (2.7) and (2.8) reduce to

$$L\left[u(x,t)\right]=\frac{x^{k}}{s}+\frac{1}{s^{\alpha}}L\left[\frac{\partial}{\partial x}\left(u^{n}(x,t)\frac{\partial u(x,t)}{\partial x}\right)\right]-\frac{1}{s^{\alpha}}L\left[\int_{0}^{t}a(t-\xi)u(x,\xi)\,d\xi\right]. \quad (2.9)$$

Now taking the Inverse Laplace transform, we have

$$u(x,t)=x^{k}+L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\frac{\partial}{\partial x}\left(u^{n}(x,t)\frac{\partial u(x,t)}{\partial x}\right)\right]\right]-L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\int_{0}^{t}a(t-\xi)u(x,\xi)\,d\xi\right]\right]. \quad (2.10)$$

To solve the equation (2.10) by HAM, we choose the linear auxiliary operator as

$$\tilde{L}[\phi(x,t,q)]=\phi(x,t,q), \quad (2.11)$$

where $\phi(x,t,q)$ is an unknown function. Furthermore, in the view of equation (2.7), we have defined the nonlinear operator as

$$N\left[\phi(x,t;q)\right]=\phi(x,t;q)-x^{k}-L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\frac{\partial}{\partial x}\left(\phi^{n}(x,t;q)\frac{\partial\phi(x,t;q)}{\partial x}\right)\right]\right]$$

$$+L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\int_{0}^{t}a(t-\xi)\phi(x,\xi;q)\,d\xi\right]\right]. \quad (2.12)$$

Now we construct the zero-order deformation equation as

$$(1-q)\widetilde{L}[\phi(x,t;q)-u_0(x,t)]=q\hbar N[\phi(x,t;q)] \quad . \tag{2.13}$$

Differentiating the zero-order equation (2.13) m-times with respect to q and then dividing it by m! and finally setting q=0 , one has the so called m-th order deformation equation as

$$\widetilde{L}[u_m(x,t)-\chi_m u_{m-1}(x,t)]=\hbar R_m[\vec{u_{m-1}}(x,t)] \quad . \tag{2.14}$$

with the initial condition

$$u_m(x,0)=0, \tag{2.15}$$

where

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases} \tag{2.16}$$

and $\hbar$ is non-zero auxiliary parameter.

For n=1,

$$R_m[u_{m-1}(x,t)]=u_{m-1}-(1-\chi_m)x^k+L^{-1}\left[\frac{1}{s^\alpha}L\left[\int_0^t a(t-\xi)u_{m-1}(x,\xi)d\xi\right]\right]$$

$$-L^{-1}\left[\frac{1}{s^\alpha}L\left[\sum_{i=0}^{m-1}\left(\frac{\partial u_i}{\partial x}\frac{\partial u_{m-1-i}}{\partial x}\right)\right]+L\left[\sum_{i=0}^{m-1}u_i\frac{\partial^2 u_{m-1-i}}{\partial x^2}\right]\right]. \tag{2.17}$$

and for n=2,

$$R_m[u_{m-1}(x,t)]=u_{m-1}-(1-\chi_m)x^k+L^{-1}\left[\frac{1}{s^\alpha}L\left[\int_0^t a(t-\xi)u_{m-1}(x,\xi)d\xi\right]\right]$$

$$-L^{-1}\left[\frac{1}{s^\alpha}L\left[2\sum_{i=0}^{m-1}\left(\sum_{j=0}^{i}\frac{\partial u_j}{\partial x}\frac{\partial u_{i-j}}{\partial x}\right)u_{m-1-i}\right]\right]$$

$$-L^{-1}\left[\frac{1}{s^{\alpha}}L\left[2\sum_{i=0}^{m-1}\left(\sum_{j=0}^{i}u_j u_{i-j}\right)\frac{\partial^2 u_{m-1-i}}{\partial x^2}\right]\right] \quad . \tag{2.18}$$

Applying the idea of homotopy- analysis method, we have

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+\hbar R_m(u_{m-1}(x,t))+c\ , \tag{2.19}$$

where c, the integration constant c is determined by the initial condition (2.15)

