



# **CHAPTER-1**

## **Introduction**

# Chapter 1

## Introduction

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### 1.1 Linear and Nonlinear PDE

A Partial differential equation (PDE) is an equation which involves an unknown function of two or more variables and certain of its partial derivatives. A second order partial differential equation in two independent variables in general is written as

$$f(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0, \text{ where, } u_x = \frac{\partial u}{\partial x}, u_{xt} = \frac{\partial^2 u}{\partial x \partial t}. \quad (1.1)$$

We consider the above equation in a suitable domain  $\Omega \subseteq R^2$ . To find the solution, we try to find a twice continuously differentiable function  $u = u(x, t)$  which when substituted in equation (1.1) converts it into identity in  $\Omega$ . The domain  $\Omega$  of  $x$  and  $t$ , where problem is defined, is known as space time domain and PDE, which involves  $t$  (time) as one of the independent variable, is known as Evolution equation whereas if both the independent variable are spatial variable like  $x, y$  then the equation is known as Equilibrium or steady-state equation. A partial differential equation is said to be linear if it is linear in the unknown function and all its derivatives with coefficients depending only on independent variables, in another way speaking a partial differential equation is linear if sum of two solutions is also a solution i. e., superposition

principal should be followed and a constant multiple of the solution is also a solution i. e., homogeneity should be followed. A partial differential equation which does not satisfy the properties of superposition and homogeneity is known as nonlinear partial differential equation.

## 1.2 Fractional differential equation

Fractional differential equations are the generalization of ordinary differential equations and integration to arbitrary non integer orders. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation. Miller and Ross (1993) defined the fractional differential equation as, let  $r_m, r_{m-1}, r_{m-2}, \dots, r_0$  be strictly decreasing sequence of non negative numbers and if  $b_1, b_2, b_3, \dots, b_m$  are constants then a fractional differential equation is defined as

$$[D^{r_m} + b_1 D^{r_{m-1}} + b_2 D^{r_{m-2}} + \dots + b_m D^{r_0}]y(t) = 0. \quad (1.2)$$

But even this equation has complexity hence some other conditions are imposed as let  $r_i$  be the rational number, and if  $q$  is the least common multiple of all the denominator of nonzero  $r_i$  we can express equation (1.2) as

$$[D^{n\nu} + a_1 D^{(n-1)\nu} + \dots + a_n D^0]y(t) = 0, \text{ where } t \geq 0 \text{ and } \nu = \frac{1}{q}. \quad (1.3)$$

On the values of  $q=1$  and  $\nu=1$ , equation (1.3) represents an ordinary differential equation, the equation (1.3) is known as fractional linear

differential equation with constant coefficients of order  $(n, q)$ . Almost four to five decades ago the paradigm started to shift from pure mathematical formulations to its applications in various fields. During the last few decades fractional calculus has been applied to almost every field of science although some of the mathematical issues remain unsolved, most of the mathematical difficulties have been resolved and some mathematical issues in this field have been resolved to a point. One of the major advantages of fractional calculus is that it can be considered as super set which contains integer order calculus and hence fractional calculus has the potential to handle even that situation which integer order calculus cannot.

### **1.3 Fractional Calculus**

Fractional calculus is the extension of calculus where the order of differentiation and integration are generalized. In fractional calculus the order of differentiation and integration can be any real number or complex number instead of only being the integer, although the idea of generalized order differentiation and integration seems to be very new but it is as old as the integer order calculus. The first formal introduction of fractional calculus is believed that it has been introduced in a letter dated September 30, 1665 between Leibniz and L'Hospital. L'Hospital asked the question that what will be the result if  $n=1/2$  in the notation for the  $n$ -th derivative which was

introduced by Leibniz. The birth of the fractional calculus comes from the Leibniz answer that "An apparent paradox, from which one day useful consequences will be drawn". The question raised by L'Hospital has been the topic of research for more than 300 years.

