CHAPTER-4

Projective Synchronization of Time-Delayed Chaotic Systems with Unknown Parameters Using Adaptive Control Method

Chapter 4

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4.1 Introduction

Inspired by the seminal works of Fujisaka and Yamada (1983a, 1983) and of Pecora and Carroll (1990) on chaos synchronization, several types of synchronization phenomena have been demonstrated and identified, such as complete synchronization, phase synchronization, anti-phase synchronization, lag synchronization, generalized synchronization, anticipatory synchronization, projective synchronization (Fujisaka and Yamada (1983a, 1983), Pecora and Carroll (1990), Mahmoud and Mahmoud (2010), Liu (2006), Rosenblum *et al.* (1997), Yang and Juan (1998), Ghosh and Bhattacharya (2010)) and so on. Amongst all kinds of chaos synchronization, projective synchronization, which is characterized by a scaling factor that helps two systems to be synchronized proportionally, is the most interesting one. The identical systems could be synchronized up to a scaling factor as affirmed by Mainieri and Rehacek (1999), and the phenomenon is known as projective synchronization. This type of synchronization is an attractive field due to its proportionality between the synchronized dynamical states. Many approaches are proposed to appreciate chaotic synchronization such as active control, adaptive control, feedback control, sliding mode control, optimal control, back stepping control and so on.

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Firstly, the synchronization of low-dimensional chaotic systems is studied, and later on, the synchronization in high-dimensional systems has become an active field of research both from the theoretical and the applied perspectives. In this regard, the research on synchronization of time-delayed chaotic systems (Pyragas (1998), He and Vaidya (1999)) has received considerable attention. Time-delayed chaotic systems are naturally related to the systems with memory that prevails for most of the physical and scientific systems such as blood production in patients with leukaemia (Mackey–Glass model), dynamics of optical systems (e.g. Ikeda system), laser physics, population dynamics, physiological model, neural networks, control system (Mackey and Glass (1977), Ikeda *et al.* (1980), Bunner *et al*. *(*1998), Yongzhen *et al.* (2011), Liao *et al.* (2007), Kwon *et al*. (2011)) and so on.

However, the theory of synchronization in chaotic systems has been intensively reviewed and studied in the last two decades; where as the better understanding of synchronization in time-delay chaotic systems is developed recently. The first study on synchronization of chaos in time-delayed system has been reported by Pyragas (1998). Later, several types of synchronization in time-delay systems have been reported, for example, lag and anticipatory synchronization, complete and generalized synchronization, phase synchronization (Sahaverdiev and Shore (2002), Sahaverdiev *et al.* (2002), Banerjee *et al.* (2013), Zhan *et al.* (2003), Senthilkumar *et al.* (2006)) and so on. A good amount of effort has been made to deal with the stability problem in delay-differential equations (DDE) system (Hale (1977)), and also to deal with the DDE problem, some of well-established tools of ordinary differential equations are used. Among these tools, the extension of the Lyapunov stability theory for DDE systems has much importance. Such extensions are known as Lyapunov–Krasovskii (Krasovskii and Brenner (1963)) and the Lyapunov–Razumikhin stability theorems, and they give the basis for the development of sufficient criteria that can be used to determine the stability of motion for DDE systems.

Adaptive control synchronization method was proposed by H. Zhang (Zhang *et al.* (2006)) in 2006 for the synchronization of two non-identical chaotic systems. In their article, an adaptive synchronization controller was developed and analytic expression of the controller and the adaptive laws of parameters are given on the basis of Lyapunov stability theory. Later the method was extended by Wang (2010), who presented the projective synchronization between hyperchaotic Lü system and Liu system with known or unknown parameters. He translated the problem of synchronization of chaotic systems with different orders into the projective synchronization of the systems with identical orders by using add-order method. Different controllers are designed for the projective synchronization of the two non-identical chaotic systems using active control when parameters are known, while the controller function and the parameter update laws are derived via adaptive control method for the unknown parameters. Recently, Agrawal *et al.* (2013) have developed a technique for serving the purpose of synchronization of fractional order as well as integer order chaotic systems. In the article, they have made some modifications on the adaptive synchronization and parameter identification method with unknown parameters for using it in fractional order chaotic systems and designed the appropriate adaptive synchronization controller and parameter identification based on Lyapunov stability method. Although all the methods mentioned earlier are appropriate to achieve the synchronization between standard and fractional order chaotic systems, the techniques are unable to synchronize the time-delayed chaotic systems. To the best of author's knowledge, the projective synchronization between time-delayed chaotic systems with unknown parameters have not yet been studied by any researcher.

