CHAPTER-6

Chaos Control and Function Projective Synchronization of Fractional Order Systems through Tracking Control Scheme

Chapter 6

Chaos control and function projective synchronization of fractional-order systems through tracking control scheme

6.1 Introduction

Study and analysis of non-linear dynamics have gained immense popularity during the last few decades due to its important feature of any real-time dynamical system. In non-linear systems, a small change in a parameter can lead to sudden and dramatic changes in both the qualitative and quantitative behaviour of the system. Chaos is an interesting phenomenon of nonlinear systems. Thus a chaotic system is applied to deterministic systems that are aperiodic and exhibit sensitive dependence on initial conditions and parameter variations. This sensitivity is popularly known as the butterfly effect. The field of chaos has grabbed the attention of the researchers and this contributes to a significant amount of the ongoing research.

In 1963, American mathematician and meteorologist Edward Norton Lorenz developed a simplified three dimensional dynamical system that exhibits lemniscates type shaped chaotic flow which shows how the state of a dynamical system evolves over time in a complex, non-repeating pattern. The system deals with the stability of fluid flows in the atmosphere. In addition to its interest in the field of nonlinear mathematics, the Lorenz model has important implications for climate and weather prediction.

The contents of this chapter have been communicated in ***Nonlinear Dynamics.***

T-system first studied by the Romanian mathematician Gheorghe Tigan in the year 2005, belongs to the class of Lorenz system (Lorenz (1963)), which is a three-dimensional autonomous nonlinear system having potential applications in secure communications. The beauty of the system is that it is a relatively simple system exhibiting a rich variety of behaviours due to the presence of three system parameters.

After the invention of the chaotic attractor in the year 1963 by the eminent scientist E. N. Lorenz, considerable research interests have been made in searching for new chaotic attractors. Recently, studying chaotic attractors in fractional order systems has become an active research area. The study of chaotic behaviour of fractional order dynamical systems viz., Chen system, Chua system, Rössler system, Lorenz system etc have given the area a new dimension.

Introduction of fractional calculus in the nonlinear systems especially those which are chaotic in nature have rendered a new dimension to the existing problems. Fractional order derivative has become a growing field of research since fractional order system response ultimately converges to the integer order system (Jafari *et al*. (2012) Jafari and Daftardar-Gejji (2006)). The fractional derivatives are used to describe the dynamics of systems due to its high accuracy. Another important point is that fractional order systems have gained popularity in the investigation of dynamical systems since they allow a greater flexibility in the model. This area of research has garnered a lot of attention and appreciation recently due to their ability to provide an exact description of different nonlinear phenomena and also they possess memory and display much more sophisticated dynamics compared to its integral order counterpart, which is of great significance in secure communication and control systems.

The applications of fractional calculus are growing rapidly. During last few years the applications can be found in physics and engineering (Hifer (2000), Podlubny (1999)). It was found that many systems in interdisciplinary fields could be described by the fractional differential equations, such as viscoelastic systems, dielectric polarization, electrode-electrolyte polarization and so on (Koeller (1984), Sun *et al*. (1984), Ichise *et al*. (1971)).

Synchronization of fractional order chaotic systems is one of the potent research areas due to its extensive applications in communication theory and control processing. The idea of synchronizing chaotic systems was first introduced by Pecora and Carroll (1990) in 1990, where they expressed that it is possible to synchronize chaotic systems through a simple coupling. Synchronization of chaotic dynamical systems has been intensively studied by many researchers and has attracted a great deal of interest in various field due to its important applications in ecological, physical, chemical systems and also in secure communications (Blasius *et al* (1999), Lakshmanan and Murali (1996), Han *et al*. (1995), Murali and Lakshmanan (2003)) etc. A variety of approaches of synchronization phenomena have been proposed to investigate chaos synchronization such as active control (Srivastava *et al*. (2012)), adaptive control (Agrawal and Das (2013)), sliding mode control (Hosseinnia (2010)) and so on, have been successfully applied to chaos synchronization. The concept of synchronization can be extended to complete synchronization (Agrawal *et al*. (2012)), anti-synchronization (Srivastava *et al.* (2014)), Projective synchronization (Ansari and Das (2013)), Function projective synchronization (Zhou and Zhu (2011)) etc. Chaos control and chaotic synchronization of fractional order differential systems have become one of the most interesting subjects in chaos theory. Recently, various effective techniques have been successfully applied to achieve chaos control and synchronization (Razminia *et al*. (2011), Abd-Elouahab *et al.* (2010), Hegazi *et al.* (2013), Wang *et al.* (2009)). Chaos control refers to manipulating the dynamical behavior of a chaotic system, in which the purpose is to enhance or create chaos when it is needed (González-Miranda (2004)).