The popularity of fractional calculus was less in the start because of the conflicting definitions for the fractional operators and inconsistent rules for the inverse operators. Fourier mentioned about the derivative of arbitrary order using the Fourier integral representation, Euler also had mentioned the arbitrary order derivative but they did not apply it anywhere. It was first Niels Henrik Abel in 1823 used fractional calculus for solving Integral equation arising in famous Tautochrone problem, which is also known as isochrones problem. During the development of fractional calculus many people have contributed among which few popular names are Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Letnikov.

Joseph Liouville defined the fractional derivative of a special class of function which can be expanded in the form of series. Because of this restriction Liouville's gave the second definition of fractional derivative which was applicable for the function of the type  $x^{-a}$  (where  $a > 0$ ). The development of Riemann -Liouville definition of fractional derivative appeared in the work of Sonin (1869) using Cauchy's Integral formula. In the recent years it is also in practice to use the Weyl formula of fractional integration. Heaviside

contribution in the development of generalized operators is in his work in the operational calculus. In twentieth century the work on fractional calculus accelerated.

The first international conference was held in university of new Haven in 1974, and just after the conference no of publication on fractional calculus increases drastically. The second conference on fractional calculus was held on 1984, where some eminent mathematician participated and few open question came on the platform of the conference .The pace of development of fractional calculus accelerated after the publication of books of Igor Podlubny in 1999 and the book by Miller and Ross (1993). These books really made tremendous popularity of fractional calculus among the researchers some other books contributed in the popularity of fractional calculus are “application of fractional calculus in physics “ by Hilfer (2000) , “The analysis of fractional differential equation” by Diethelm (2004) and “Theory and application of fractional differential equation” by Kilbas *et al.*(2006). Since fractional calculus is finding it applications in many field of Engineering and Science, some of the areas where fractional calculus has made a profound impact include Rheology, Viscoelasticity, Electrical Engineering, Electrochemistry, Biology, Signal and image processing, Mechanics, Physics, Control theory, Fluid dynamics, Diffusive transport etc. Presently it is hard to find the area of Science, where use of fractional calculus is not found.

## 1.4 Special Functions

In fractional calculus we use some special functions like Gamma function, Mittag-leffler function etc.

### 1.4.1 Gamma function

The Euler's Gamma function generalizes  $n!$  by  $\Gamma(n+1)$  and it allows  $n$  to take non-integer value or even complex values ( Kilbas *et al.* , 2006 )

Gamma function is defined by the integral as

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx . \quad (1.4)$$

Gamma function follows the reduction formula

$$\Gamma(z+1) = z\Gamma(z) . \quad (1.5)$$

### 1.4.2 Mittag-Laffler function

One parameter Mittag-Leffler function  $E_{\alpha}(z)$  is the generalization of exponential function and it is also known as Mittag- Leffler function of first kind (Mittag- Leffler, 1903) defined as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} , \quad \alpha > 0, \quad (1.6)$$

The two parameters Mittag-Leffler function  $E_{\alpha,\beta}(z)$ , also known as Mittag-Leffler function of second kind is defined as (Kilbas *et al.*, 2006)

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} , \quad \alpha > 0, \beta > 0, \quad (1.7)$$

It is very obvious that for  $\alpha = 1$  and  $\beta = 1$ , we get

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z, \quad (1.8)$$

and for  $\beta = 1$ ,

$$E_{\alpha,1}(z) = E_{\alpha}(z). \quad (1.9)$$

## 1.5 Popular definitions of fractional order derivatives and fractional order integrals

Fractional calculus is as old as conventional calculus but the beauty of the fractional calculus is the fractional derivatives and integrals has non-local property. The fractional derivatives and integrals considers the history as well as nonlocal distributed effect. After the L'Hospital's question it was first Leibniz (1695) who started in the direction of fractional calculus. Then many scientist have tried to define the fractional derivative but the first serious attempt to give a logical definition of a fractional derivative is due to Liouville, he gave two formulae for fractional order derivative. His first formula explains that the arbitrary order derivative of a function  $f(x)$  which can be expanded in the series form

$$f(x) = \sum_{n=0}^{\infty} c_n e^{ax}, \quad \operatorname{Re} a > 0, \quad (1.10)$$

is



$$D^\nu f(x) = \sum_{n=0}^{\infty} c_n a^\nu e^{ax} . \quad (1.11)$$

Liouville gave the second formula for fractional derivative for the class of the function of type  $x^{-a}$  where  $a > 0$  as

$$D^\nu x^{-a} = \frac{(-1)^\nu \Gamma(a + \nu)}{\Gamma(a)} x^{-a-\nu}, \quad a > 0. \quad (1.12)$$

