CHAPTER-7

In-Phase and Anti-Phase Synchronization between Fractional-Order Chaotic Systems with Uncertainties and External Disturbances

Chapter 7

In-phase and anti-phase synchronization between fractional order chaotic systems with uncertainties and external disturbances

7.1 Introduction

During the last few decades, study of nonlinear phenomena occurring in various areas of scientific fields has gained immense popularity amongst the scientists and engineers for the development of the models using non-linear differential equations. Introduction of fractional order time derivative in nonlinear models has rendered a differenrt dimension to the existing problems. Nowadays fractional order derivative has become one of the potent areas of research to the scientists and engineers since fractional order system response ultimately converges to the integer order system. For high accuracy, fractional order derivatives are used to describe the dynamics of systems. A wide range of problems in different branches of science and engineering have already been studied by a number of researchers of different parts of the world to explore the potential of the fractional derivative. The usage of first order time derivative with a fractional order time derivative is not only applicable for non-Gaussian but also for Non-Markovian systems. The attribute of fractional order derivative in linear and non-linear dynamical systems is the increased degrees of flexibilities in the model. The fractional order differential operator is non local for which it takes into account the fact that the future state not only depends upon the present state but also upon all

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of the history of its previous states. For this realistic property, the fractional order systems are being attractive to the researchers.

Chaos is an interesting nonlinear phenomenon which has tremendous applications in different areas of science and engineering. Synchronization is an important phenomenon in chaotic system that may occur when two or more chaotic systems are coupled or one chaotic system drives the other. The pioneering work of Pecore and Corrall (1990) on synchronization between the drive (master) and response (slave) systems of two identical or non identical systems with different initial conditions, has attracted a great deal of interest to the researchers in various fields due to its important applications in ecological systems, physical systems, chemical systems, modeling brain activity, system identification, pattern recognition phenomena and secure communications etc. (Blasius *et al.* (1999), Lakshmanan and Murali (1996), Han *et al.* (1995), Cuomo and Oppenheim (1993), Kocarev and Parlitz (1995), Murali and Lakshmanan (2003)). In recent years various synchronization schemes, such as linear and nonlinear feedback synchronization, time delay feedback approach, adaptive control, active control etc. (Chen and Lu (2002), Huang *et al.* (2004), Park and Kwon (2005), Park (2005), Ho *et al.* (2005), Huang *et al.* (2009)) have been successfully applied to chaos synchronization. Chaos synchronization using active control method is an efficient technique. In 1999, Park *et al.* (1999) have studied the phase synchronization in the forced Lorenz system. Ho *et al.* (2002) designed the phase and anti-phase synchronization of two chaotic systems by using the same method. Bai and Lonngren (1997) have presented synchronization of two Lorenz systems and Austin *et al.* (2009) have studied chaos synchronization between the Genesio system and the unified system using this effective method.

The important feature of the synchronization is that the difference of states of chaotic systems converges to zero for large time, while during anti-synchronization the state vectors have the same absolute values but opposite in sign. Mathematically, the synchronization is achieved when $\lim_{t\to\infty}\|x_1(t)-x_2(t)\|=0$ and anti-synchronization is obtained when $\lim_{t\to\infty}\|x_1(t)+x_2(t)\|=0$, where $x_1(t)$ and $x_2(t)$ are the state vectors of the drive and response systems respectively. In 2009, Al-sawalha and Noorani (2009) have investigated anti-synchronization between two different hyperchaotic systems. Al-sawalha and Noorani (2009) have proposed anti-synchronization of chaotic systems with uncertain parameters via adaptive control. In 2007, Li and Zhou (2007) have studied anti-synchronization in different chaotic systems.

From the literature survey it is seen that synchronization and anti-synchronization between fractional order systems are few in numbers and therefore, this are of research is yet to be explored. Synchronization between fractional order chaotic systems is already being investigated in the research article of Yan and Li (2007), Deng and Li (2005), Li and Yan (2007), Xu *et al.* (2008). Yan and Li (2007) have proposed on chaos synchronization of fractional differential equations. Erjaee and Taghvafard (2011) have designed phase and anti-phase synchronizations of fractional order chaotic systems via active control. Bhalekar and Gejji (2011) have done anti-synchronization of non identical fractional-order chaotic systems using active control method.