In the present chapter, the author has investigated the necessary conditions for the existence of chaotic attractors in the commensurate and incommensurate fractional-order T-system applying the stability theory of fractional-order systems, chaos control of the commensurate fractional order T-system with proper feedback control method and also the synchronization between identical T-system and non-identical chaotic systems viz., T-system and Lorenz system of fractional orders using function projective synchronization (FPS) based on tracking control method. The beauty of FPS system is that drive and response systems could be synchronized up to a scaling function, but not a constant. Thus, FPS is considered to be more general case of projective synchronization (PS), and it is used to obtain higher unpredictability of the error dynamical system which can enhance the security of communications. This proportional feature can be used to extend binary digital to M-nary digital communication (Chee and Xu (2005)) for achieving fast communication compared to projective synchronization. Here numerical simulations are carried out using Adams-Bashforth-Moulton method (Diethelm *et al*. (2004), Diethelm and Ford (2004)) for different fractional order derivatives which are displayed graphically to demonstrate the efficiency of the proposed approach for different particular cases.

6.2 Stability analysis of fractional order T-system

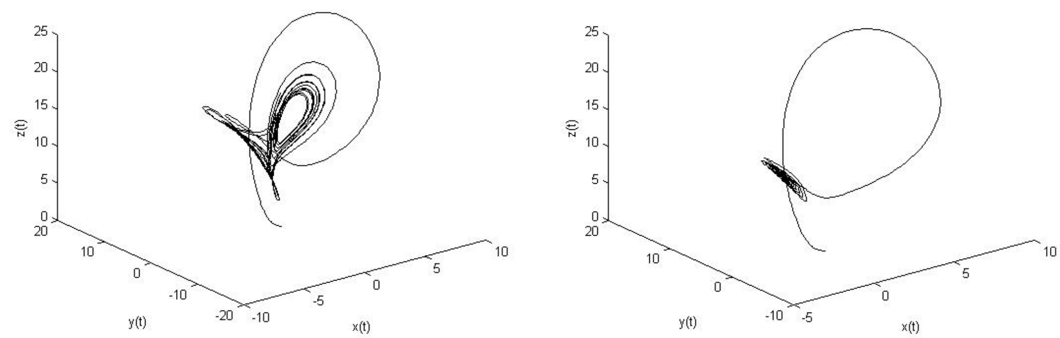
The fractional-order T-system is defined by

$$
\begin{aligned}
D_t^{\alpha_1} x &= a\,(y - x) \\
D_t^{\alpha_2} y &= (c - a)x - a\,x\,z \\
D_t^{\alpha_3} z &= -b\,z + x\,y,
\end{aligned}
\tag{6.1}
$$

where $0 < \alpha_i < 1$, $i = 1, 2, 3$ and x, y, z are state variables of the system. For the parameters $(a, b, c) = (2.1, 0.6, 30)$ the system exhibits chaos.

To find the necessary conditions for the existence of chaotic attractors in fractional-order system, the three equilibrium points of the system (6.1) as $a(y - x) = 0$, $(c - a)x - a\,x\,z = 0$ and $-b\,z + x\,y = 0$ are obtained. Substituting the values of (a, b, c), we get the equilibrium points of the system (6.1) as $E_0 = (0, 0, 0)$, $E_1 = (2.823, 2.823, 13.2857)$, $E_2 = (-2.823, -2.823, 13.2857)$. For these equilibrium points, the corresponding eigenvalues of Jacobian matrix are $(-8.7761, 6.6761, -0.60)$, $(-3.4294, 0.3647 \pm 4.5132i)$ and $(-3.4294, 0.3647 \pm 4.5132i)$ respectively. It is seen that equilibrium point E_0 is a saddle point of index 1, E_1 and E_2 are saddle points of index 2. In chaotic systems, it is proved that scrolls are generated only around the saddle points of index 2. Moreover, saddle points of index 1 are responsible only for connecting scrolls (Chua *et al.* (1986), Silva (1993), Cafagna and Grassi (2003), Lu *et al.* (2004), Tavazoei and Haeri (2007), Mohammad and Delavari (2012)).