2.2.1 Case-I (for n=1) : Taking $u_0(x,\ t) = x^k$,

then from equation (2.19), the values $u_m(x,t)$ for m=1, 2, 3, . . .can be obtained

as

$$u_1(x,t)=\hbar\left(-k(2k-1)x^{2k-2}\frac{t^{\alpha}}{\Gamma(\alpha+1)}+a_0x^k\frac{t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right),$$

$$\begin{aligned}u_2(x,t)=&\ \hbar(1+\hbar)\left(-k(2k-1)x^{2k-2}\frac{t^{\alpha}}{\Gamma(\alpha+1)}+a_0x^k\frac{t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right)\\ &+3\hbar^2k\left(6k^3-13k^2+9k-2\right)x^{3k-4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &+(a_0\hbar)^2x^k\frac{t^{2\alpha-2\beta+2}}{\Gamma(2\alpha-2\beta+3)}-3a_0\hbar^2k(2k-1)x^{2k-2}\frac{t^{2\alpha-\beta+1}}{\Gamma(2\alpha-\beta+2)},\end{aligned}$$

and so on.

2.2.2 Case-II (for n=2) : Taking $u_0(x,\ t) = x^k$,

then from equation (2.19), the values $u_m(x,t)$ for m=1,2,3,...can be obtained as

$$u_1(x,t)=\hbar\left(-k(3k-1)x^{3k-2}\frac{t^{\alpha}}{\Gamma(\alpha+1)}+a_0x^k\frac{t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right),$$

$$u_2(x,t)=\hbar(1+\hbar)\left(-k(3k-1)x^{3k-2}\frac{t^{\alpha}}{\Gamma(\alpha+1)}+a_0x^k\frac{t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right)$$
$$+\hbar^2k\left(75k^3-100k^2+43k-6\right)x^{5k-4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+(a_0\hbar)^2x^k\frac{t^{2\alpha-2\beta+2}}{\Gamma(2\alpha-2\beta+3)}-4a_0\hbar^2k(3k-1)x^{3k-2}\frac{t^{2\alpha-\beta+1}}{\Gamma(2\alpha-\beta+2)},$$

and so on.

Finally, the m-th order approximation series solution is given as

$$\tilde{u}_m(x,t)=\sum_{k=0}^{m}u_k(x,t). \tag{2.23}$$

2.3 Numerical results and discussion

In this section, numerical results of the field variable *u(x,t)* for different values of fractional order derivative α are calculated for the parameters' values $\beta=0.5,\ k=2,\ \hbar=-1$ at *x*=0.5 and these results are depicted through Fig. 2.1 and Fig. 2.2 at $\alpha=0.6(0.1)0.9$ for *n*=1 and *n*=2 respectively, Fig. 2.3 and Fig.2. 4 at $\alpha=1.2(0.1)1.5$ for *n*=1 and *n*=2 when $a_0=10$ and also through Figs. 2.5 – 2.8 with similar conditions when $a_0=-10$, when the degree of non-linearity is one i.e., *n*=1, it is seen from Fig. 2.3 that even for $\alpha>1$ due to the effect of sink term $a_0>0$ the sub-diffusions are observed with lesser overshoots than those for $\alpha<1$ (Fig.2.1). If the degree of non-linearity increases then similar types of results are found from Fig. 2.2 and Fig. 2.4 with much greater overshoots of sub-diffusion. It is also observed from Figs. 2.5 -2.8 that even for $\alpha<1$ the super-diffusions are found due to the effect of source term $a_0 < 0$. Figs. 2.9 – 2.12 demonstrate the variations of Mean square displacement $\langle X^2(t) \rangle$ with time t for linear fractional order system n=0 in the presence of sink (death) and source (birth) terms. It is seen from the figures that there are behavioral changes of $\langle X^2(t) \rangle$ for death and birth processes. The figures clearly justify the occurrence of anomalous behavior of linear diffusion equation in the fractional order system. Tables 2.1 and 2.2 show comparison between the approximate and exact values for n=1 and n=2 in the absence of reaction term i.e., $a_0 = 0$, which clearly exhibit the fact that even six order terms of the approximation of the solutions are sufficient to get good approximation to the exact solution. It is evident that the accuracy can further be enhanced by computing few more terms of the approximate solutions.