In the present endeavour, a new analytic treatment of projective synchronization of a time-delay system is investigated using adaptive control method. Some results for the particular class of time-delayed chaotic systems are achieved. Either because of the intrinsic characteristic of adaptive laws of parameters or because of the dynamics of controller, the effect of time delay on dynamical behavior of system are quite remarkable. In this research field, some attempts have been reported in recent years. Jinde Cao *et al.* (2009) studied projective synchronization of a class of delayed chaotic systems. Grassi and Miller (2007) introduced projective synchronization of time-delay, continuous-time and discrete-time systems via a linear observer. Cun-Fang Feng (Feng (2010), Feng *et al.* (2008)) presented the analytic investigation of projective synchronization and generalized projective synchronization between time-delayed chaotic systems. For other results reported in the literature, see (Ghosh (2009), Feng and Wang (2012), Feng *et al.* (2006)). Criteria for global and asymptotical stability of error system are obtained by consideration of a suitable Lyapunov–Krasovskii functional. Theoretical results are illustrated through numerical simulations of examples, namely, advanced Lorenz system, multiple delay Rössler system and time-delayed Chua’s oscillator. The numerical simulation results which are depicted through graphs for different particular cases exhibit the flexibility and effectiveness of the proposed scheme.

4.2 Problem formulation

Consider the drive system in the form of

$$x'(t) = f_1(x(t)) + g_1(x(t))A + h_1(x(t-\tau))B, \qquad t > 0,$$
$$x(t) = \phi(t), \qquad -\tau \le t \le 0 \qquad (4.1)$$

and the response system in the form of

$$y'(t) = f_2(y(t)) + g_2(y(t))C + h_2(y(t-\tau))D + U, \qquad t > 0,$$
$$y(t) = \psi(t), \qquad -\tau \le t \le 0, \qquad (4.2)$$

where $x, y \in R^n$ are the state vectors of systems (4.1) and (4.2), respectively; $A \in R^{m_1}$, $B \in R^{m_2}$, $C \in R^{m_3}$ and $D \in R^{m_4}$ are the parameter vectors of the systems; $f_1(x), f_2(y) \in R^n, g_1(x) \in R^{n\times m_1}, h_1(x(t-\tau)) \in R^{n\times m_2}, g_2(y) \in R^{n\times m_3}$ and $h_2(y(t-\tau)) \in R^{n\times m_4}$ are nonlinear functions; $\phi(t)$ and $\psi(t)$ represent the trajectories of the solutions in the past; τ is the time delay; and U is the controller.

It is known that two time-delayed chaotic systems coupled in a drive and response configuration can exhibit projective synchronization if there exists an error vector e such that

$$\lim_{t\to\infty}\|e(t)\| = \lim_{t\to\infty}\|y(t) - \lambda\, x(t)\| = 0,$$

where $\lambda = diag(\lambda_1, \lambda_2, ...\lambda_n)$ is a scaling constant matrix such that λ_i's are constants for all $i \in N$.

Considering $e(t) = y(t) - \lambda\, x(t)$ as the synchronization error vector, the error dynamical system becomes

$$e'(t) = y'(t) - \lambda\, x'(t)$$
$$= f_2(y(t)) - \lambda f_1(x(t)) + g_2(y(t))C - \lambda g_1(x(t))A + h_2(y(t-\tau))D - \lambda h_1(x(t-\tau))B + U. \qquad (4.3)$$

Next aim is to find a suitable and effective control function U and parameter estimation update laws to ensure that the drive and response systems with uncertain parameters approach towards the projective synchronization.