The necessary condition of existence of chaos in fractional-order system is that the eigenvalues λ of Jacobian matrix of the system lies in the unstable region $|\arg(\lambda)| < \frac{\alpha\pi}{2}$, as shown in (Matignon (1996), Ahmed *et al.* (2007)). Through calculation, it may be shown that the lowest fractional order at which T-system shows the regular chaotic behavior for the commensurate case when $\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 0.95$. Thus the order of the system (6.1) is sum of all fractional orders derivatives in the system i.e., 2.85 for which T-system exhibits chaotic behavior. Fig. 6.1(a) displays the chaotic attractor for the same values of parameters for the standard order system and Fig. 6.1(b) shows that fractional-order T-system system is not chaotic at the commensurate fractional orders $(\alpha_1, \alpha_2, \alpha_3) = (0.92,\ 0.92,\ 0.92)$.



(a) (b)

Fig.6.1. Phase portraits of the fractional-order T-system for commensurate fractional orders (a) $(\alpha_1, \alpha_2, \alpha_3) = (0.95, 0.95, 0.95)$ (b) $(\alpha_1, \alpha_2, \alpha_3) = (0.92,\ 0.92,\ 0.92)$ in x-y-z space.

It is known that for incommensurate fractional order system, $\alpha_i = k_i / m_i$, $(k_i, m_i) = 1,\ \ k_i, m_i \in IN,\ \ i = 1,2,3$. Let m be the least common multiple of the denominators $m_i's$ of $\alpha_i's$. Then the equilibrium point E of a nonlinear fractional order system is asymptotically stable if all the roots λ of the equation

$$\det(diag\,(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, .\lambda^{m\alpha_3}) - J\big|_E) = 0,$$

satisfy $\left|\arg(\lambda)\right| > \dfrac{\pi}{2m}$ (Tavazoei and Haeri (2009)).

For incommensurate one, the following four different cases are considered.

Case 1: If $\alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = 0.90$, we get

$\det(diag(\lambda^{10}\lambda^{10}\lambda^{9}) - J(E_1)) = 0,$ for $m = 10$, which gives

$\lambda^{29} + 0.6\lambda^{20} + 2.1\lambda^{19} + 18\lambda^{10} - 0.000063\lambda^{9} + 70.3094 = 0$.

Thus, $\frac{\pi}{20} - \min_i\{|\arg(\lambda_i)|\} = 0.000424625 > 0$, which clearly shows that the system is chaotic as shown in Fig. 6.2 (a).

Case 2: If $\alpha_1 = 0.93,\ \alpha_2 = 0.99,\ \alpha_3 = 0.91$, we obtain as in Case 1,

$\det(diag(\lambda^{93}\lambda^{99}\lambda^{91}) - J(E_1)) = 0,\quad$ for $m = 100$.

i.e., $\lambda^{283} + 0.6\lambda^{192} + 2.1\lambda^{190} + 1.26\lambda^{99} + 16.7403\lambda^{93} - 0.000063\lambda^{91} + 70.3094 = 0.$

Now, $\frac{\pi}{200} - \min_i\{|\arg(\lambda_i)|\} = \frac{\pi}{200} > 0$ shows that the chaotic attractor occurs and is depicted through Fig. 6.2 (b).

Case 3: If $\alpha_1 = 0.93,\ \alpha_2 = 0.95,\ \alpha_3 = 0.96,$ we have

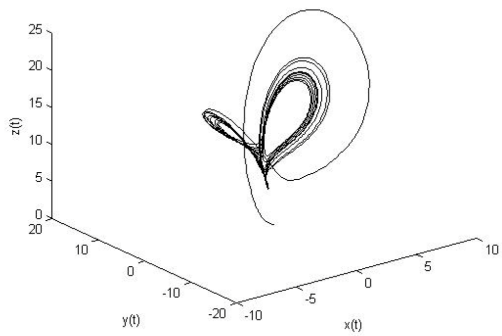
$\lambda^{284} + 0.6\lambda^{188} + 2.1\lambda^{191} + 1.26\lambda^{95} + 16.7403\lambda^{93} - 0.000063\lambda^{96} + 70.3094 = 0,$

which gives $\frac{\pi}{200} - \min_i\{|\arg(\lambda_i)|\} = \frac{\pi}{200} > 0.$ The chaotic attractor for this case is shown through Fig. 6.2 (c).