t	x	u(x,t) at $\alpha=\frac{1}{2}$	u(x,t)at $\alpha=1$	Exact solution at $\alpha=1$
0.0	-1.0	-1.0	-1.0	-1.0
0.0	-0.5	-0.5	-0.5	-0.5
0.0	0.0	0.0	0.0	0.0
0.0	0.5	0.5	0.5	0.5
0.0	1.0	1.0	1.0	1.0
0.5	-1.0	-0.2021	-0.5	-0.5
0.5	-0.5	0.2979	0.0	0.0
0.5	0.0	0.7979	0.5	0.5
0.5	0.5	1.2979	1.0	1.0
0.5	1.0	1.7979	1.5	1.5
1.0	-1.0	0.1284	0.0	0.0
1.0	-0.5	0.6284	0.5	0.5
1.0	0.0	1.1284	1.0	1.0
1.0	0.5	1.6284	1.5	1.5
1.0	1.0	2.1284	2.0	2.0

Table 2.1. Comparison of the HAM solution with the exact solution for n = 1.

t	x	$u(x,t)$ at $\alpha = \frac{1}{2}$	$u(x,t)$ at $\alpha = 1$	Exact solution at $\alpha = 1$
0.0	-1.0	-1.0	-1.0	-1.0
0.0	-0.5	-0.5	-0.5	-0.5
0.0	0.0	0.0	0.0	0.0
0.0	0.5	0.5	0.5	0.5
0.0	1.0	1.0	1.0	1.0
0.1	-1.0	-52.945	-1.2905	-1.2910
0.1	-0.5	-26.473	-0.6452	-0.6455
0.1	0.0	0.0	0.0	0.0
0.1	0.5	26.473	0.6452	0.6455
0.1	1.0	52.945	1.2905	1.2910
0.2	-1.0	-341.268	-2.0515	-2.2361
0.2	-0.5	-170.634	-1.0259	-1.1180
0.2	0.0	0.0	0.0	0.0
0.2	0.5	170.634	1.0259	1.1180
0.2	1.0	341.268	2.0518	2.2361

Table 2.2. Comparison of the HAM solution with the exact solution for n = 2.

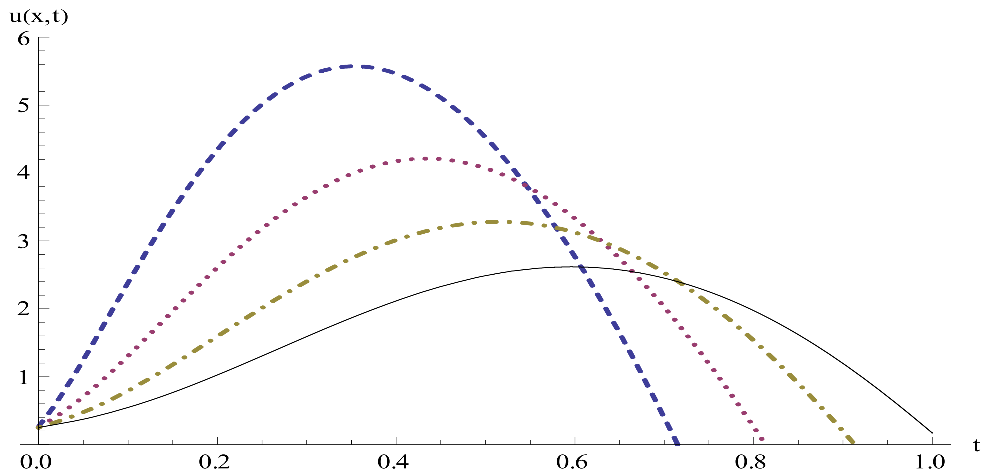

Fig. 2.1. Plots of $u(x, t)$ vs. t for a_0 = 10, β = 0.5, k = 2, n = 1, x = 0.5 and for different values of α.