Theorem 4.1 If the nonlinear controller is designed as

$$U = -f_2(y(t)) + \lambda f_1(x(t)) - g_2(y(t))\hat{C} + \lambda g_1(x(t))\hat{A} - h_2(y(t-\tau))\hat{D} + \lambda h_1(x(t-\tau))\hat{B} \\ -(1/2+k)e(t)$$

and the adaptive laws of parameters are taken as

$$\begin{aligned} \hat{A}' &= -[\lambda g_1(x(t))]^T e(t) - \overline{A} \\ \hat{B}' &= -[\lambda h_1(x(t-\tau))]^T e(t) - \overline{B} \\ \hat{C}' &= [g_2(y(t))]^T e(t) - \overline{C} \\ \hat{D}' &= [h_2(y(t-\tau))]^T e(t) - \overline{D}, \end{aligned}$$

where parameter error vectors are $\overline{A} = \hat{A} - A$, $\overline{B} = \hat{B} - B$, $\overline{C} = \hat{C} - C$ and $\overline{D} = \hat{D} - D$, then the response system will be synchronized by the drive system globally and asymptotically, and satisfies $\lim_{t\to\infty}(\hat{A} - A) = 0$, $\lim_{t\to\infty}(\hat{B} - B) = 0$, $\lim_{t\to\infty}(\hat{C} - C) = 0$ and $\lim_{t\to\infty}(\hat{D} - D) = 0$ for any positive constant k, the vectors $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are the estimations of parameters A, B, C and D, respectively.

Proof: The error dynamical system is given by

$$e'(t) = f_2(y(t)) - \lambda f_1(x(t)) + g_2(y(t))C - \lambda g_1(x(t))A + h_2(y(t-\tau))D - \lambda h_1(x(t-\tau))B + U.$$

Now using the controller as designed in the preceding text, we obtain

$$e'(t) = -\lambda g_1(x(t))(A - \hat{A}) - \lambda h_1(x(t-\tau))(B - \hat{B}) + g_2(y(t))(C - \hat{C}) + h_2(y(t-\tau))(D - \hat{D}) \\ -(1/2+k)e(t).$$

According to Lyapunov stability theorem, the error system is asymptotically stable. In other words, if the error system becomes zero, drive and response systems are regarded as projective synchronized. Let us construct the Lyapunov–Krasovskii functional V to carry out stability analysis as

$$\mathrm{V} = \frac{1}{2}e^T(t)e(t) + \frac{1}{2}\int_{-\tau}^{0} e^T(t+\theta)e(t+\theta)d\theta + \frac{1}{2}(\overline{A}^T\overline{A} + \overline{B}^T B + \overline{C}^T\overline{C} + \overline{D}^T\overline{D}).$$

The derivative of V along the trajectory of error dynamic system is

$$\mathrm{V}' = e^T(t)e'(t) + \frac{1}{2}(e^T(t)e(t) - e^T(t-\tau)e(t-\tau)) + (\overline{A}'^T\,\overline{A} + \overline{B}'^T\,\overline{B} + \overline{C}'^T\,\overline{C} + \overline{D}'^T\,\overline{D}),$$

where $\overline{A}'^T = \hat{A}'^T$, $\overline{B}'^T = \hat{B}'^T$, $\overline{C}'^T = \hat{C}'^T$ and $\overline{D}'^T = \hat{D}'^T$.

