CHAPTER-5

A New Modified Adaptive Function Projective Synchronization Method to Synchronize the Time-Delayed Chaotic Systems

Chapter 5

A new modified adaptive function projective synchronization method to synchronize the time-delayed chaotic systems

5.1 Introduction

Synchronization of chaotic systems is carried out when two or more systems are coupled or one system drives the other. It is a difficult phenomenon due to the extreme dependence on initial conditions. During coupling the trajectories of the systems emerging from two different initial conditions will spread exponentially with time caused due to the transition between system variables, which encourages researchers to take challenges for the study of synchronization of coupled chaotic systems. There are different types of synchronization schemes viz., complete, hybrid, phase, anti-phase, projective synchronizations etc. (Sudheer and Sabir (2011), Fujisaka and Yamada (1983a, 1983), Pecora and Carroll (1990), Mahmoud and Mahmoud (2010), Liu (2006), Rosenblum et al. (1997), Agrawal et al. (2012a, 2012), Das et al. (2013), Ghosh and Bhattacharya (2010)). Function projective synchronization between two systems is a generalization of projective synchronization, which is synchronized upto a scaling factor. This interesting phenomenon is firstly handled by Mainieri and Rehacek (1999). Delay differential equations have potential applications in science and engineering (Mackey and Glass (1997), Ikeda et al. (1980), Bunner et al. (1998), Yongzhen et al. (2011), Liao et al. (2007), Kwon et al. (2011)) due to the presence of factors like process time

---

The contents of this chapter have been communicated in ***Pramana - Journal of Physics***.

existence of some stage structure, modelling via high dimensional compartmental models, estimation of parameters involved in the models etc. Research on synchronization of time delayed chaotic systems (Pyragas (1998), He and Vaidya (1999)) has received considerable attention of the researchers working in population dynamics, laser physics, physiological model, neural networks, control theory etc. (Sahaverdiev and Shore (2002), Sahaverdiev et al. (2002), Banerjee et al. (2013), Zhan et al. (2003), Senthilkumar et al. (2006)) due to their natural relation to the systems with memory. In 2011, Sudheer and Sabir (2011) have proposed adaptive function projective synchronization with some modifications to synchronize the time-delayed chaotic systems. They considered estimated parameters in response system. During stabilization of error system, they have used Lyapunov stability theory while in the present article the author has designed controller function with appropriate estimated parameters and used Lyapunov-Krasovskii Functional approach (Krasovskii and Brenner (1963)) to stabilize the error system. During comparison of effectiveness of the methods, the Rossler system having the same parameters' values and scaling function factors as in Sudheer and Sabir (2011) are taken for Function Projective Synchronization. It is seen that the time of synchronization through numerical simulation, which are carried out using Raunga-Kutta method for delay differential equation for the proposed method, is less as compared to the exiting method (Sudheer and Sabir (2011)). To validate the proposed method another two cases for different initial conditions are accomplished with graphical plots along with the demonstrations of graphs obtained using the method described in Sudheer and Sabir (2011), which also establishes the fact that the proposed method gives the faster synchronization for different considered cases.

5.2 Proposed modified adaptive function projective synchronization method (PMAFPS)

Consider the drive system in the form

$$\begin{aligned} x'(t) &= f(x(t)) + g(x(t))A + h(x(t-\tau))B, \qquad t>0, \\ x(t) &= \phi(t), \quad -\tau \le t \le 0 \end{aligned} \tag{5.1}$$

and the response system in the form

$$\begin{aligned} y'(t) &= f(y(t)) + g(y(t))A + h(y(t-\tau))B + U, \qquad t>0, \\ y(t) &= \psi(t), \quad -\tau \le t \le 0, \end{aligned} \tag{5.2}$$

where $x, y \in R^n$ are the state vectors of systems (5.1) and (5.2), respectively; $A \in R^{m_1}$ and $B \in R^{m_2}$ are the unknown parameter vectors of the systems; $f(x), f(y) \in R^n$, $g(x), g(y) \in R^{n\times m_1}$, and $h(x(t-\tau)), h(y(t-\tau)) \in R^{n\times m_2}$ are nonlinear functions; $\phi(t)$ and $\psi(t)$ represent the trajectories of the solutions in the past; τ is the time delay; and $U \in R^n$ is the controller.