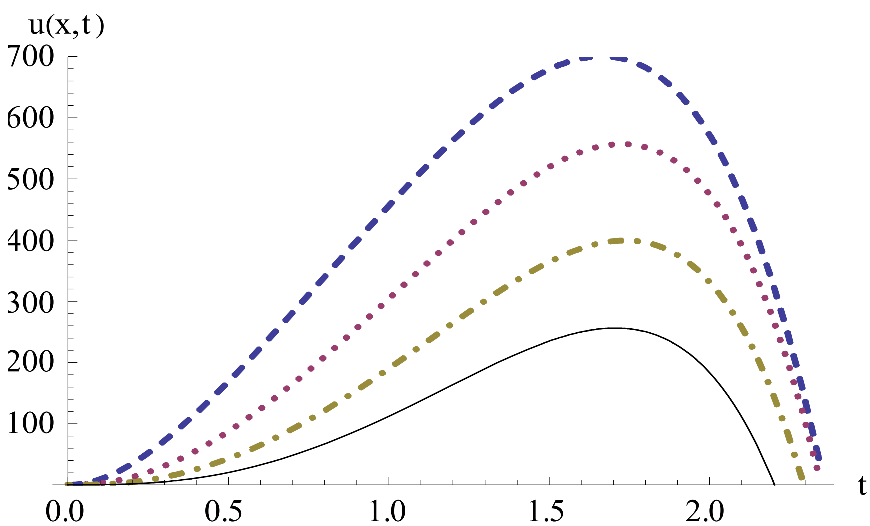

Fig. 2.2. Plots of $u(x, t)$ vs. t for a_0 = 10, β = 0.5, k = 2, n = 2, x = 0.5 and for different values of α.

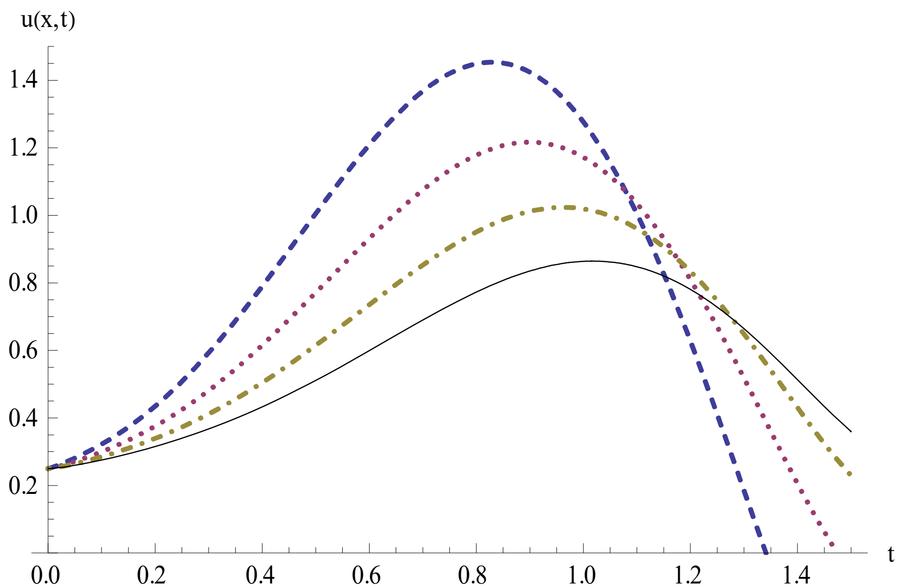

Fig. 2.3. Plots of $u(x, t)$ vs. t for a_0 = 10, β = 0.5, k = 2, n = 1, x = 0.5 and for different values of α .

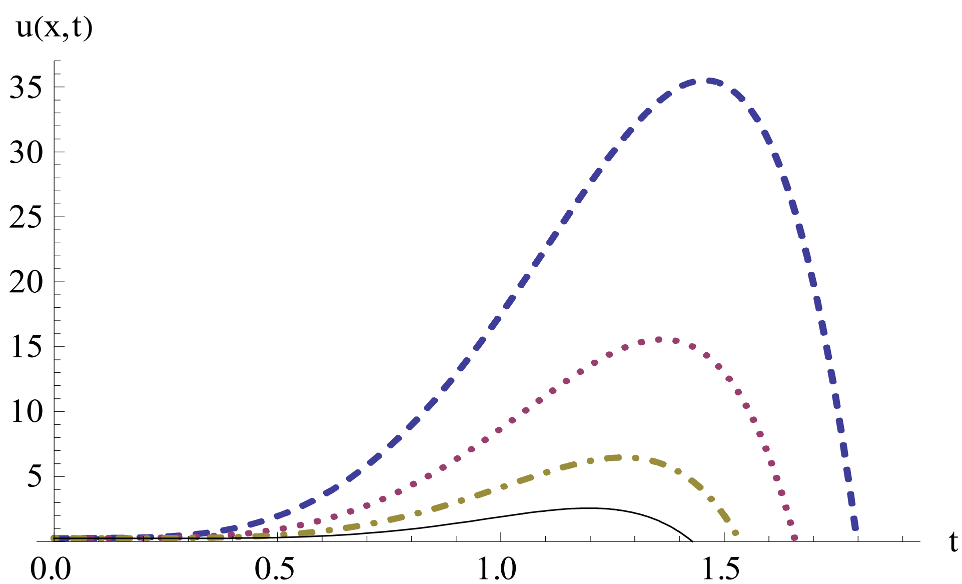

Fig. 2.4. Plots of $u(x, t)$ vs. t for a_0 = 10, β = 0.5, k = 2, n = 2, x = 0.5 and for different values of α .