Another approach starts with the generalization of repeated integration which says that if  $f$  is locally integrable in  $(a, \infty)$  then its  $n$ -fold integral is given by

$${}_a I_x^n f(x) = \int_a^x du_1 \int_a^{u_1} du_2 \dots \int_a^{u_{n-1}} f(u_n) du_n = \frac{1}{(n-1)!} \int_a^x (x-u)^{n-1} f(u) du \quad (1.13)$$

For almost all  $x$  with  $-\infty \leq a < x < \infty$  and  $n \in \mathbb{N}$ . On generalizing  $(n-1)! = \Gamma(n)$ , we can get a formula for the integration of arbitrary order as

$${}_a I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-u)^{\alpha-1} f(u) du, \quad \alpha > 0. \quad (1.14)$$

This formula leads to the definition of Riemann- Liouville fractional order integration, which says that

$$J_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-u)^{\alpha-1} f(u) du, \quad \alpha > 0, x > 0, \quad (1.15)$$

and  $J_x^0 f(x) = f(x)$ .

Until now many scientists had given many definitions of fractional derivative among them few are very popular as Riemann-Liouville definition, Caputo definition (Hilfer, 2000), Grunwald-Letnikov definition.

**Definition** A real function  $f(x)$ ,  $x > 0$  is said to be in a space  $C_\mu$ ,  $\mu \in R$ , if there exists a real number  $p > \mu$  such that  $f(t) = t^p f_1(t)$ , where  $f_1(t) \in C[0, \infty)$ , and is said to be in space  $C_\mu^n$  if  $f^{(n)} \in C_\mu$ ,  $n \in N$ .

### 1.5.1 Riemann-Liouville fractional derivative

The fractional order derivative of order  $\alpha > 0$ , of a function  $f(x)$  is defined as

$${}_a D_x^\alpha f(x) = \left( \frac{d}{dx} \right)^n (J_x^{n-\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^n \int_a^x (x-u)^{n-\alpha-1} f(u) du, \quad (1.16)$$

where  $x > 0, a > 0, n-1 \leq \alpha < n, n \in N$ .

### 1.5.2 Properties of Riemann-Liouville operator

The following are few properties of Riemann-Liouville fractional integral operator  $J_x^\alpha$  and fractional order differential operator  ${}_a D_x^\alpha$  for  $f \in C_\mu, \mu \geq -1, \alpha > 0, \beta > 0$  and  $\gamma \geq -1$ :

- (i)  $J_x^\alpha J_x^\beta f(x) = J_x^{\alpha+\beta} f(x)$ ,
- (ii)  $J_x^\alpha J_x^\beta f(x) = J_x^\beta J_x^\alpha f(x)$ ,
- (iii)  $J_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\gamma+1}$ .

### 1.5.3 Caputo fractional order derivative

In the Caputo sense, the Fractional order derivative of a function  $f(x)$  is defined by (Caputo, 1967; Caputo and Mainardi, 1971)

$${}_a^c D_x^\alpha f(x) = (J_x^{n-\alpha} f^{(n)})(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-u)^{n-\alpha-1} f^{(n)}(u) du, \quad (1.17)$$

where  $x > 0, a > 0, n - 1 \leq \alpha < n, n \in N$ .