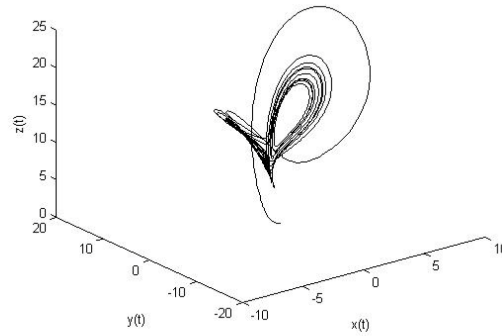
Case 4: If $\alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = 0.70$,

$\lambda^{27} + 0.6\lambda^{20} + 2.1\lambda^{17} + 18\lambda^{10} - 0.000063\lambda^{7} + 70.3094 = 0.$

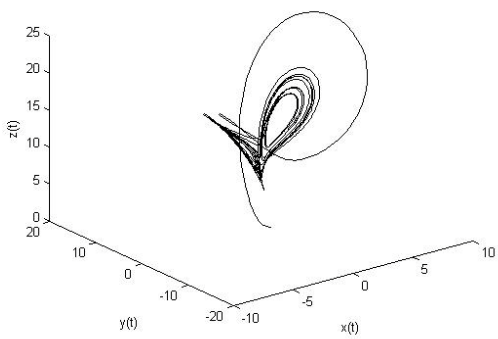
Thus $\frac{\pi}{20} - \min_i\{|\arg(\lambda_i)|\} = -0.0175357 < 0,$ which implies that system is asymptotically stable and it does not exhibit chaos as shown in Fig. 6.2 (d).



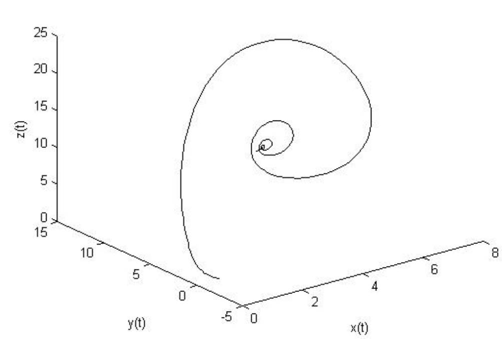
(a) $\alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = 0.90.$



(b) $\alpha_1 = 0.93,\ \alpha_2 = 0.99,\ \alpha_3 = 0.91.$



(c) $\alpha_1 = 0.93,\ \alpha_2 = 0.95,\ \alpha_3 = 0.96.$



(d) $\alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = 0.70.$

Fig. 6.2. Phase portraits of the fractional-order T-system system with incommensurate fractional orders at (a) $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0.90)$ (b) $(\alpha_1, \alpha_2, \alpha_3) = (0.93, 0.99, .91)$ (c) $(\alpha_1, \alpha_2, \alpha_3) = (0.93, 0.95, 0.96)$ and (d) asymptotic stable at $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0.70)$.

6.3 The control of fractional order chaotic T-System

In this section, the control law is applied to the commensurate fractional order chaotic T-system. In this method, controller output is directly exerted to the fractional order chaotic system as

$$\begin{aligned} D_t^{\alpha} x &= a(y-x) \\ D_t^{\alpha} y &= (c-a)x - a\,x\,z + \mu_1(t) \\ D_t^{\alpha} z &= -b\,z + x\,y + \mu_2(t), \end{aligned} \tag{6.2}$$

where $\mu_1(t)$ and $\mu_2(t)$ are control functions. For suitable stabilization, we use the control functions as $\mu_1(t) = -k_1 x - k_2 y - k_3 z$ and $\mu_2(t) = -g_1 x - g_2 y - g_3 z,$ so that the system (6.2) becomes asymptotically stable, where $k_1, k_2, k_3, g_1, g_2, g_3$ are all positive feedback control gains.

Using the above control laws in equations (6.2), we get

$$\begin{aligned} D_t^{\alpha} x &= a(y-x) \\ D_t^{\alpha} y &= (c-a)x - a\,x\,z - k_1 x - k_2 y - k_3 z \\ D_t^{\alpha} z &= -b\,z + x\,y - g_1 x - g_2 y - g_3 z. \end{aligned} \tag{6.3}$$

Now, the Jacobian matrix of equation (6.3) can be written as

$$J = \begin{pmatrix} -a & a & 0 \\ -(c-a) - az & -1 & -ax \\ -y & x & -b-1 \end{pmatrix}.$$

Taking the values of control parameters as $k_1 = 0, k_2 = 1, k_3 = 0,$ $g_1 = 0, g_2 = 0, g_3 = 1.$ For the equlibrium point E1 or E2 the eigenvalues λ_i of the matrix J of the feedback system are $-0.51172 \pm 4.44699i$ and $-3.67656,$ which satisfy the condition $\left|\arg(\lambda_i)\right| > \frac{\alpha\pi}{2}, i = 1,2,3.$ Thus the system (6.3) is asymptotically stable. From Fig. 6.3, it is clear that the feedback controllers have stabilized the chaotic systems.