Let, $e(t) = \lambda(t)x(t) - y(t)$ represents the synchronization error vector, where $\lambda(t)$ is the scaling function matrix. The error dynamical system is

$$\begin{aligned} e'(t) &= \lambda'(t)x'(t) + \lambda(t)x'(t) - y'(t) \\ &= \lambda'(t)x'(t) + (\lambda(t)f(x(t)) - f(y(t))) + (\lambda(t)g(x(t)) - g(y(t)))A \\ &\quad + (\lambda h(x(t-\tau)) - h(y(t-\tau)))B - U. \end{aligned} \tag{5.3}$$

Let us design the nonlinear controller function and adaptive update laws as

$$\begin{aligned} U &= \lambda'(t)x'(t) + (\lambda(t)f(x(t)) - f(y(t))) + (\lambda(t)g(x(t)) - g(y(t)))\hat{A}(t) \\ &\quad + (\lambda h(x(t-\tau)) - h(y(t-\tau)))\hat{B}(t) + (1/2 + k)e(t), \qquad k>0 \end{aligned} \tag{5.4}$$

and

$$\hat{A}' = -[\lambda(t)g(x(t)) - g(y(t))]^T e(t) - \overline{A}(t)$$
$$\hat{B}' = -[\lambda(t)h(x(t-\tau)) - h(y(t-\tau))]^T e(t) - B(t), \tag{5.5}$$

where vectors $\hat{A}(t)$ and $\hat{B}(t)$ are the estimated values of unknown parameters A and B respectively, $\overline{A}(t) = A - \hat{A}(t)$ and $\overline{B}(t) = B - \hat{B}(t)$ are estimate errors. Using equation (5.4), the error dynamical system (5.3) is reduced to

$$e'(t) = (\lambda(t)g(x(t)) - g(y(t)))(A - \hat{A}(t)) + (\lambda h(x(t-\tau)) - h(y(t-\tau)))(B - \hat{B}(t)) \\ -(1/2 + k)e(t). \tag{5.6}$$

Let us consider the Lyapunov-Krasovskii Functional (Krasovskii and Brenner (1963)) to carry out stability analysis as

$$\mathrm{V} = \frac{1}{2}e^T(t)e(t) + \frac{1}{2}\int_{-\tau}^{0} e^T(t+\theta)e(t+\theta)d\theta + \frac{1}{2}(\overline{A}^T(t)\overline{A}(t) + \overline{B}^T(t)B(t)). \tag{5.7}$$

The time derivative of V along the trajectory of error dynamical system is given by

$$\mathrm{V}' = e^T(t)e'(t) + \frac{1}{2}(e^T(t)e(t) - e^T(t-\tau)e(t-\tau)) + (\overline{A}'^T(t)\overline{A}(t) + \overline{B}'^T(t)\overline{B}(t)),$$

where $\overline{A}'^T = -\hat{A}'^T$ and $\overline{B}'^T = -\hat{B}'^T$.

Now

$$V' = e^T(t)((\lambda(t)g(x(t)) - g(y(t)))(A - \hat{A}(t)) + (\lambda h(x(t-\tau)) - h(y(t-\tau)))(B - \hat{B}(t)) \\ -(1/2 + k)e(t)) + \frac{1}{2}(e^T(t)e(t) - e^T(t-\tau)e(t-\tau)) - (\hat{A}'^T(t)\overline{A}(t) + \hat{B}'^T(t)\overline{B}(t)).$$

Using adaptive parameters update laws, we get

$$V' = -ke^T(t)e(t) - \frac{1}{2}e^T(t-\tau)e(t-\tau) - \overline{A}^T\overline{A} - \overline{B}^T\overline{B} < 0, \tag{5.8}$$

where $V \in R^n$ is positive definite function and $V' \in R^n$ is negative definite function. Thus $e_i(t) \to 0$ as $t \to \infty, i = 1,2,3$. Therefore, the error system is asymptotically stable which means that PMAFPS between the systems (5.1) and (5.2) is achieved and it is also seen that the parameters' estimation errors $\overline{A}(t)$ and $\overline{B}(t)$ decay to zero as time goes to infinity.

5.3 Systems' Descriptions

A double delayed Rössler System is given by (Sudheer and Sabir (2011), Ghosh *et al.* (2008))

$$\begin{aligned} x'_1(t) &= -x_2(t) - x_3(t) + a_1 x_1(t-\tau_1) + a_2 x_1(t-\tau_2) \\ x'_2(t) &= x_1(t) + b_1 x_2(t) \\ x'_3(t) &= b_2 + x_1(t)x_3(t) - c x_3(t), \end{aligned} \tag{5.9}$$

where a_1, a_2 are the geometric factors, while b_1, b_2 and c are the usual parameters of a classical Rössler system, τ_1 and τ_2 are time delays. The double delayed Rössler system exhibits the chaotic trajectories for the parameter values a_1 = 0.2, a_2 = 0.5, $b_1 = b_2$ = 0.2, c = 5.7, $\tau_1 = 1.0$ and $\tau_2 = 2.0$ with initial condition $(x_1(t), x_2(t), x_3(t)) = (0.5, 1, 1.5)$ with $-\tau \le t \le 0$ as shown in Fig. 5.1.