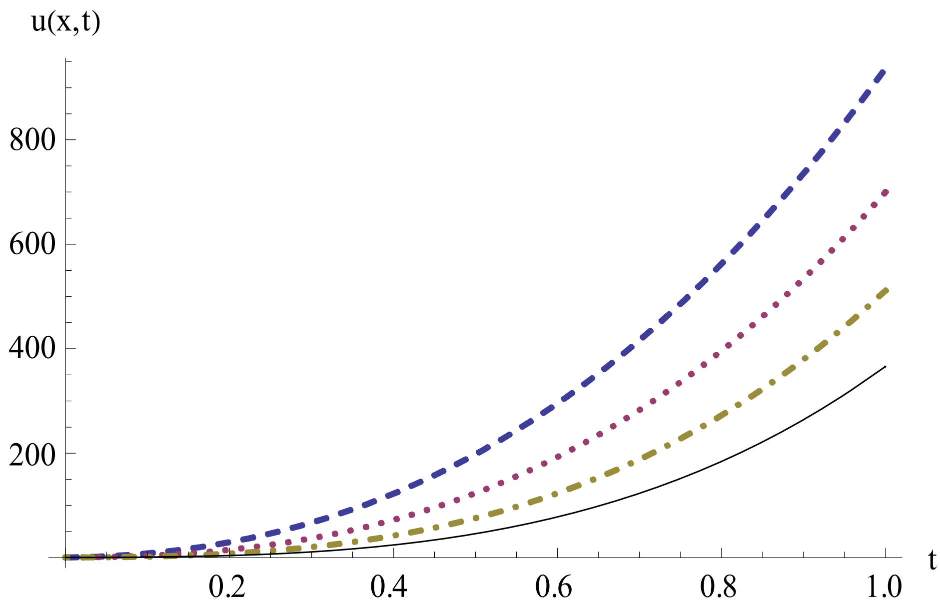

Fig. 2.5. Plots of $u(x, t)$ vs. t for a_0 = -10, β = 0.5, k = 2, n = 1, x = 0.5 and for different values of α .

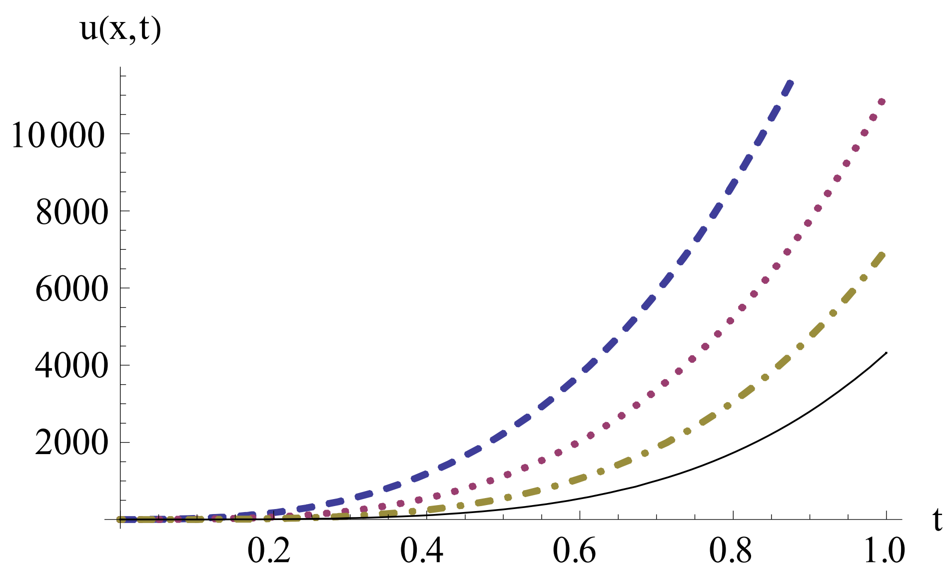

Fig.2.6. Plots of $u(x, t)$ vs. t for a_0 = -10, β = 0.5, k = 2, n = 2, x = 0.5 and for different values of α .