### 1.5.4 Properties of Caputo operator

The following are some basic properties of Caputo fractional derivative:

(i) If  $f \in C_{-1}^n, n \in N$ , then  $D_x^\alpha f(x), 0 < \alpha < n$  is well defined and  $D_x^\alpha f(x) \in C_{-1}$ ,

(ii)  $D_x^\alpha J_x^\alpha f(x) = f(x)$ ,

(iii)  $J_x^\alpha D_x^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{x^k}{k!}, x > 0$ , for  $f \in C_\mu^n, n - 1 \leq \alpha < n, n \in N$ .

### 1.5.5 Grunwald-Latnikov definition of fractional order derivative

It says that (Podlubny, 1999)

$${}_a D_x^\alpha f(x) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{n=0}^{\left[ \frac{x-a}{h} \right]} (-1)^n \binom{\alpha}{n} f(x - nh), \text{ where } \left[ \frac{x-a}{h} \right] \rightarrow \text{Integer}. \quad (1.18)$$

### 1.6 Laplace transform of fractional order derivative

The Laplace transform of a function  $f(t)$  defined for all real no  $t \geq 0$ , is the function  $F(s)$  which is defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt, s \in C. \quad (1.19)$$

The necessary and sufficient conditions for the existence of Laplace transform are

(i) Function should be piecewise smooth in every finite interval in  $[0, \infty)$ ,

(ii) Function should be of exponential order i.e., there exists a constant  $M > 0$  and  $T > 0$  such that  $|f(t)| \leq Me^{at}$ ,  $t > T$ .

The original function  $f(t)$  can also be regained from its Laplace transform by taking the inverse Laplace transform such as

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds, \quad c = \text{Re}(s) > c_0, \quad (1.20)$$

where  $c_0$  lies in the right half plane of absolute convergence of the Laplace integral.

### 1.6.1 Laplace transform of Reimann-Liouville's fractional derivative and integral

Let  $\alpha > 0$  and  $F(s)$  is the Laplace transform of  $f(t)$ , then following definitions hold according to Podlubny (1999).

(i) The Laplace transform of Reimann-Liouville fractional integral of order  $\alpha$  is defined as

$$L[{}_0D_t^{-\alpha} f(t); s] = s^{-\alpha} F(s). \quad (1.21)$$

(ii) Laplace transform of Reimann-Liouville fractional derivative of order  $\alpha$  is defined as

$$\begin{aligned} L[{}_0D_t^\alpha f(t); s] &= s^\alpha F(s) - \sum_{k=0}^{n-1} s^k [D^{\alpha-k-1} f(t)]_{t=0} \\ &= s^\alpha F(s) - \sum_{k=0}^{n-1} s^{n-k-1} [D^k J^{n-\alpha} f(t)]_{t=0}, \quad \text{where } n-1 < \alpha < n. \end{aligned} \quad (1.22)$$

### 1.6.2 Laplace transform of Caputo fractional derivative

The Laplace transform of Caputo fractional derivative of order  $\alpha$  is defined as

$$L\left[{}_0 D_t^\alpha f(t); s\right] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} \left[ f^{(k)}(0) \right], \text{ where } n-1 < \alpha < n. \quad (1.23)$$

If the Reimann-Liouville and Caputo both fractional derivative exist for a function  $f(t)$  then in the Laplace transform of Reimann-Liouville fractional derivative the initial value of fractional integral  $J^{n-\alpha} f(t)$  and its integer order derivative of order  $k=1,2,3, \dots, n-1$  are required whereas in the case of Caputo fractional order derivative only initial value of the function and its integer order derivative of order  $k=1,2,3, \dots, n-1$  are required. This gives an advantage to Caputo derivative over the Reimann-Liouville fractional derivative while applying in the physical problems.