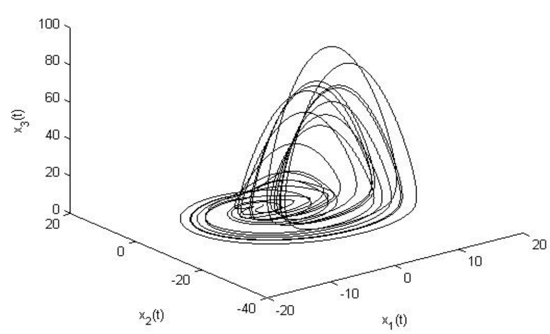


Fig. 5.1. Phase portraits of Rössler system in $x_1(t) - x_2(t) - x_3(t)$ space.

5.4 Proposed modified adaptive function projective synchronization between identical Rössler systems

In order to achieve PMAFPS behaviour the drive system is taken as (5.9) and the response system is given by

$$\begin{aligned} y'_1(t) &= -y_2(t) - y_3(t) + a_1 y_1(t-\tau_1) + a_2 y_1(t-\tau_2) + u_1(t) \\ y_2(t) &= y_1(t) + b_1 y_2(t) + u_2(t) \\ y'_3(t) &= b_2 + y_1(t) y_3(t) - c y_3(t) + u_3(t), \end{aligned} \tag{5.10}$$

where $u_1(t), u_2(t), u_3(t)$ are controllers and the parameters (a_1, a_2, b_1, b_2, c) of drive and response systems are unknown.

Defining the error states as $e_i(t) = \lambda_i(t) x_i(t) - y_i(t),\ i = 1,2,3$, we get

$$\begin{aligned} e'_1(t) &= \lambda'_1(t) x_1(t) - \lambda_1(t)(x_2(t) + x_3(t)) + \lambda_1(t)(a_1 x_1(t-\tau_1) + a_2 x_1(t-\tau_2)) \\ &\quad + y_2(t) + y_3(t) - a_1 y_1(t-\tau_1) - a_2 y_1(t-\tau_2) - u_1(t) \\ e'_2(t) &= \lambda'_2(t) x_2(t) + \lambda_2(t)(x_1(t) + b_1 x_2(t)) - y_1(t) - b_1 y_2(t) - u_2(t) \\ e'_3(t) &= \lambda'_3(t) x_3(t) + \lambda_3(t)(b_2 + x_1(t) x_3(t) - c x_3(t)) - b_2 - y_1(t) y_3(t) + c y_3(t) - u_3(t). \end{aligned} \tag{5.11}$$

According to our PMAFPS method, we take the synchronization controller as

$$\begin{aligned} u_1(t) &= \lambda'_1(t) x_1(t) - \lambda_1(t)(x_2(t) + x_3(t)) + y_2(t) + y_3(t) + \hat{a}_1(t)(\lambda_1(t) x_1(t-\tau_1) - y_1(t-\tau_1)) \\ &\quad + \hat{a}_2(t)(\lambda_1(t) x_1(t-\tau_2) - y_1(t-\tau_2)) + (1/2 + k_1) e_1(t) \\ u_2(t) &= \lambda'_2(t) x_2(t) + \lambda_2(t) x_1(t) - y_1(t) + \hat{b}_1(t)(\lambda_2(t) x_2(t) - y_2(t)) + (1/2 + k_2) e_2(t) \\ u_3(t) &= \lambda'_3(t) x_3(t) + \hat{b}_2(t)(\lambda_3(t) - 1) + \lambda_3(t) x_1(t) x_3(t) - y_1(t) y_3(t) - \hat{c}(t)(\lambda_3(t) x_3(t) - y_3(t)) \\ &\quad + (1/2 + k_3) e_3(t) \end{aligned} \tag{5.12}$$