Initially researchers involve only in doing synchronization and anti-synchronization between identical and non-identical chaotic systems using various methods viz., Active, Adaptive and Sliding mode methods, but later the influences of the uncertainties during synchronization and anti-synchronization are been considered. In the real world applications e.g., in secure communication, the receiver plants suffer from the various uncertainties including parameter perturbation and external disturbance, which may influence the accuracy of the communication. Thus, the synchronization between chaotic systems with uncertainties and disturbances are tough jobs for researchers since there are always possibilities of destroying synchronization with the effects of those parameters especially for fractional order systems. Aghababa (2012) designed robust stabilization and synchronization of a class of fractional-order chaotic systems via a novel fractional sliding mode controller. Chen *et al.* (2012) have studied disturbance-observer-based robust synchronization control of uncertain chaotic systems. Jawaadaa *et al.* (2012) have done robust active sliding mode anti-synchronization of hyperchaotic systems with uncertainties and external disturbances. Yang *et al.* (2009) proposed the robust synchronization of fractional chaotic systems via adaptive sliding mode control. Fu and Li (2011) have studied robust adaptive anti-synchronization of two different hyperchaotic systems with external uncertainties. But to the best of author's knowledge the synchronization and anti-synchronization between non-identical fractional order Gensio-Tesi and Qi systems with parametric uncertainties and external disturbances using active control method have not yet been studied by any researcher.

In the present chapter, a sincere attempt has been taken to study phase and anti-phase synchronizations between non-identical fractional order chaotic systems viz., Genesio-Tesi and Qi chaotic systems using active control method in the presence of parametric uncertainties and external disturbances. Numerical simulation results are carried out using Adams–Bashforth–Moulton method (Diethelm et al. (2004), Diethelm and Ford (2004)) and are displayed graphically, which clearly exhibit that the active control method is effective, easy to implement and reliable for both the phase and anti-phase synchronizations of two nonlinear fractional order uncertain chaotic systems.

7.2 System description and Problem formulation

7.2.1 System description

The fractional-order Qi system (Yang *et al.* (2009)) is described as

$$\begin{aligned} D_t^{\alpha} x &= a(y-x) + yz \\ D_t^{\alpha} y &= cx - y - xz \\ D_t^{\alpha} z &= -bz + xy, \end{aligned} \tag{7.1}$$

where α $(0 < \alpha \leq 1)$ is the fractional order time derivative. The largest Lyapunov exponent of Qi system (Wu and Yang (2010)) is 4.2134. This system has large chaotic region when the parameters are varied. The lowest value of α for which the system remains chaotic is 0.915. The chaotic attractor in x-y-z space is depicted through Fig. 7.1 for $\alpha = 0.915$ at $a = 35,\ b = 8/3,\ c = 80$.

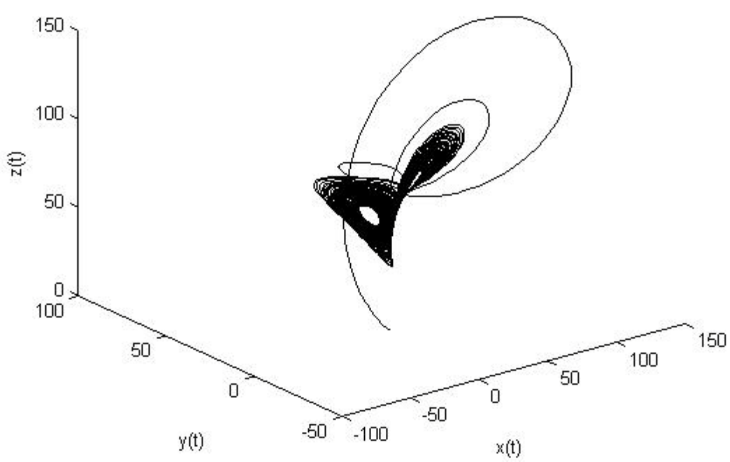


Fig. 7.1. Phase portrait of Qi system for $\alpha = 0.915$.

The fractional-order Genesio–Tesi system (Faieghi and Delavari (2011)) is described as

$$\begin{aligned} D_t^{\alpha} x &= y \\ D_t^{\alpha} y &= z \\ D_t^{\alpha} z &= -p\,x - q\,y - r\,z + m\,x^2. \end{aligned} \tag{7.2}$$

The largest Lyapunov exponent of Genesio–Tesi system is 0.0022. Lowest value of α for which the system remains chaotic is 0.93. The chaotic attractors in x-y-z space is depicted through Fig. 7.2 for $\alpha = 0.93$ at $p = 6,\ q = 2.92,\ r = 1.2$ and $m = 1$.

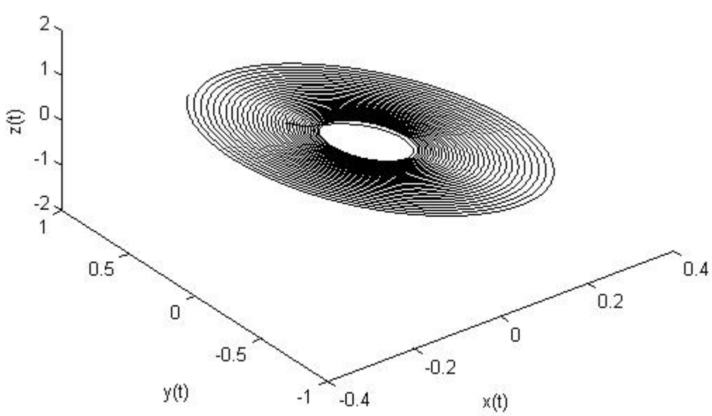


Fig. 7.2. Phase portrait of Genesio-Tesi system for $\alpha = 0.93$.