### 1.7 Homotopy Analysis Method (HAM)

The HAM technique was introduced by S. J. Liao in 1992 for the linear and nonlinear problems. This technique is the combination of classical perturbation technique and Homotopy, a concept of Topology. HAM is the unification of Luyapunov artificial small parameter method, Delta expansion method, Adomian decomposition method. On theoretical background HAM works on the concept that a nonlinear equation can be split into infinite number of linear sub equations. Let us consider here one nonlinear equation in a general form of two variables as

$$N[u(x,t)] = 0, \quad (1.24)$$

where  $N$  is the nonlinear operator,  $u(x,t)$  is the unknown function to be evaluated with the spatial and temporal variable  $x, t$ . Let us consider  $u_0(x,t)$ , an initial guess for the solution and construct the zero order deformation equation as

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = \hbar H(x,t)N[\phi(x,t;q)], \quad (1.25)$$

where  $L$  is an auxiliary linear operator,  $\hbar \neq 0$  is the convergence control parameter,  $H(x,t) \neq 0$  is the auxiliary function,  $q$  is known as embedding parameter and  $q \in [0,1]$ . In HAM there is freedom to choose the auxiliary linear operator  $L$ , auxiliary function  $H(x,t)$  and initial approximation  $u_0(x,t)$  for  $u(x,t)$ . Since proper base function is selected for solution by HAM, therefore this method is very popular among the researchers to solve linear and nonlinear differential equations. The auxiliary linear operator has the property that  $L[0] = 0$ . It is obvious from the equation (1.25) that at  $q = 0$ , then  $\phi(x,t;0) = u_0(x,t)$  and at  $q = 1$ ,  $\phi(x,t;1) = u(x,t)$ . Hence as embedding parameter  $q$  varies from 0 to 1, then  $\phi(x,t;q)$  varies from initial guess  $u_0(x,t)$  to the solution of the differential equation  $u(x,t)$ . Considering the Taylor's series of  $\phi(x,t;q)$  as

$$\phi(x,t;q) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t) q^k, \quad (1.26)$$

where  $u_k(x,t) = \frac{1}{k!} \frac{\partial^k \phi}{\partial q^k} \Big|_{q=0}$ , which is named as Homotopy derivative by S. J.

Liao. If the above series (1.26) is convergent then at  $q = 1$  it converges to the exact solution of the differential equation (1.24) and the convergence of the series (1.26) is controlled by  $\hbar$  parameter. Substituting  $q = 1$  in the equation (1.26) we get

$$\phi(x,t;1) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t) , \quad (1.27)$$

which is one of the solutions of the differential equation (1.24) as proved by Liao (1992). Let us consider the vector

$$\vec{u}_n(x,t) = \{u_0(x,t), u_1(x,t), u_2(x,t), \dots, u_n(x,t)\}, \quad (1.28)$$

hence the m-th order deformation equation is obtained as

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\vec{u}_{m-1}(x,t)), \quad (1.29)$$

with the initial condition

$$u_m(x,0) = 0, \quad (1.30)$$

where

$$R_m(\vec{u}_{m-1}(x,t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}} \Big|_{q=0}. \quad (1.31)$$

$$\text{and } \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

On solving the m-th order deformation equation (1.29) we get

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\bar{u}_{m-1}(x,t))] + c,$$

where  $c$  is the integration constant determined by the initial condition in (1.30). In this way we obtain  $u_m(x,t)$  for  $m \geq 1$ . Therefore to get the final solution we have

$$u(x,t) = \lim_{N \rightarrow \infty} \Phi_N(x,t), \quad (1.32)$$

$$\text{where } \Phi_N(x,t) = \sum_{m=0}^{N-1} u_m(x,t) .$$

The speciality of HAM is the finding of convergence control parameter  $\hbar$ . As done by researchers during finding the solutions of a nonlinear problems the convergence region is controlled by plot of  $\hbar$  and then choosing the proper value of  $\hbar$  from this region for getting convergence of the series solution. Since HAM is good mathematical tool to solve nonlinear problems if we have the idea about the structure of the solution of the problem so that a proper base function can be selected and for finding the solution as we know that any real continuous function can be represented by so many type of base function viz., algebraic, periodic, exponential. Thus for the some physical nonlinear problems in physical world, sometime it is difficult to approximate the solution when there will be lack of knowledge about proper set of base functions.