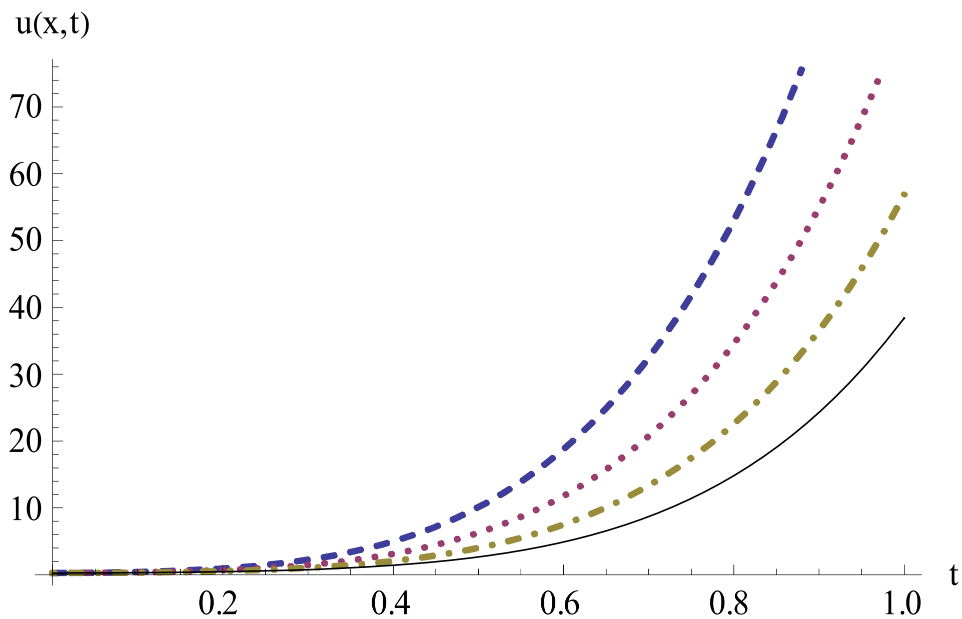

Fig. 2.7. Plots of $u(x, t)$ vs. t for a_0 = -10, β = 0.5, k = 2, n = 1, x = 0.5 and for different values of α .

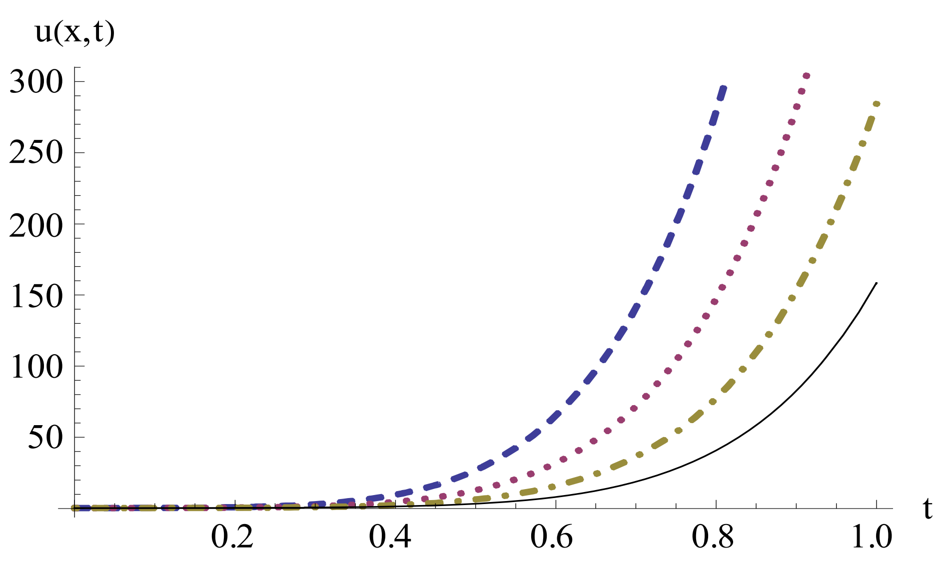

Fig. 2.8. Plots of $u(x, t)$ vs. t for a_0 = -10, β = 0.5, k = 2, n = 2, x = 0.5 and for different values of α .