and the estimated parameters as

$$\begin{aligned}
\hat{a}_1'(t) &= (\lambda_1(t)x_1(t-\tau_1) - y_1(t-\tau_1))e_1(t) + (a_1 - \hat{a}_1(t)) \\
\hat{a}_2'(t) &= (\lambda_1(t)x_1(t-\tau_2) - y_1(t-\tau_2))e_1(t) + (a_2 - \hat{a}_2(t)) \\
\hat{b}_1'(t) &= (\lambda_2(t)x_2 - y_2(t))e_2(t) + (b_1 - \hat{b}_1(t)) \\
\hat{b}_2'(t) &= (\lambda_3(t) - 1)e_3(t) + (b_2 - \hat{b}_2(t)) \\
\hat{c}'(t) &= -(\lambda_3(t)x_3(t) - y_3(t))e_3(t) + (c - \hat{c}(t)),
\end{aligned} \tag{5.13}$$

which helps to accomplish the error system as

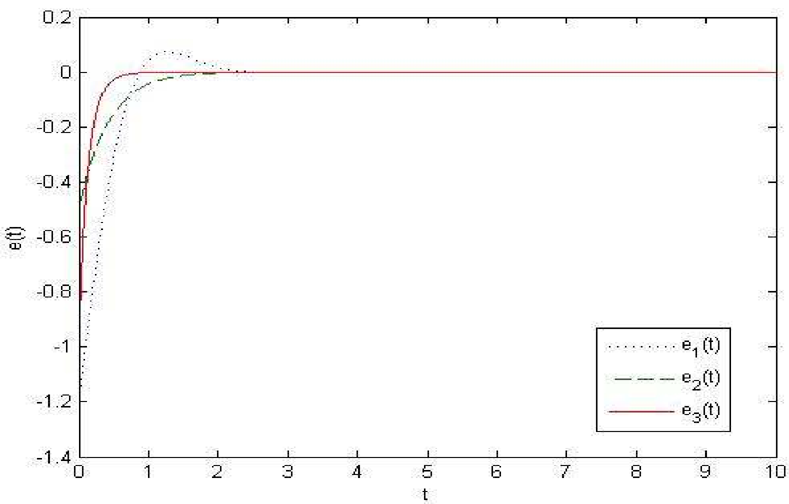
$$\begin{aligned}
e'_1(t) &= (\lambda_1(t)x_1(t-\tau_1) - y_1(t-\tau_1))(a_1 - \hat{a}_1(t)) + (\lambda_1(t)x_1(t-\tau_2) - y_1(t-\tau_2))(a_2 - \hat{a}_2(t)) \\
&\quad - (1/2 + k_1)e_1(t) \\
e'_2(t) &= (\lambda_2(t)x_2(t) - y_2(t))(b_1 - \hat{b}_1(t)) - (1/2 + k_2)e_2(t) \\
e'_3(t) &= (\lambda_3(t) - 1)(b_2 - \hat{b}_2(t)) - (\lambda_3(t)x_3(t) - y_3(t))(c - \hat{c}(t)) - (1/2 + k_3)e_3(t).
\end{aligned} \tag{5.14}$$

Now proceeding as section 5.2 with proper choices of controller and estimation of unknown parameters using parameter update laws, it may be concluded that the PMAFPS between systems (5.9) and (5.10) is achieved.