## 1.8 Variation Iteration Method (VIM)

It was J.H. He who first proposed the variation iteration method(1998; 1999b; 2000b),which was later followed by many mathematicians such as wazwaz (2001), Bildik (2006), Ates and Yildirim (2009), Sweilam *et al.* (2007), Odibat and Momani (2008b) etc. during the solutions of various linear and nonlinear differential equations. It is also applied to solve the fractional differential equations. In the approach of variational iteration method, we get an approximate analytical solution of linear and nonlinear problems without linearization and discretization. The proposed method is very simple, attractive and a concise mathematical tool. The implementation of this method reduces the computational work as compared to traditional techniques. He (1998) first applied VIM to solve fractional order differential equation. After that many scientists applied this technique to solve fractional order differential equation. Some of them are odibat and Momani(2009), Khan *et al.*( 2011), Wu(2012) who used VIM for solving the time fractional diffusion equations in porous medium.

This approach can be used in an efficient and reliable way to handle the nonlinear differential equation such as follows

$$L[u(x)] + N[u(x)] - g(x) = 0, \tag{1.33}$$

where  $L[.]$  represents the linear operator defined as  $L[.] = \frac{d^m(.)}{dx^m}$ ,  $m \in N$ ,  $N[.]$  represents the nonlinear operator and  $g(x)$  is known analytic function, with the initial conditions as

$$u^{(n)} = c_n, n = 0, 1, 2, \dots, m-1, \quad (1.34)$$

where  $c_m$ 's are real numbers. In the variational iteration method, we construct a correctional functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) \{L(u_n) + N(u_n) - g(\xi)\} d\xi. \quad (1.35)$$

In the above expression  $\lambda(\xi)$  is the Lagrange multiplier which is calculated optimally using variational theory. To calculate the Lagrange multiplier we use the concept of restricted variation for nonlinear term  $N(u)$ , constructing the functional (1.35) stationary. In that case  $\delta \tilde{u}_n = 0$ , where  $\tilde{u}_n$  represents a restricted variable. Hence,

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(\xi) \{L(u_n) + N(\tilde{u}_n) - g(\xi)\} d\xi, \quad (1.36)$$

which gives the Lagrange multipliers as  $\lambda(\xi) = f(\xi, x)$  and on substituting it in the equation(1.35), we get an iterative formula as

$$u_{n+1}(x) = u_n(x) + \int_0^x f(\xi, x) \{L(u_n) + N(u_n) - g(\xi)\} d\xi. \quad (1.37)$$

The initial approximation for the solution  $u_0(x)$  can be chosen with the help of the initial and boundary conditions of the problem. This algorithm is more successful with proper selection of the initial approximation  $u_0(x)$ . Usually the initial values  $u^{(n)}(0) = c_n, n = 0, 1, 2, \dots, m-1$  are taken for the selection of initial approximation  $u_0(x)$ . In this method if we construct the initial approximation  $u_0(x)$  as

$$u_0(x) = \sum_{j=0}^{m-1} \frac{c_j}{j!} x^j, \quad (1.38)$$

then the higher order approximations  $u_n(x)$  for  $n \geq 1$ , can be calculated and the final solution can be obtained as

$$u(x) = \lim_{n \rightarrow \infty} u_n(x). \quad (1.39)$$

## 1.9 Variable order fractional differential equations

Variable order systems are the extension of fractional order to the functional order system, where the order of derivative and integral changes, and are the functions of either independent variable or dependent variable or both. A variable order system will be like

$$D_t^{\alpha(y,t)} f(t) = y(t). \quad (1.40)$$

The introduction of constant order calculus from its inception has dominated in the modeling of the physical processes. Consider the problem of a sliding

box on a surface which is partially wet. Then the motion of the box first governed by the dry friction and afterward it is governed by the viscous friction. One question rises here that what is the best mathematical framework to describe this kind of transition. Another physical problem which can be considered is that the force acting on a particle which is oscillating in the viscous fluid and this complex dynamics is represented by the differential equation such as Navier stokes equation which can be represented by simple variable order equation. One more example is that when mass is moving from a viscous medium to a viscoelastic medium the motion of the mass cannot be captured by the constant order calculus. Variable order equation leads to a new paradigm in Science. It is very obvious that variable order calculus is a very natural option for the modeling of the dynamical problems with transitional situation.