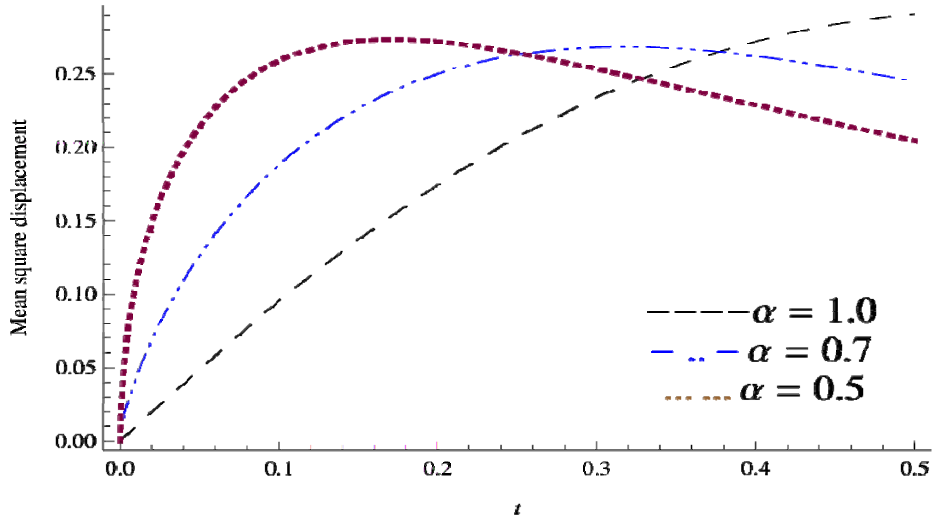

Fig. 2.9. Plots of $\langle X^2(t) \rangle$ vs. t for $a_0 = 5, \beta = 0.5$ and for different values of α .

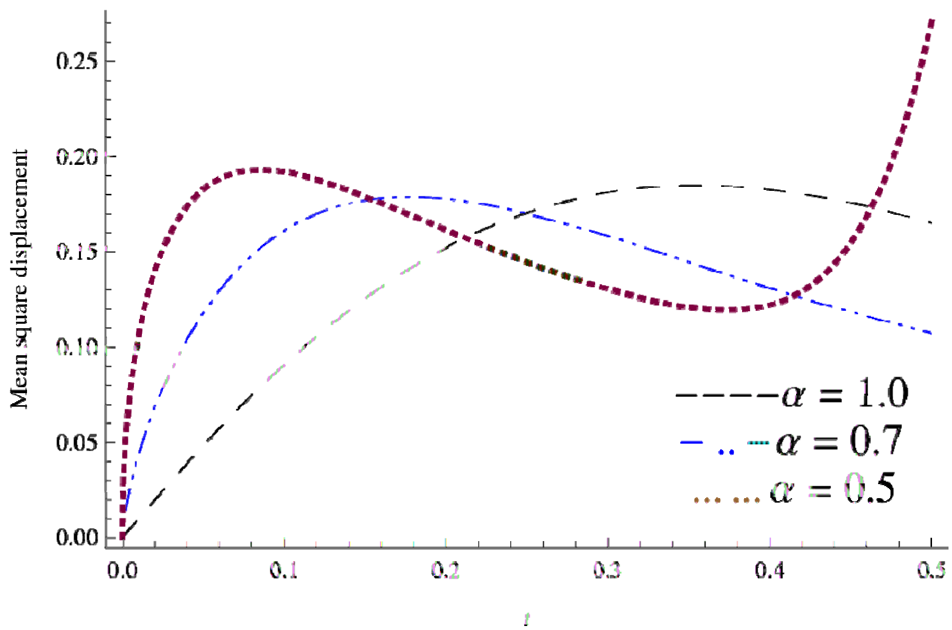

Fig. 2.10. Plots of $\langle X^2(t) \rangle$ vs. t for $a_0 = 10, \beta = 0.5$ and for different values of α