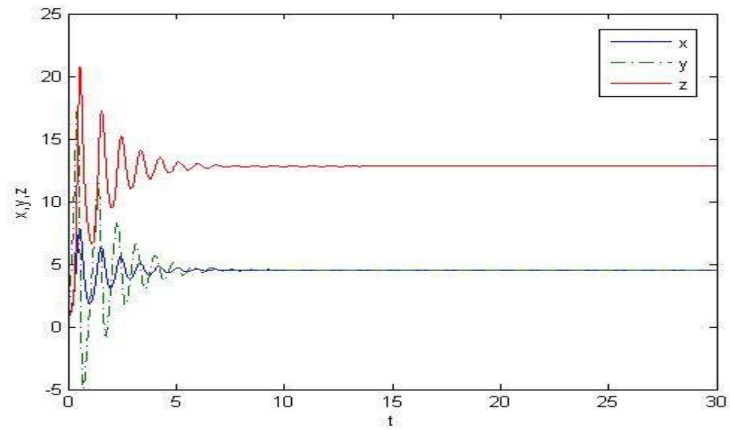


Fig. 6.3. State trajectories of the controlled chaotic system (6.3).

6.4 Fractional order function projective synchronization between identical chaotic systems

Consider fractional order T-system as the drive system as

$$\begin{aligned} D_t^{\alpha} x_1 &= a(x_2 - x_1) \\ D_t^{\alpha} x_2 &= (c-a)x_1 - a\, x_1 x_3 \\ D_t^{\alpha} x_3 &= -b\, x_3 + x_1\, x_2 \end{aligned} \tag{6.4}$$

and fractional order T-system as response system as

$$\begin{aligned} D_t^{\beta} y_1 &= a(y_2 - y_1) + u_1 \\ D_t^{\beta} y_2 &= (c-a)y_1 - a\, y_1 y_3 + u_2 \\ D_t^{\beta} y_3 &= -b\, y_3 + y_1\, y_2 + u_3, \end{aligned} \tag{6.5}$$

where u_1, u_2, u_3 are control functions. Then, the error states become

$$e_i = y_i - k_i(x_1, x_2, x_3)x_i, \quad i = 1, 2,3. \tag{6.6}$$

With the suitable controller, we can get the fractional order error dynamical system as

$$D_t^{\beta} e = (M(x,y) + N(x,y))e, \tag{6.7}$$

where $e = [\,e_1\, e_2\, e_3\,]^T$,

$$M(x,y) = \begin{pmatrix} -a & a & 0 \\ (c-a) - ay_3 & 0 & -ak_1x_1 \\ y_2 & k_1x_1 & -b \end{pmatrix} \text{ and } N(x,y) = \begin{pmatrix} 0 & -c+ay_3 & -y_2 \\ 0 & -1 & (a-1)k_1x_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Choosing real symmetric positive definite matrix $P = diag(1,1,1),$ we obtain

$$[M(x,y) + N(x,y)]P + P[M(x,y) + N(x,y)]^T = -Q, \tag{6.8}$$

where $Q = diag(2a,\, 2,\, 2b)$ is also a real symmetric positive definite matrix. Thus, two identical chaotic T-system for different fractional order derivatives achieves the FPS.

6.4.1 Numerical simulation and results

Taking $a = 2.1,\ b = 0.6,\ \ c = 30$, we obtain

$$M(x,y) + N(x,y) = \begin{pmatrix} -2.1 & -27.9 + 2.1y_3 & -y_2 \\ 27.9 - 2.1y_3 & -1 & -k_1x_1 \\ y_2 & k_1x_1 & -0.6 \end{pmatrix}.$$