7.2.2 Problem Formulation

Consider an uncertain fractional order chaotic system as the master system as

$$D_t^{\alpha} x = (A_1 + \Delta A_1)x + f_1(x) + d_1(t), \qquad 0 < \alpha \le 1 \tag{7.3}$$

and another uncertain fractional order chaotic system as the response system as

$$D_t^{\alpha} y = (A_2 + \Delta A_2)y + f_2(y) + d_2(t) + \mu(t), \qquad 0 < \alpha \le 1 \, , \tag{7.4}$$

where $x(t), y(t) \in R^n$, A_1, $A_2 \in R^{n\times n}$ is a known constant matrices with proper dimension, $f_1, f_2 : R^n \to R^n$ are the nonlinear part of the system, ΔA_1, $\Delta A_2 \in R^{n\times n}$ are parametric uncertainties of chaotic systems with $|\Delta A_1| \le \delta_2$, $|\Delta A_2| \le \delta_2$, δ_1, δ_2 are positive constants and $d_1(t), d_2(t)$ are the external disturbances of uncertain chaotic systems with $|d_1(t)| \le \rho_1$, $|d_2(t)| \le \rho_2$, ρ_1, $\rho_2 > 0$ and $\mu(t) \in R^n$ is the control input vector of the uncertain chaotic system (7.4). Now controller $\mu(t)$ is to be designed in such a way that the states of the master and response systems are synchronized.

The corresponding error dynamics can be obtained as

$$\begin{aligned} D_t^{\alpha} e &= (A_2 + \Delta A_2)y + f_2(y) + d_2(t) - (A_1 + \Delta A_1)x - f_1(x) - d_1(t) + \mu(t) \\ &= (A_2 + \Delta A_2 + \Delta A_1)e + d_2(t) + F(x, y) - d_1(t) + \mu(t), \end{aligned} \tag{7.5}$$

where $e = y - x$ and $F(x, y) = f_2(y) - f_1(x) + ((A_2 + \Delta A_2) - A_1)x - \Delta A_1 y.$ Next design an appropriate feedback control $\mu(t)$ which stabilizes the system so that the error system (7.5) converges to zero as time *t* tends to infinity, i.e., $\lim_{t\to\infty} \|e\| = 0$. This implies that the systems (7.3) and (7.4) are synchronized.

If there is any eigen value of the error system is equal to zero, then another type of synchronization phenomenon called phase synchronization may occur, in which the difference between various states of synchronized systems may not necessarily converge to zero, but is less than or equal to a constant. The same procedure may be used for anti-phase synchronization process, in which the state vectors have the same absolute values but opposite in sign.

7.3 Phase synchronization between fractional order uncertain Genesio-Tesi and uncertain Qi systems using active control method

In this section, the phase synchronization between two different fractional order uncertain Genesio-Tesi and Qi systems is studied. It is assumed that uncertain Genesio-Tesi system drives the uncertain Qi system. We define the Genesio-Tesi as a drive system as

$$\begin{aligned} D_t^{\alpha} x_1 &= y_1 + 0.5 x_1 + 0.1\cos(100t) \\ D_t^{\alpha} y_1 &= z_1 - 0.3 y_1 + 0.1\sin(100t) \\ D_t^{\alpha} z_1 &= -p x_1 - q y_1 - r z_1 + m x_1^2 - 0.5 x_1 + 0.1\sin(100t), \end{aligned} \tag{7.6}$$

where uncertain parameter $\Delta A_1 = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & -0.3 & 0 \\ -0.5 & 0 & 0 \end{pmatrix}$ and disturbance term

$$d_1(t) = \begin{pmatrix} 0.1\cos(100t) \\ 0.1\sin(100t) \\ 0.1\sin(100t) \end{pmatrix}.$$

Fig. 7.3 shows the chaotic attractor of the Genesio-Tesi system with uncertainties and disturbances for the order of the derivative $\alpha = 0.95$.