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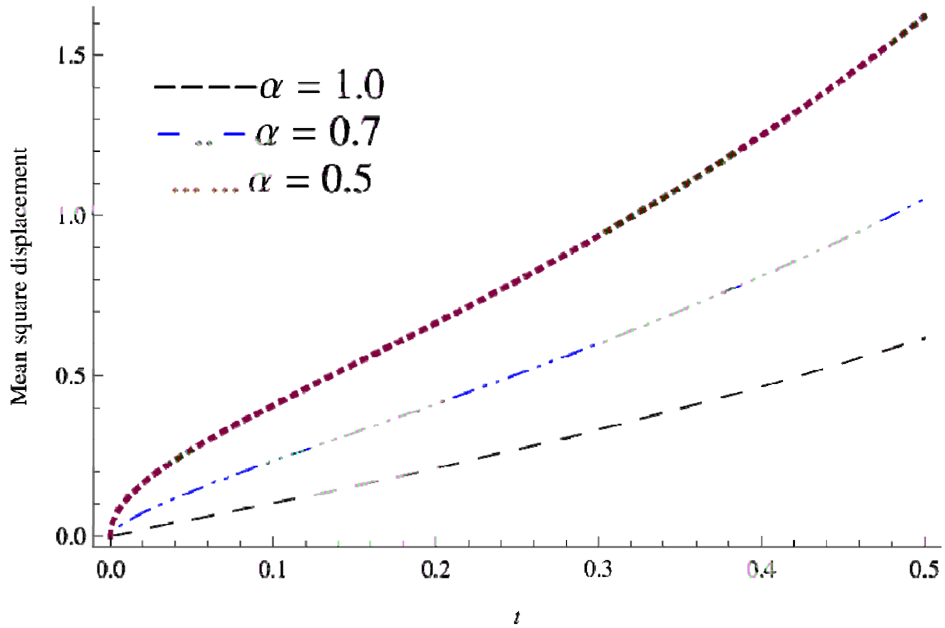

Fig. 2.11. Plots of $\langle X^2(t) \rangle$ vs. t for a_0 = -2, β = 0.5 and for different values of α .

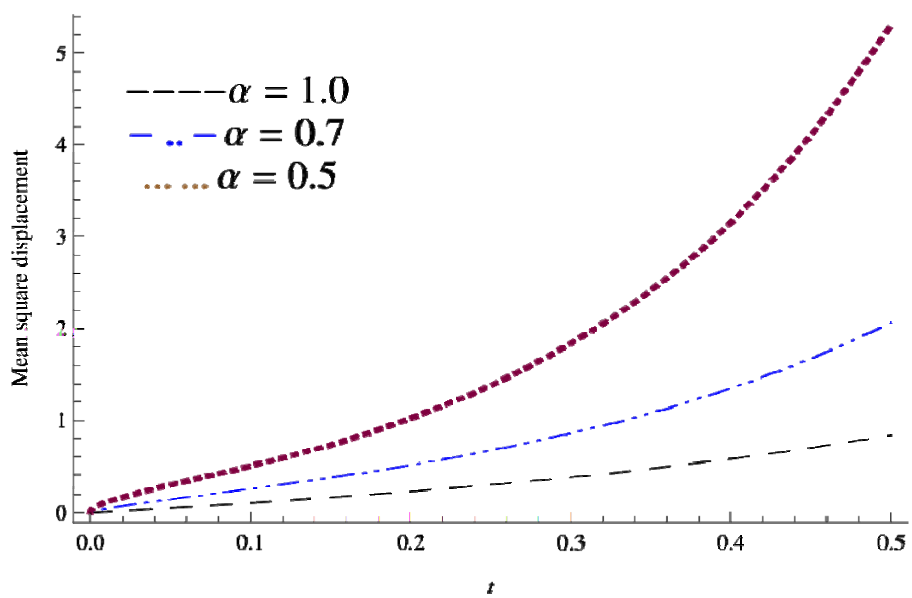

Fig. 2.12. Plots of $\langle X^2(t) \rangle$ vs. t for a_0 = -5, β = 0.5 and for different values of α .

2.4 Conclusion

There are four important goals that have been achieved through the study of the present chapter. First one is the successful presentation of the effects of the reaction term on the nonlinear fractional order diffusion-wave solutions. Second one is the graphical presentations of the sub-diffusion and super-diffusion for different particular cases for both birth and death processes. Third one is the study of mean square displacement which justifies the anomalous nature of fractional order diffusion processes for linear as well as nonlinear cases. Fourth one is the tabular presentation of the comparison of the approximate solutions for some particular cases with the exact solutions, which clearly reveals the reliability and effectiveness of our considered method HAM.