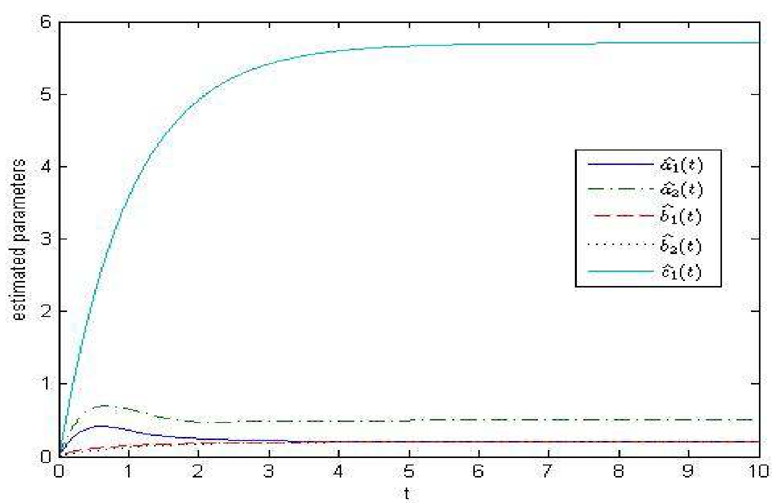
5.4.1 Numerical Simulation and Results

To demonstrate the effectiveness of PMAFPS method, during the numerical simulation, the initial conditions of state vectors of drive and response systems are taken as (0.5, 1, 1.5) and (2.5, 2, 2.5) respectively. The true values of unknown parameter vectors are selected as $a_1 = 0.2,\ a_2 = 0.5,\ b_1 = 0.2,$ $b_2 = 0.2,\ c = 5.7$. The initial value of estimated unknown parameter vector is chosen as $(\hat{a}_1(t), \hat{a}_2(t), \hat{b}_1(t), \hat{b}_2(t), \hat{c}(t)) = (0,0,0,0,0)$. To compare the results with the result proposed by Sudheer and Sabir (2011), the parametric values of Rössler system are chosen as given in section 5.3, the scaling function factors are taken as $\lambda_1(t) = 2 + \sin(t+35)$, $\lambda_2(t) = 1.5 + \sin(t)$ and $\lambda_3(t) = 2 - \cos(t)$ and the control input as $(k_1, k_2, k_3) = (2,2,2)$ as considered in (Sudheer and Sabir (2011)). Figs. 5.2(a) and 5.2(b) represent that the errors

$e_i(t) \to 0,\ i = 1,2,3$ and the convergence of estimated parameters to the original values after small duration of time, which clearly show that in both the occasions it takes much lesser time in comparison with the results as obtained in Figs. 5.3(a) and 5.3(b) through the method described in (Sudheer and Sabir (2011)). This validates the feasibility and effectiveness of the new proposed method. Figs. 5.4(a), 5.4(b), 5.5(a) and 5.5(b) depict the numerical simulation results of errors and estimated parameters using the proposed method and the existing method respectively for the initial conditions as (1, 1, 1) and (1.5, 1.5, 1.5) and also for scaling function factors $\lambda_1(t) = 2 + \sin(t + 35)$, $\lambda_2(t) = 1.5 + \sin(t)$ and $\lambda_3(t) = 2 - \cos(t)$. Figs. 5.6(a), 5.6(b), 5.7(a) and 5.7(b) describe those for initial conditions of drive and response systems as (2.5, 2, 2.5) and (0.5, 1, 1.5) respectively and for scaling function factors $\lambda_1(t) = 1 + \cos(0.05t)$, $\lambda_2(t) = 2 - \sin(t)$ and $\lambda_3(t) = 3 + \cos(t + 10)$.

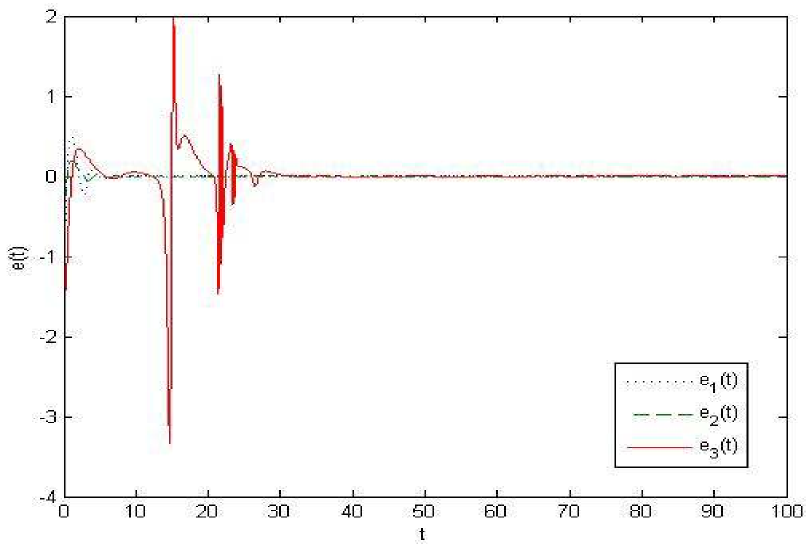


(a)

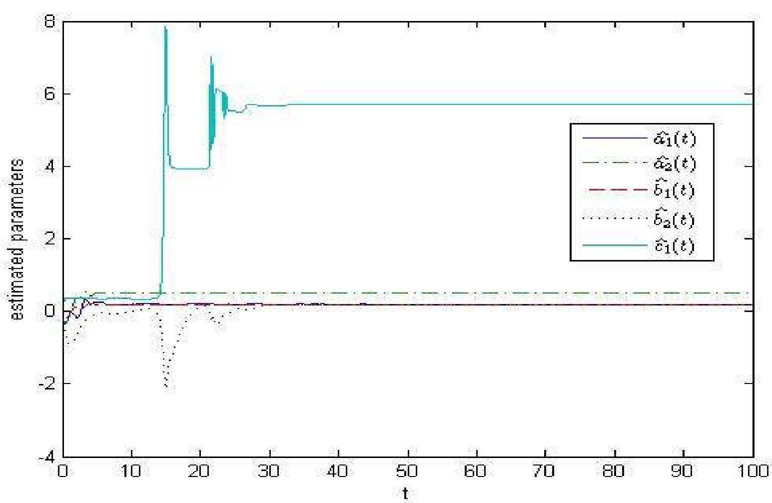


(b)

Fig. 5.2. (a) State trajectories of errors system and (b) The estimated parameters obtained by using the proposed method for the initial conditions of drive and response systems as (0.5, 1, 1.5) and (2.5, 2, 2.5) respectively.