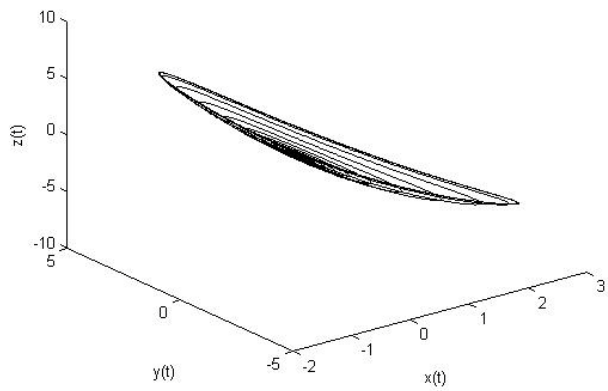


Fig. 7.3. Phase portrait of uncertain Genesio-Tesi system for $\alpha = 0.95$.

The Qi-system is defined as the response system as

$$\begin{aligned} D_t^{\alpha} x_2 &= a(y_2 - x_2) + y_2 z_2 + z_2 + 0.5\cos(10t) + \mu_1(t) \\ D_t^{\alpha} y_2 &= c x_2 - y_2 - x_2 z_2 - 0.02 z_2 - x_2 - 0.5\sin(10t) + \mu_2(t) \\ D_t^{\alpha} z_2 &= -b z_2 + x_2 y_2 - 2y_2 + 0.5\cos(10t) + \mu_3(t), \end{aligned} \tag{7.7}$$

where $\Delta A_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & -0.02 \\ 0 & -2 & 0 \end{pmatrix}$ is uncertain parameter $d_2(t) = \begin{pmatrix} 0.5\cos(10t) \\ -0.5\sin(10t) \\ 0.5\cos(10t) \end{pmatrix}$ is disturbance term and $\mu(t) = [\mu_1(t),\ \mu_2(t),\ \mu_3(t)]^T$ is the controller to be designed. The chaotic attractor of the Qi-system with uncertainties and disturbances is depicted through Fig. 7.4.

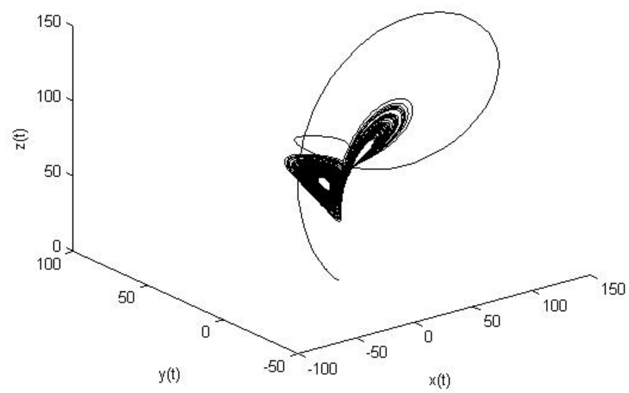


Fig. 7.4. Phase portrait of uncertain chaotic Qi system for $\alpha = 0.93$.

To investigate the synchronization of systems (7.6) and (7.7), the error states are defined as $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$ and $e_3 = z_2 - z_1$. The corresponding error dynamics are obtained by subtracting equation (7.6) from equation (7.7), which is given by

$$\begin{aligned} D_t^{\alpha} e_1 &= a(e_2 - e_1) + e_3 - x_1(0.5 + a) + y_1(a-1) + z_1 + y_2 z_2 \\ &\quad + 0.5\cos(10t) - 0.1\cos(100t) + \mu_1(t) \\ D_t^{\alpha} e_2 &= (c-1)e_1 - e_2 - 0.02 e_3 + (c-1)x_1 - 0.7 y_1 - 1.02 z_1 \\ &\quad - x_2 z_2 - 0.5\sin(10t) - 0.1\sin(100t) + \mu_2(t) \\ D_t^{\alpha} e_3 &= -b e_3 - 2e_2 + (0.5+p)x_1 + (q-2)y_1 + (r-b)z_1 \\ &\quad + x_2 y_2 - m x_1^2 + 0.5\cos(10t) - 0.1\sin(100t) + \mu_3(t). \end{aligned} \tag{7.8}$$

Choosing the control functions as

$$\begin{aligned} \mu_1(t) &= x_1(0.5+a) - y_1(a-1) - z_1 - y_2 z_2 - 0.5\cos(10t) + 0.1\cos(100t) + V_1(t) \\ \mu_2(t) &= -(c-1)x_1 + 0.7 y_1 + 1.02 z_1 + x_2 z_2 + 0.5\sin(10t) + 0.1\sin(100t) + V_2(t) \\ \mu_3(t) &= -(0.5+p)x_1 - (q-2)y_1 - (r-b)z_1 - x_2 y_2 + m x_1^2 - 0.5\cos(10t) \\ &\quad + 0.1\sin(100t) + V_3(t), \end{aligned} \tag{7.9}$$

equation (7.8) reduces to

$$
\begin{aligned}
D_t^{\alpha} e_1 &= a(e_2 - e_1) + e_3 + V_1(t) \\
D_t^{\alpha} e_2 &= (c-1)e_1 - e_2 - 0.02e_3 + V_2(t) \\
D_t^{\alpha} e_3 &= -be_3 - 2e_2 + V_3(t),
\end{aligned}
\tag{7.10}
$$

where $V_1(t)$, $V_2(t)$, $V_3(t)$ are the linear control inputs, which are defined as

$$
\begin{bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{bmatrix} = M \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},
$$