Considering, $\alpha = 0.96$ and $\beta = 0.97$ for drive and response systems respectively, the scaling function becomes $K(x_1, x_2, x_3) = diag(k_1, k_2, k_3)$ $= diag(x_1 + x_3, 2 + x_1x_2, x_2 + 1)$. The initial values of the drive and the response systems are taken as $(0.1, 1.2, -0.5)$ and $(0.7, -0.2, 0.3)$ respectively. The error states between drive and response systems are shown in Figs. 6.4. Next we compute the root mean square error (RMSE) e as $e = \left(\sum_{i=1}^{3} (y_i - k_i(x)x_i)^2 \right)^{\frac{1}{2}}$ to predict the magnitude of all errors given in Figs. 6.4 in a single measure, which seems to be a good measure of accuracy and this is displayed through Fig. 6.5. Taking the same values of parameters, initial conditions and scaling functions for synchronization of the considered systems for standard order ($\alpha = 1$ and $\beta = 1$), it is seen from Fig. 6.6 and Fig. 6.7 that it takes more time to synchronize as compared to the fractional order systems.

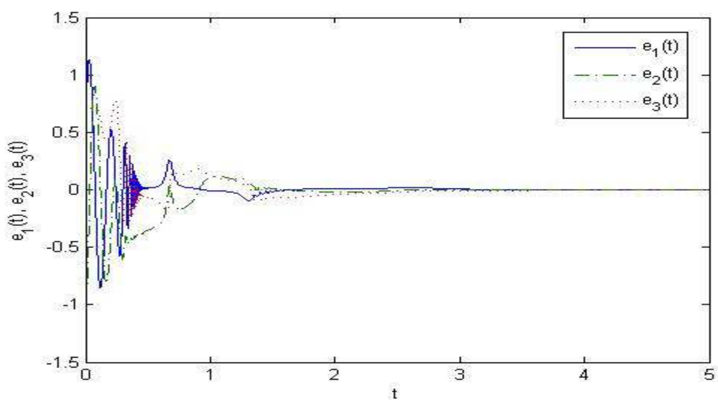


Fig. 6.4. The error states of the FPS between the systems (6.4) and (6.5) for $\alpha = 0.96$ and $\beta = 0.97$.

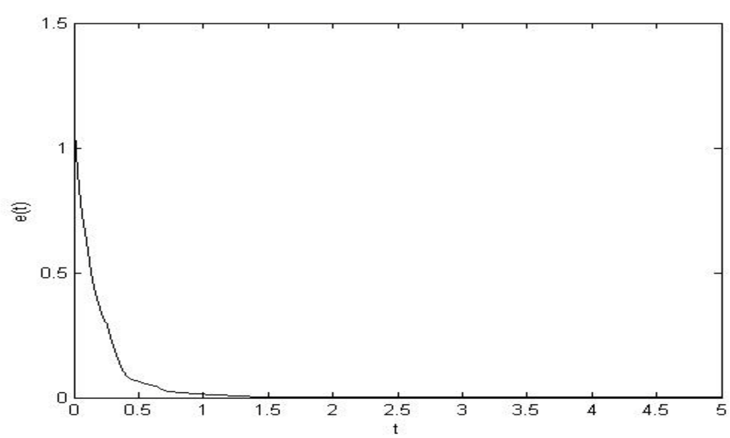


Fig. 6.5. Root mean square error (RMSE) for $\alpha = 0.96$ and $\beta = 0.97$.

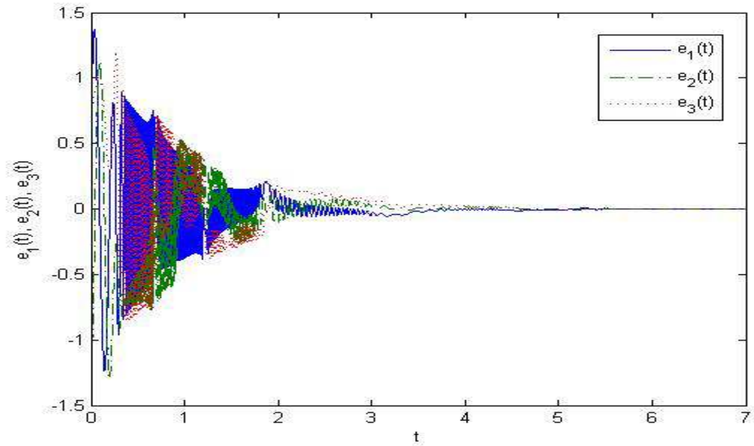


Fig. 6.6. The error states of the FPS between the systems (6.4) and (6.5) for $\alpha = 1$ and $\beta = 1$.