Now

$$\begin{aligned}\mathrm{V}' = {} & e^T(t)(-\lambda g_1(x(t))(A-\hat{A}) - \lambda h_1(x(t-\tau))(B-\hat{B}) + g_2(y(t))(C-\hat{C}) \\ & + h_2(y(t-\tau))(D-\hat{D}) - (1/2+k)e(t)) + \frac{1}{2}(e^T(t)e(t) - e^T(t-\tau)e(t-\tau)) \\ & + (\hat{A}'^T\,\overline{A} + \hat{B}'^T\,\overline{B} + \hat{C}'^T\,\overline{C} + \hat{D}'^T\,\overline{D}).\end{aligned}$$

Using adaptive update laws, we obtain

$$V' = -ke^T(t)e(t) - \frac{1}{2}e^T(t-\tau)e(t-\tau) - \overline{A}^T\overline{A} - \overline{B}^T\overline{B} - \overline{C}^T\overline{C} - \overline{D}^T\overline{D} < 0.$$

Thus, it may be concluded that if only the control parameter k > 0, $V \in R$ is positive definite function and $V' \in R$ is negative definite function, the error systemis globally and asymptotically stable according to Lyapunov–Krasovskii stability theory (Hale (1977), Krasovskii and Brenner (1963)). Consequently, the state of response and drive systems will be synchronized globally and asymptotically. It is also seen that the synchronization error e

and the parameters' estimation errors $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ decay to zero as time becomes large. This completes the proof.

Again for the case of projective synchronization when the drive and response systems are identical, the systems are defined as

$$\begin{aligned} x'(t) &= f(x(t)) + g(x(t))A + h(x(t-\tau))B, \quad t > 0, \\ x(t) &= \phi(t), \; -\tau \le t \le 0 \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} y'(t) &= f(y(t)) + g(y(t))A + h(y(t-\tau))B + U, \quad t > 0, \\ y(t) &= \psi(t), \quad -\tau \le t \le 0. \end{aligned} \tag{4.5}$$

Theorem 4.2 If the nonlinear controller is chosen as

$$\begin{aligned} U = &-(f(y(t)) - \lambda f(x(t))) - (g(y(t)) - \lambda g(x(t)))\hat{A} - (h(y(t-\tau)) - \lambda h(x(t-\tau)))\hat{B} \\ &-(1/2 + k)e(t) \end{aligned}$$

and the adaptive laws of parameters are taken as

$$\begin{aligned} \hat{A}' &= [g(y(t)) - \lambda g(x(t))]^T e(t) - \overline{A} \\ \hat{B}' &= [h(y(t-\tau)) - \lambda h(x(t-\tau))]^T e(t) - \overline{B}, \end{aligned}$$

where $\overline{A} = \hat{A} - A$ and $\overline{B} = \hat{B} - B$, then the response system will be synchronized by the drive system globally and asymptotically, and satisfies $\lim_{t\to\infty}(\hat{A} - A) = 0$ and $\lim_{t\to\infty}(\hat{B} - B) = 0$, for any positive constant *k*, the vectors $\hat{A}$ and $\hat{B}$ are the estimations of parameters A and B, respectively.

Proof: It can be proved proceeding as Theorem 1 through considering $A = C$, $B = D$, $f_1 = f_2$, $g_1 = g_2$ and $h_1 = h_2$.

4.3 Systems' descriptions

4.3.1 Advanced Lorenz system

The advanced Lorenz system given in (Zhang *et al.* (2009)) is described as

$$\begin{aligned} x'(t) &= a(y(t) - x(t)) + px(t-\tau) \\ y'(t) &= -x(t)z(t) + bx(t) + cy(t) \\ z'(t) &= x^2(t) - dz(t), \end{aligned} \tag{4.6}$$

where a, b, c, d and p are the parameters. For the parameters' values a=20, b=14, c=10.6, p=3 and time delay $\tau = 0.001$, the aforementioned system shows the chaotic behaviour at initial values (x(t), y(t), z(t)) = (-20, 8, 20) where $-\tau \le t \le 0$. The chaotic behaviour of system (4.6) is depicted through Fig. 4.1.