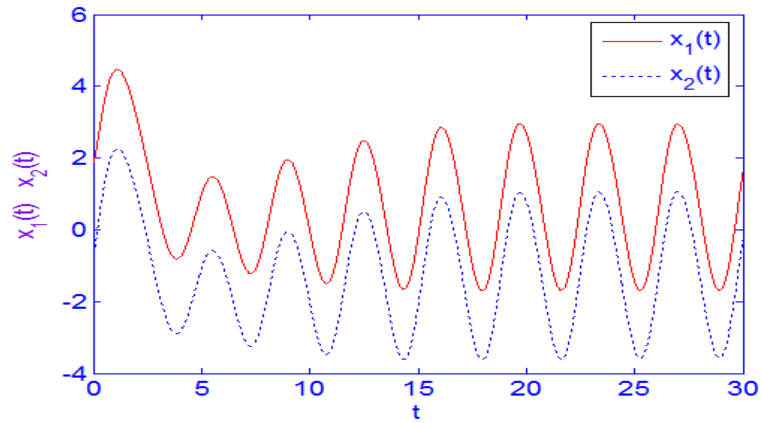
where M is a 3×3 constant matrix. In order to make the closed loop system stable, the matrix M should be selected in such a way that the eigenvalues λ_i of error dynamical system satisfy the control $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$, $i = 1,2,3$. Choosing the matrix M as

$$
M = \begin{pmatrix} 0 & -a & -1 \\ -(c-1) & 0 & 0.02 \\ 0 & 2 & 0 \end{pmatrix},
$$

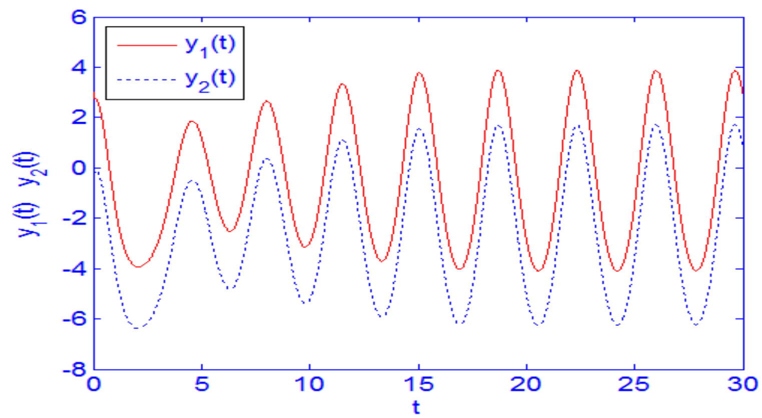
the error system is reduced to

$$
\begin{aligned}
D_t^{\alpha} e_1 &= -ae_1 \\
D_t^{\alpha} e_2 &= -e_2 \\
D_t^{\alpha} e_3 &= -be_3.
\end{aligned}
\tag{7.11}
$$

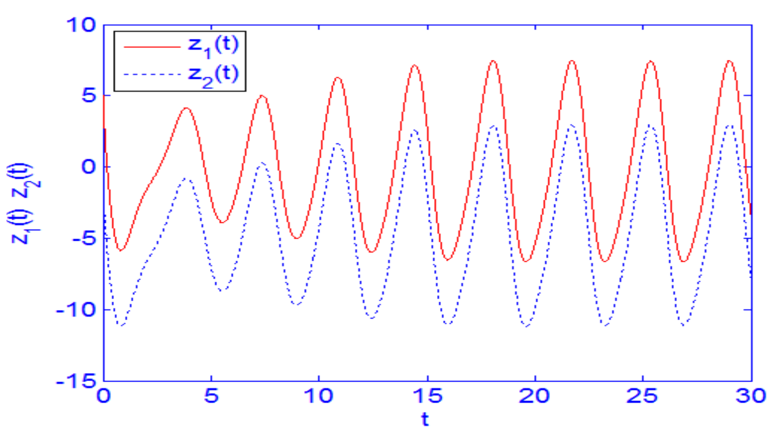
Here all the three eigenvalues of the system are negative which clearly shows that the error system (7.11) is stable and thus the chaotic systems (7.6) and (7.7) are synchronized.



(a) State trajectories between x_1 and x_2.



(b) State trajectories between y_1 and y_2.



(c) State trajectories between z_1 and z_2.

Fig. 7.5. Phase synchronization for fractional order $\alpha = 0.95$.