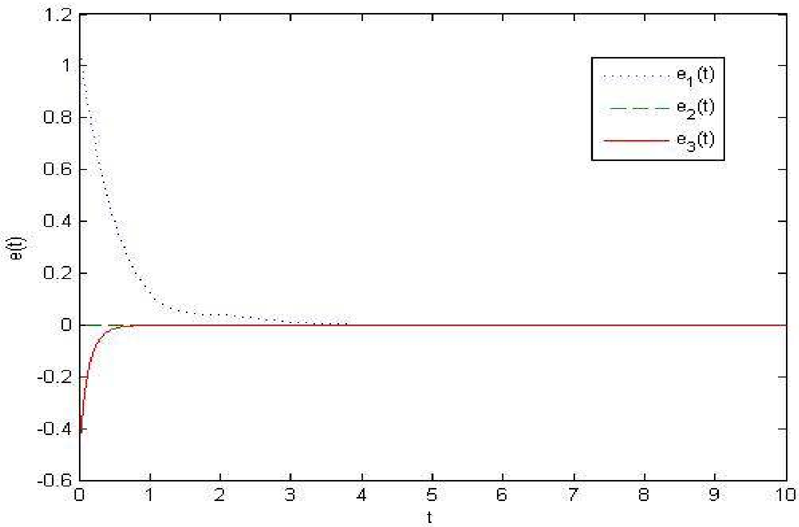


(a)

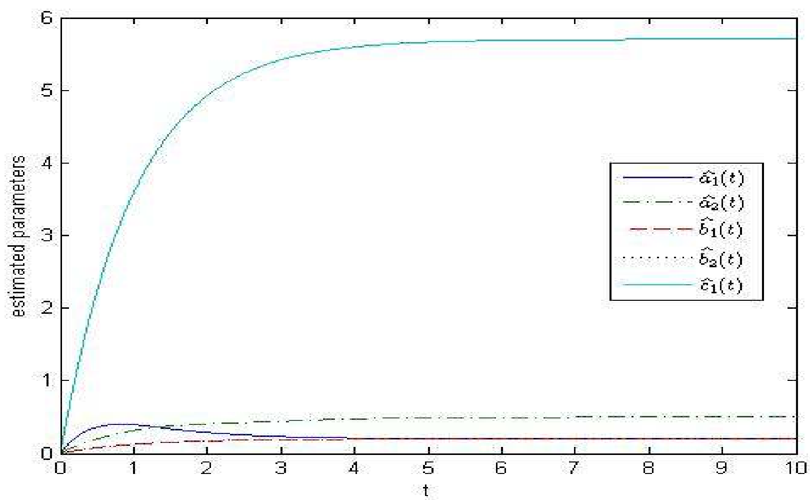


(b)

Fig. 5.3. (a) State trajectories of errors system and (b) The estimated parameters obtained by using the method described by Sudheer and Sabir (2011) for the initial conditions of drive and response systems as (0.5, 1, 1.5) and (2.5, 2, 2.5) respectively.

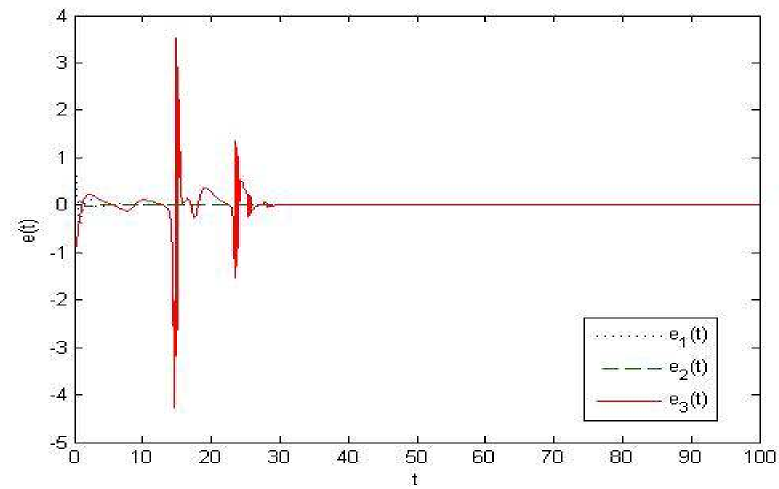


(a)