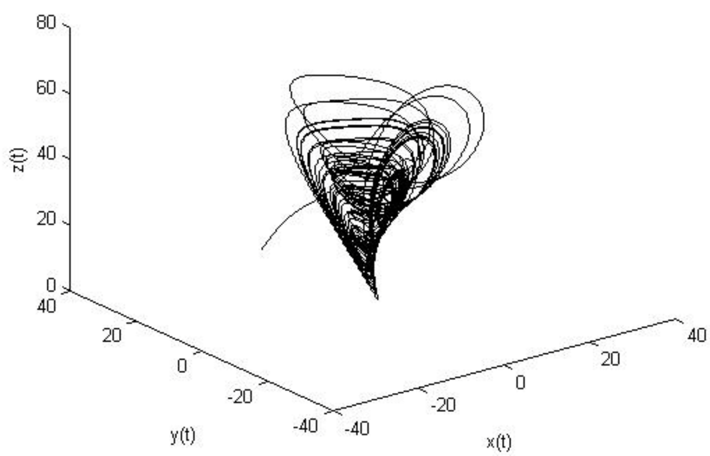


Fig. 4.1. Phase portraits of Advanced Lorenz system in *x*-*y*-*z* space.

4.3.2 Multiple delay Rössler system

A double delayed Rössler system (Ghosh *et al.* (2008)) is given by

$$\begin{aligned} x'(t) &= -y(t) - z(t) + a_1 x(t-\tau_1) + a_2 x(t-\tau_2) \\ y'(t) &= x(t) + b_1 y(t) \\ z'(t) &= b_2 + x(t)z(t) - c_1 z(t), \end{aligned} \tag{4.7}$$

where a_1, a_2 are the geometric factors, while b_1, b_2 and c are the usual parameters of a classical Rössler system, τ_1 and τ_2 are time delays. The double delayed Rössler system exhibits the chaotic trajectories for the parameter values a_1 = 0.2, a_2 = 0.5, b_1 = b_2 = 0.2, c = 5.7, $\tau_1 = 1.0$ and $\tau_2 = 2.0$ with initial condition. (x(t), y(t), z(t)) = (0.5, 1, 1.5) where $-\tau \le t \le 0$ and shown through Fig. 4.2.

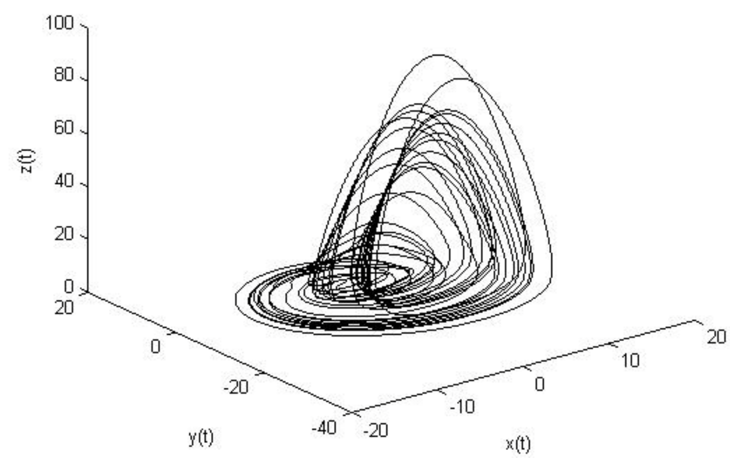


Fig. 4.2. Phase portraits of Multiple delay Rössler system in *x-y-z* space.

4.3.3 Time-delay Chua's oscillator

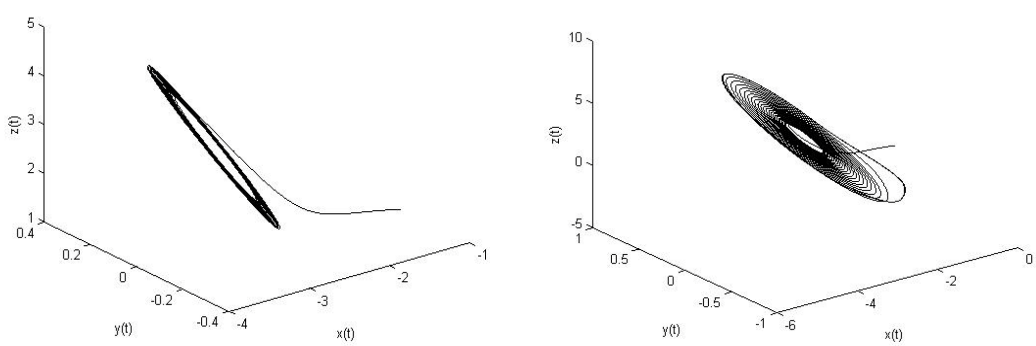
Time-delay feedback Chua's oscillator (Cruz-Hernández (2004)) is given by