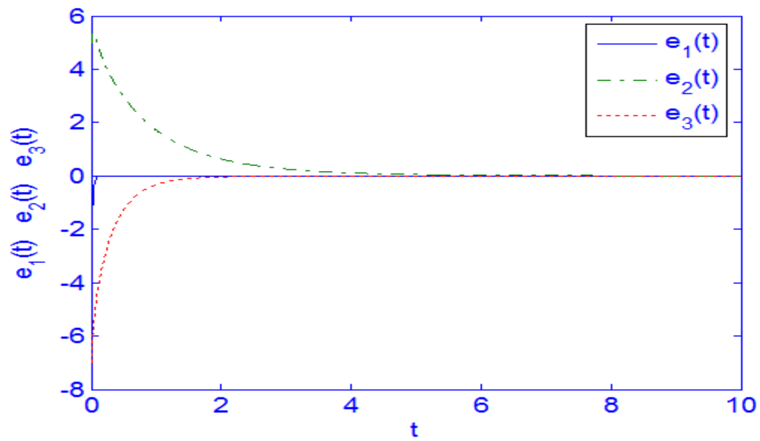


Fig. 7.6. The evolution of the error functions of uncertain chaotic systems.

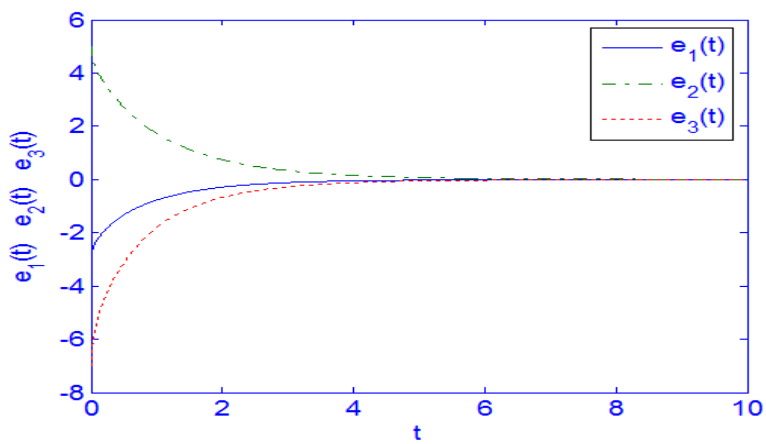


Fig. 7.7. The evolution of the error functions of chaotic systems.

7.4 Anti-phase synchronization between fractional order uncertain Qi and Genesio-Tesi systems using active control method

Here the fractional order uncertain Qi system is taken as the drive system as

$$\begin{aligned} D_t^{\alpha} x_1 &= a(y_1 - x_1) + y_1 z_1 + z_1 + 0.5\cos(10t) \\ D_t^{\alpha} y_1 &= c x_1 - y_1 - x_1 z_1 - 0.02 z_1 - x_1 - 0.5\sin(10t) \\ D_t^{\alpha} z_1 &= -b z_1 + x_1 y_1 - 2y_1 + 0.5\cos(10t), \end{aligned} \tag{7.12}$$

where uncertain parameters and disturbance terms are defined in the Section 7.3

The fractional order uncertain Genesio-Tesi system is considered as the response system as

$$
\begin{aligned}
D_t^{\alpha} x_2 &= y_2 - 0.02 x_2 + 0.1 z_2 + 0.1\cos(10t) + \mu_1(t) \\
D_t^{\alpha} y_2 &= z_2 - 0.5 y_2 - 0.5\sin(10t) + \mu_2(t) \\
D_t^{\alpha} z_2 &= -p x_2 - q y_2 - r z_2 + m x_2^2 - 10 x_2 + 0.2\sin(10t) + \mu_3(t),
\end{aligned}
\tag{7.13}
$$

where uncertain parameter $\Delta A_2 = \begin{pmatrix} -0.02 & 0 & 0.1 \\ 0 & -0.5 & 0 \\ 10 & 0 & 0 \end{pmatrix}$, disturbance term $d_2(t) = \begin{pmatrix} 0.1\cos(10t) \\ -0.5\sin(10t) \\ 0.2\cos(10t) \end{pmatrix}$. Defining the error states as $e_1 = x_2 + x_1$, $e_2 = y_2 + y_1$, $e_3 = z_2 + z_1$ and choosing the control functions as

$$
\begin{aligned}
\mu_1(t) &= -(a-1) y_1 - (0.02 - a) x_1 - 0.9 z_1 - y_1 z_1 - 0.6\cos(10t) + V_1(t) \\
\mu_2(t) &= -(1-c) x_1 + 0.5 y_1 + 1.02 z_1 + x_1 z_1 + \sin(10t) + V_2(t) \\
\mu_3(t) &= -(p+10) x_1 - (q-2) y_1 - (r-b) z_1 - m x_2^2 - x_1 y_1 - 0.7\cos(10t) + V_3(t),
\end{aligned}
\tag{7.14}
$$