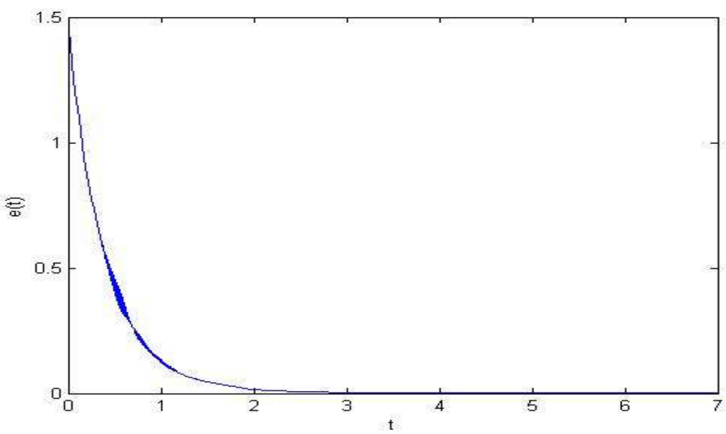


Fig. 6.7. Root mean square error (RMSE) for $\alpha = 1$ and $\beta = 1$.

6.5 Function projective synchronization between different fractional order chaotic systems

Consider fractional order T-system as a drive system as given in equation (6.4) and also taking the fractional order Lorenz system (Zhou and Zhu (2011)) as a response system as

$$D_t^{\beta} y_1 = l(y_2 - y_1) + u_1$$
$$D_t^{\beta} y_2 = my_1 - y_1 y_3 - y_2 + u_2 \qquad (6.9)$$
$$D_t^{\beta} y_3 = -ny_3 + y_1 y_2 + u_3.$$

The system (6.9) shows the chaotic behavior for the parameter values *l=10, m=28* and *n=8/3*. Using the equation (6.6) with the suitable control functions, we get the fractional order error dynamical system as

$$D_t^{\beta} e = (M(x,y) + N(x,y))e,$$

where $e = [e_1\, e_2\, e_3]^T$,

$$M(x,y) = \begin{pmatrix} -l & l & 0 \\ m - y_3 & -1 & -k_1(x)x_1 \\ y_2 & k_1(x)x_1 & -n \end{pmatrix} \text{ and } N(x,y) = \begin{pmatrix} 0 & -(l+m)+y_3 & -y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Choosing real symmetric positive definite matrix $P = diag(1, 1, 1)$, we obtain

$$[M(x,y) + N(x,y)]P + P[M(x,y) + N(x,y)]^T = -Q,$$

where $Q = diag(2l, 2, 2n)$ real symmetric positive definite matrix, which clearly exhibits that the chaotic T-system and Lorenz system for different fractional orders achieve the FPS.

6.5.1 Numerical simulation and results

In this section, taking *l=10, m=28, n=8/3* and $\alpha = 0.96$ for drive system and $\beta = 0.995$ for response systems, we get scaling function as $K(x_1, x_2, x_3) = diag(k_1, k_2, k_3) = diag(x_1 x_3,\ 2 + x_1,\ x_2 + x_3)$. The initial values of the drive and response systems are taken as $(1,2,1)$ and $(-5, -2, -5)$ respectively. The numerical results of different combination of error states between the systems and also the RMSE for good measure of accuracy are depicted through Figs. 6.8 and 6.9 Synchronization for the standard order case i.e., for $\alpha = 1, \beta = 1$ for considered chaotic systems are depicted through Figs. 6.10 and 6.11.

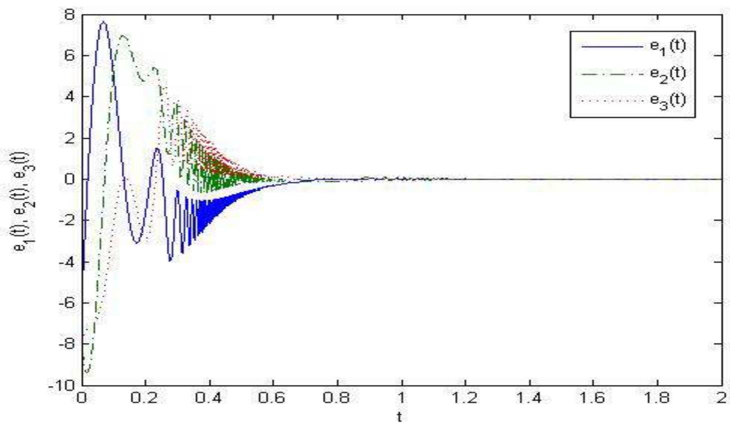


Fig. 6.8. The error states of the FPS between the systems (6.4) and (6.9) for $\alpha = 0.96$ and $\beta = 0.995$.