$$\begin{aligned} x'(t) &= \alpha(y(t) - x(t) - f(x(t))) \\ y'(t) &= x(t) - y(t) + z(t) \\ z'(t) &= -\beta y(t) - \gamma z(t) - \mu\varepsilon \sin(\sigma\, x(t-\tau)), \end{aligned} \qquad (4.8)$$

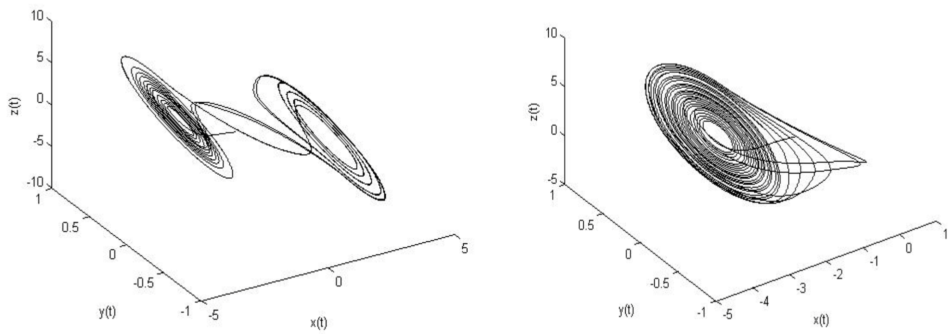
where the nonlinear function is described as

$$f(x(t)) = bx(t) + \frac{1}{2}(a-b)(|x(t)+1| - |x(t)-1|).$$

Taking the values of parameters are $\alpha = 10, \beta = 19.53, \gamma = 0.1636, \mu = 19.53$ $a = -1.4325, b = -0.7831$ and time delay $\tau = 0.001$, chaotic behaviour of the system (4.8) is depicted through Fig. 4.3 for the initial condition (x(t), y(t), z(t)) = (-1, -0.1, 1) where $-\tau \le t \le 0$ and also for different sets of parameters ε and σ.



(a) (b)



(c) (d)

Fig. 4.3. Phase portrait of time-delayed feedback Chua's oscillator in *x*-*y*-*z* space for different sets of parameters: (a) $\varepsilon = 0.07$ and $\sigma = 0.4$, (b) $\varepsilon = 0.2$ and $\sigma = 2$, (c) $\varepsilon = 0.5$ and $\sigma = 3$ and (d) $\varepsilon = 1$ and $\sigma = 1$.

4.4 Projective synchronization between identical systems

In this section, an example of projective synchronization between two identical advanced Lorenz systems is presented.

4.4.1 Projective synchronization between identical time-delayed advanced Lorenz systems

The drive system is described as

$$\begin{aligned} x_1'(t) &= a(y_1(t) - x_1(t)) + px_1(t-\tau) \\ y_1'(t) &= -x_1(t)z_1(t) + bx_1(t) + cy_1(t) \\ z_1'(t) &= x_1^2(t) - dz_1(t) \end{aligned} \tag{4.9}$$

and the response system as

$$\begin{aligned} x_2'(t) &= a(y_2(t) - x_2(t)) + px_2(t-\tau) + u_1(t) \\ y_2'(t) &= -x_2(t)z_2(t) + bx_2(t) + cy_2(t) + u_2(t) \\ z_2'(t) &= x_2^2(t) - dz_2(t) + u_3(t), \end{aligned} \tag{4.10}$$