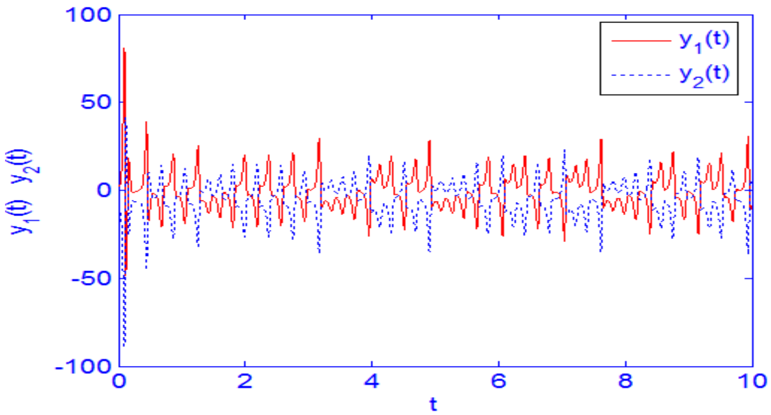
where the linear control inputs are expressed as

$$
\begin{bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{bmatrix} = \begin{pmatrix} 0 & -1 & -0.1 \\ 0 & 0 & -1 \\ (p+10) & q & 0 \end{pmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},
$$

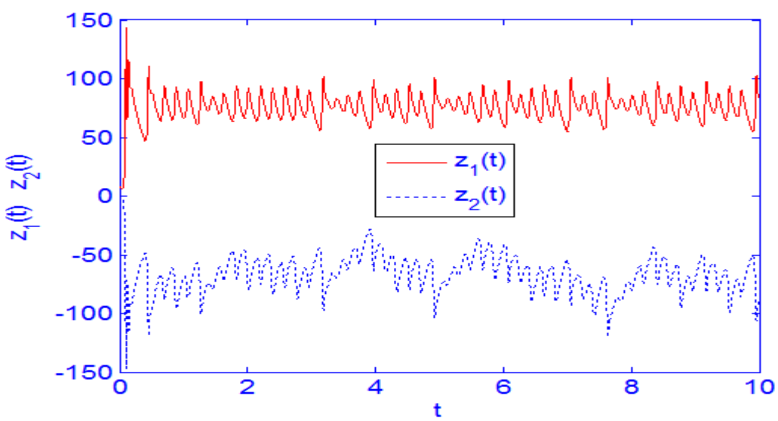
we get the error system as

$$
\begin{aligned}
D_t^{\alpha} e_1 &= -0.02 e_1 \\
D_t^{\alpha} e_2 &= -0.5 e_2 \\
D_t^{\alpha} e_3 &= -r e_3.
\end{aligned}
\tag{7.15}
$$

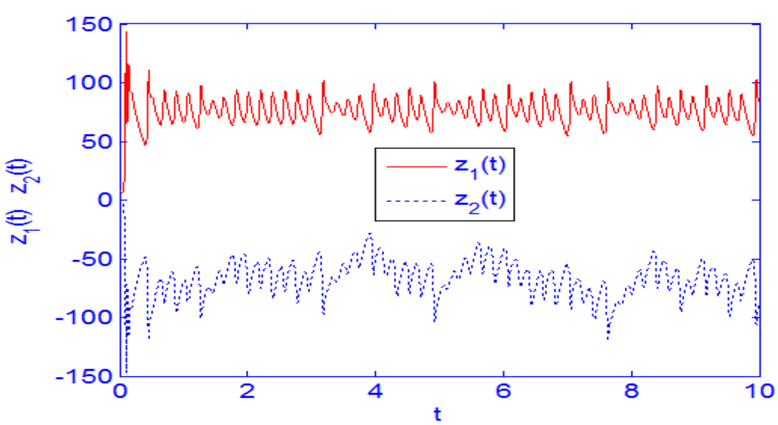
Thus the error system (7.15) is stable and anti-synchronization between the systems (7.12) and (7.13) is achieved.



(a) State trajectories between x_1 and x_2.



(b) State trajectories between y_1 and y_2.



(b) State trajectories between z_1 and z_2.

Fig. 7.8. Anti-phase synchronization for fractional-order $\alpha = 0.95$.