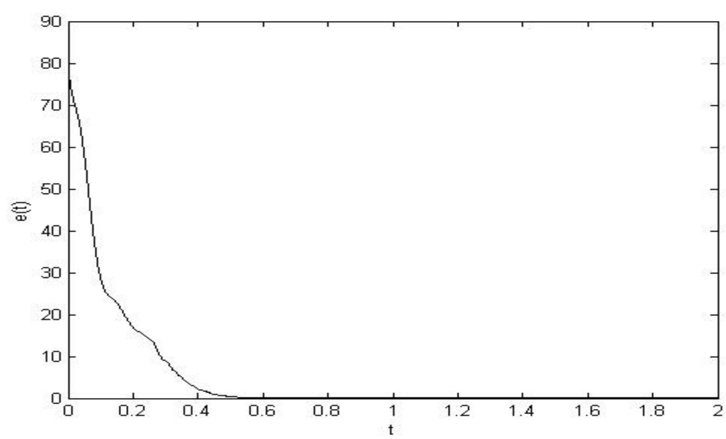


Fig. 6.9. Root mean square error (RMSE) for $\alpha = 0.96$ and $\beta = 0.995$.

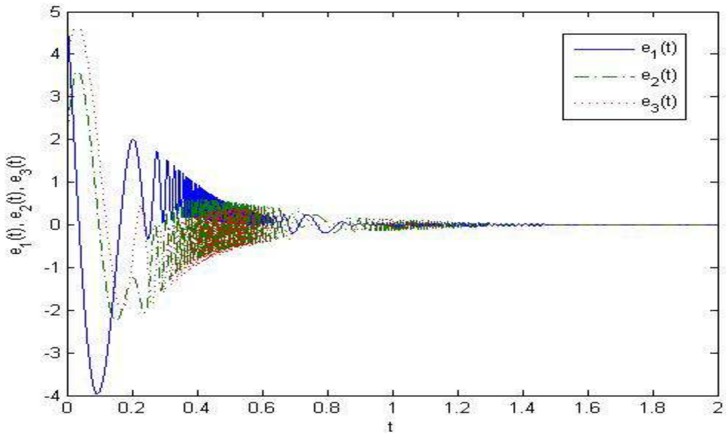


Fig. 6.10. The error states of the FPS between the systems (6.4) and (6.9) for $\alpha = 1$ and $\beta = 1$.

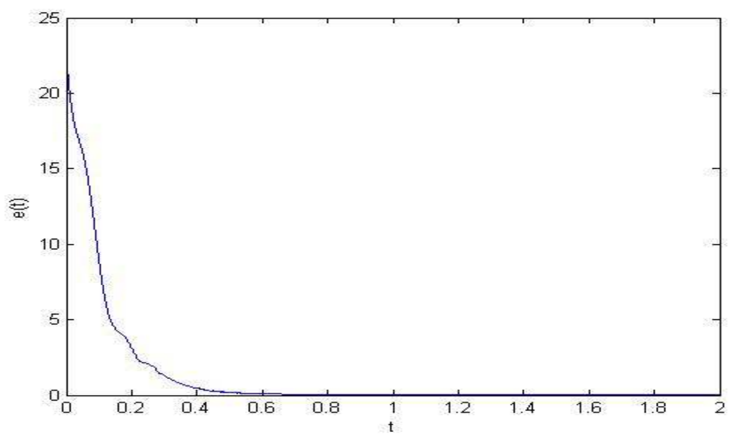


Fig. 6.11. Root mean square error (RMSE) for $\alpha = 1$ and $\beta = 1$.

6.6 Conclusion

Three important goals are achieved through the present study. First one is the successful presentation of the stability criterion for commensurate and incommensurate fractional-order T-system based on the stability theory of fractional-order systems. Second one, control laws are proposed to stabilize the fractional order chaotic T-system with proper feedback control method. Third one is the successful graphical presentations of synchronization between identical T-system and also between non-identical T-system &

Lorenz system in fractional order case using numerical simulations. The author is optimist that the simulation results of the present research work will be appreciated and utilized by the researchers involved in the field of mathematical modelling of fractional order nonlinear dynamical systems. The salient feature of the article is the graphical presentations of less time requirement for synchronization of identical and non-identical chaotic systems when it approaches from standard order to fractional order.