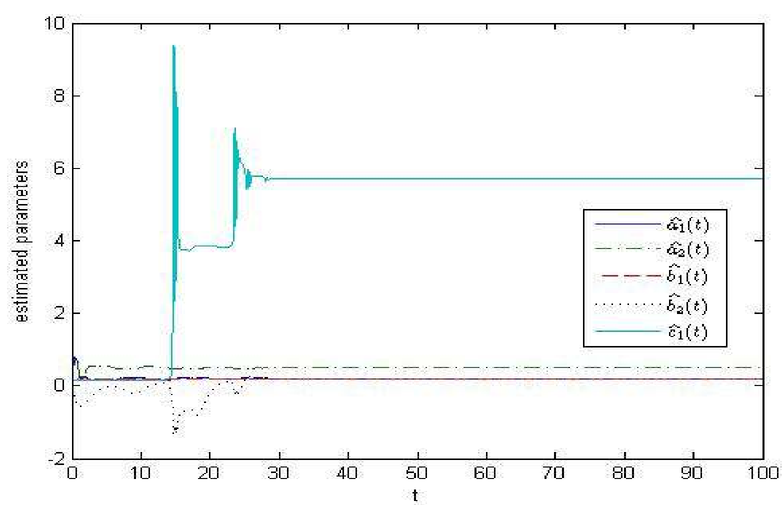


(b)

Fig. 5.4. (a) State trajectories of errors system and (b) The estimated parameters obtained by using the proposed method for the initial conditions of drive and response systems as (1, 1, 1) and (1.5, 1.5, 1.5) respectively.

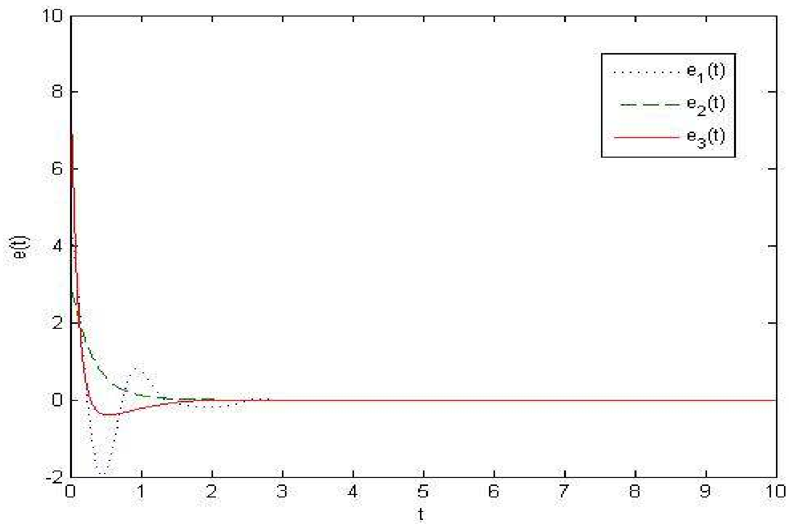


(a)

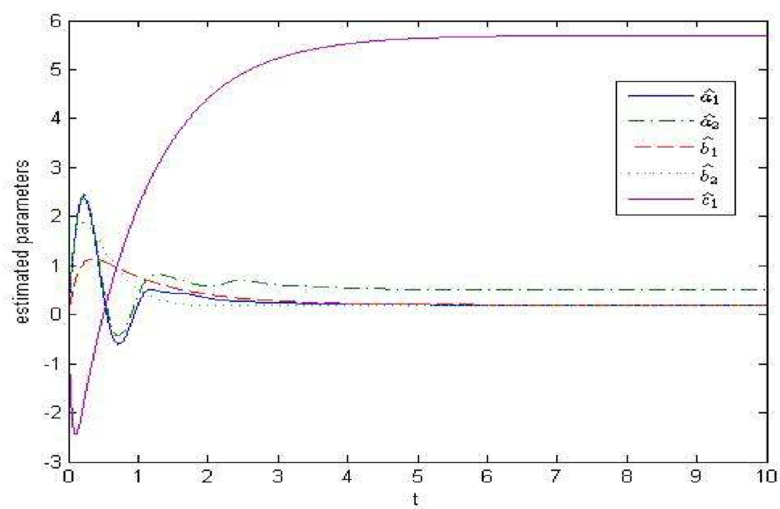


(b)

Fig. 5.5. (a) State trajectories of errors system and (b) The estimated parameters obtained by using the method described by Sudheer and Sabir (2011) for the initial conditions of drive and response systems as (1, 1, 1) and (1.5, 1.5, 1.5) respectively.