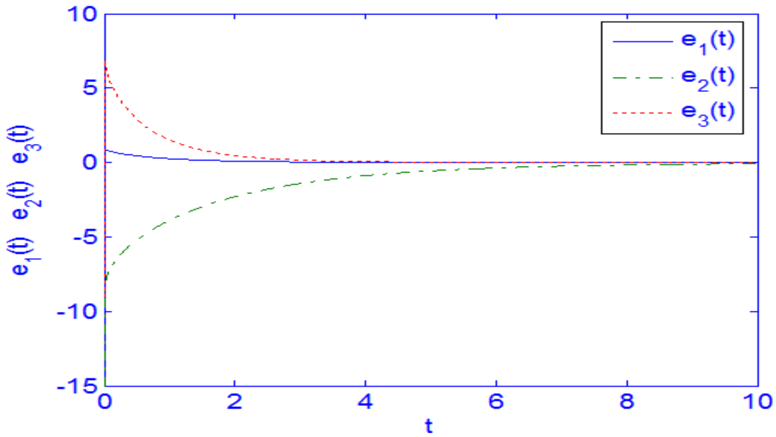


Fig. 7.9. The evolution of the error functions of uncertain chaotic systems.

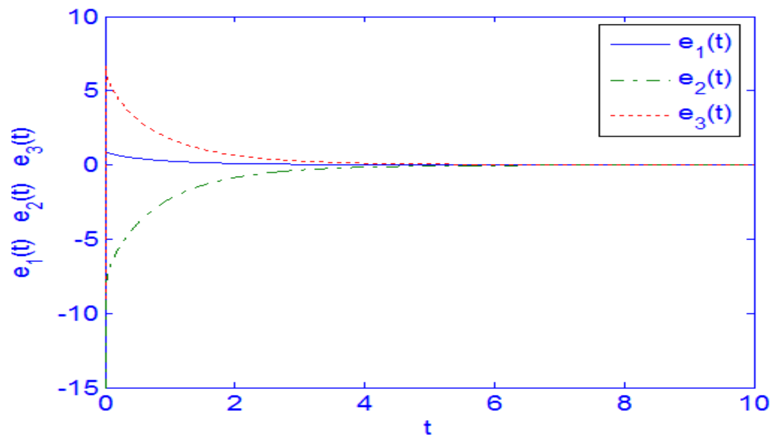


Fig. 7.10. The evolution of the error functions of chaotic systems.

7.5 Numerical simulation and results

In numerical simulation the parameters of the Genesio-Tesi and Qi systems are taken as $p = 6,\ q = 2.92,\ r = 1.2$ and $m = 1$ and $a = 35, b = 8/3, c = 80$ respectively and time step size is taken as 0.005. The initial values of the drive and response systems are taken as $(-2, 3, 5)$ and $(-1, -1, -2)$ respectively. Thus the initial errors are $(1, -4, -7)$. Now choosing $\lambda_1 = 0,\ \lambda_2 = -1,\ \lambda_3 = -b,$ the control function is obtained and phase synchronization between signals x_1 and x_2 is achieved. It should be noted that, when $\lambda_1 = 0,\ \lambda_2 = -1,\ \lambda_3 = -b,$

signals y_1 and y_2, and z_1 and z_2 become synchronized. If $\lambda_1 = -a,\ \lambda_2 = 0,\ \lambda_3 = -b$ and $\lambda_1 = -a,\ \lambda_2 = -1,\ \lambda_3 = 0$ are taken, phase synchronizations between signals y_1 and y_2, and z_1 and z_2 are also been obtained. State trajectories of the phase synchronization of drive and response systems are depicted through Fig. 7.5 for the order of the derivative $\alpha = 0.95$.

During anti-synchronization, the initial values of the drive and response systems are taken as (-2, 3, 8) and (3, -12, -1) respectively. Thus the initial errors are (5, -15, -9). Now proceeding as before, with proper choices of eigen values, the obtained state trajectories during the anti-phase synchronizations of drive and response systems are displayed through Fig. 7.8 at $\alpha = 0.95$.

Fig. 7.6 shows that time taken for synchronization of uncertain chaotic systems is less after removing the uncertainty parameters and disturbances as compared to that of the simple chaotic systems depicted in Fig. 7.7. While during anti-synchronization of uncertain chaotic systems it is found that it takes more time during the first one Fig. 7.9 than the later one Fig. 7.10.

7.6 Conclusion

In the present chapter three important objectives are achieved during the study of phase and anti-phase synchronizations between non-identical fractional order chaotic systems using active control method in presence of parametric uncertainties and external disturbances. First one, using stability analysis suitable conditions for phase and anti-phase synchronization of fractional order chaotic systems through linear controller input parameters on the respective systems have been achieved. Second one is the successful implementation of the powerful active control method which provides a simple way for phase synchronization and anti-phase synchronization between a pair of fractional order uncertain chaotic systems. Third one is the comparison of time for synchronization and anti-synchronization with and without the presence of uncertain and disturbance parameters. Numerical simulation results show that the method is easy to implement and reliable for the phase and anti-phase synchronization of the nonlinear fractional order chaotic systems.