### **1.10 Diffusion equations**

Diffusion is macroscopically related with the change of concentration. On contrary to the mass flow of liquids, diffusion involves random and spontaneous movements of individual molecules. In diffusion, flux represents the number of particles traversing a unit area per unit time and the concentration is the number of particles per unit volume. The classical diffusion equation can be derived using Fick's Law and conservation law.

According to Fick's law, "the rate of diffusion per unit area in a direction perpendicular to the area is proportional to the gradient of concentration of the solute in that direction", which can be presented mathematically as

$$J = -D \frac{\partial u}{\partial x}, \quad (1.41)$$

where  $J$  is the diffusion flux,  $D$  is diffusion constant. According to continuity equation which states that "change in density in any part of system is due to inflow and outflow of the material (i.e., no material can be created or destroyed)" and mathematically presented as

$$\frac{\partial u}{\partial t} + \frac{\partial J}{\partial x} = 0. \quad (1.42)$$

Hence the classical diffusion equation can be derived from (1.41) and (1.42) as

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}. \quad (1.43)$$

Consider the fractional Fick law (Norwood, 1972; Moodi and Tait, 1983)

$$J = -D \int_0^t K(t-\tau) \cdot \nabla u(\tau) d\tau \quad (1.44)$$

Now taking the kernel  $K(t-\tau)$  as

$$K(t-\tau) = \frac{1}{\Gamma(\alpha-1)} (t-\tau)^{\alpha-2}, \quad 0 < \alpha < 2, \quad (1.45)$$

the equation (1.41) together with the equation (1.42) and the equation (1.44) leads us to fractional diffusion equation as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D \frac{\partial^2 u}{\partial x^2} . \quad (1.46)$$

The parameter  $D$  appearing in the diffusion equation signifies the strength or weakness of the underlying diffusion process. There are multiple mechanisms responsible for the transport. Diffusion is one of the transport phenomena and one of the mechanisms responsible for the diffusion is Brownian motion. This is observed as early as 1785, by Robert Brown.

### 1.11 Bernstein basis polynomial

Bernstein basis polynomial was introduced in 1912 by Sergei Natanovich Bernstein in the effort to express the continuous function by the polynomials in any interval. Bernstein polynomial is the linear combinations of Bernstein basis polynomial. The Bernstein basis polynomial of degree  $n$  form a basis for the vector space of polynomials of degree at most  $n$ . Any Bernstein polynomial of degree  $n$  can be recursively written in terms of two Bernstein basis polynomial of degree  $n-1$  as

$$B_{i,n}(x) = (1-x)B_{i,n-1}(x) + xB_{i-1,n-1}(x) . \quad (1.47)$$

In Bernstein polynomial partitions, the unity property

$$\sum_{i=0}^n B_{i,n}(x) = \sum_{i=0}^{n-1} B_{i,n-1}(x) = \dots = \sum_{i=0}^1 B_{i,1}(x) = 1, \quad (1.48)$$

are very useful in computer graphics. Its intrinsic numerical stability makes it more advantageous while applying. Corresponding to any continuous function  $f(x)$  in  $[0, 1]$ , there is a sequence of Bernstein polynomials as follows:

$$B_n(f; x) = \sum_{r=0}^n f\left(\frac{r}{n}\right) \binom{n}{r} x^r (1-x)^{n-r}, \quad (1.49)$$

which converges uniformly to  $f(x)$ . Another interesting property of Bernstein polynomial is that the approximation of any function  $f(x)$  by Bernstein polynomial is at least as smooth as function  $f(x)$  is i.e., if  $f(x)$  has  $C^r$  instead of  $C^0$  continuity all derivatives of  $B_n(f; x)$  up to order  $r$  converge uniformly to the corresponding derivative of the function  $f(x)$  (Floater, 2005).