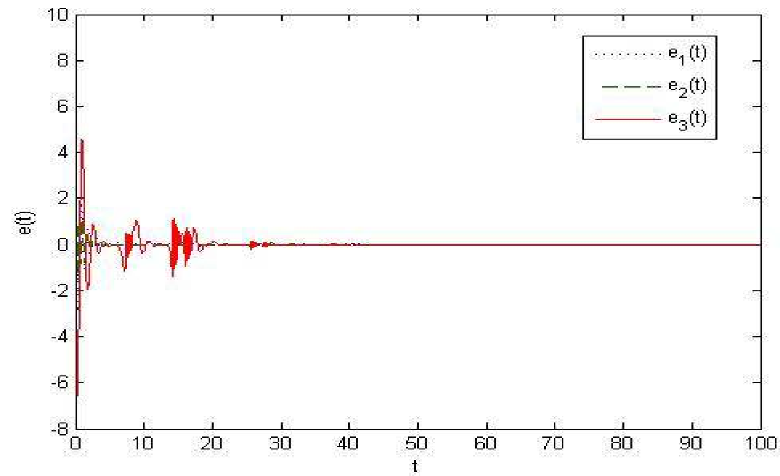


(a)

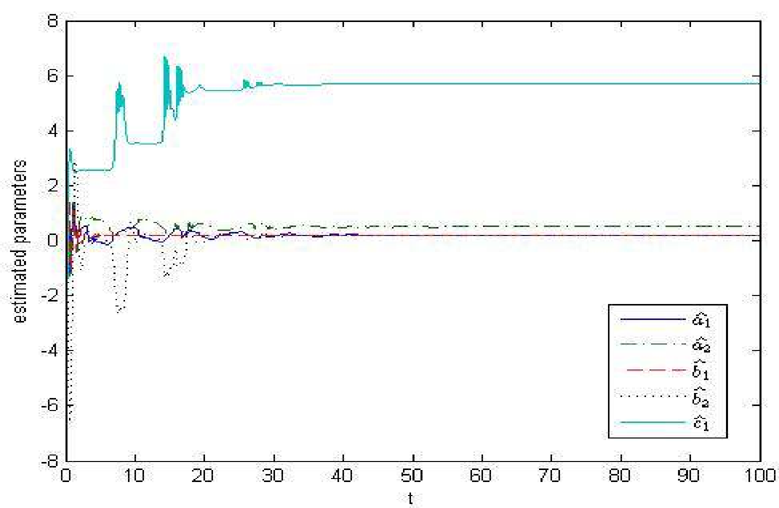


(b)

Fig. 5.6. (a) State trajectories of errors system and (b) The estimated parameters obtained by using the proposed method for scaling functions $\lambda_1(t) = 1 + \cos(0.05t)$, $\lambda_2(t) = 2 - \sin(t)$ and $\lambda_3(t) = 3 + \cos(t + 10)$ and the initial conditions of drive and response systems as (2.5, 2, 2.5) and (0.5, 1, 1.5) respectively.



(a)



(b)

Fig. 5.7. (a) State trajectories of errors system and (b) The estimated parameters obtained by using the method described by Sudheer and Sabir (2011) for scaling functions $\lambda_1(t) = 1 + \cos(0.05t)$, $\lambda_2(t) = 2 - \sin(t)$ and $\lambda_3(t) = 3 + \cos(t + 10)$ and the initial conditions of drive and response systems as (2.5, 2, 2.5) and (0.5, 1, 1.5) respectively.

5.5 Conclusion

In the present chapter the author has developed a new method for function projective synchronization of time-delayed chaotic systems through proper design of controller functions with corresponding parameter identification laws based on Lyapunov-Krasovskii stability theory. During numerical simulation the method is applied for function projective synchronization of identical Rössler system and compared the results with the results described in (Sudheer and Sabir (2011)). The main feature of the chapter is the demonstration of minimum time requirement for synchronization by applying the new method as compared to the earlier results for three different cases. The author is optimist that the new proposed method will be useful to the engineers and scientists working in the field of dynamical system especially those involved in synchronization of time-delayed chaotic systems.