where $u_1(t), u_2(t), u_3(t)$ are controllers to be designed so that both the systems are synchronized. According to projective synchronization technique, consider the constant scaling matrix $\lambda = diag(\lambda_1, \lambda_2, \lambda_3)$, such that the error states are defined as $e_1 = x_2(t) - \lambda_1 x_1(t)$, $e_2 = y_2(t) - \lambda_2 y_1(t)$, and $e_3 = z_2(t) - \lambda_3 z_1(t)$. The corresponding error dynamical system is obtained as

$$\begin{aligned}
e_1'(t) &= (y_2(t) - x_2(t) - \lambda_1(y_1(t) - x_1(t)))a + (x_2(t-\tau) - \lambda_1 x_1(t-\tau))p + u_1(t) \\
e_2'(t) &= -x_2(t)z_2(t) + \lambda_2 x_1(t)z_1(t) + (x_2(t) - \lambda_2 x_1(t))b + (y_2(t) - \lambda_2 y_1(t))c + u_2(t) \\
e_3'(t) &= x_2^2(t) - \lambda_3 x_1^2(t) - (z_2(t) - \lambda_3 z_1(t))d + u_3(t).
\end{aligned}$$

According to Theorem 2, the synchronous controller functions are

$$\begin{aligned}
u_1(t) &= -(y_2(t) - x_2(t) - \lambda_1(y_1(t) - x_1(t)))\hat{a} - (x_2(t-\tau) - \lambda_1 x_1(t-\tau))\hat{p} - (1/2 + k)e_1(t) \\
u_2(t) &= x_2(t)z_2(t) - \lambda_2 x_1(t)z_1(t) - (x_2(t) - \lambda_2 x_1(t))\hat{b} - (y_2(t) - \lambda_2 y_1(t))\hat{c} - (1/2 + k)e_2(t) \\
u_3(t) &= -(x_2^2(t) - \lambda_3 x_1^2(t)) + (z_2(t) - \lambda_3 z_1(t))\hat{d} - (1/2 + k)e_3(t)
\end{aligned}$$

and the adaptive laws of parameters are

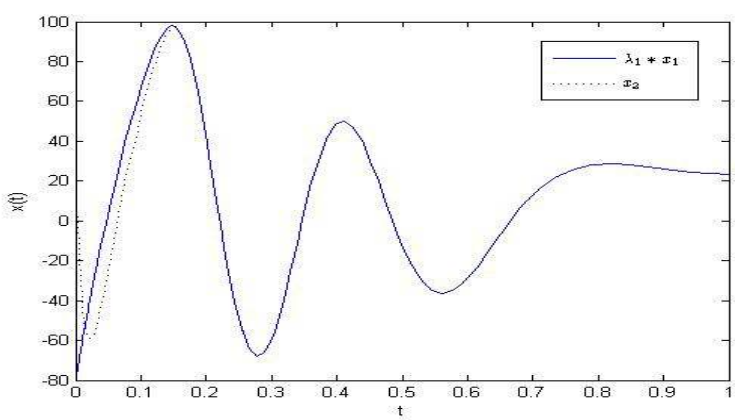
$$\begin{aligned}
\hat{a}' &= (y_2(t) - x_2(t) - \lambda_1(y_1(t) - x_1(t)))e_1(t) - (\hat{a} - a) \\
\hat{b}' &= (x_2(t) - \lambda_2 x_1(t))e_2(t) - (\hat{b} - b) \\
\hat{c}' &= (y_2(t) - \lambda_2 y_1(t))e_2(t) - (\hat{c} - c) \\
\hat{d}' &= -(z_2(t) - \lambda_3 z_1(t))e_3(t) - (\hat{d} - d) \\
\hat{p}' &= (x_2(t-\tau) - \lambda_1 x_1(t-\tau))e_1(t) - (\hat{p} - p).
\end{aligned}$$

Finally, to accomplish the projective synchronization, the error system is given by

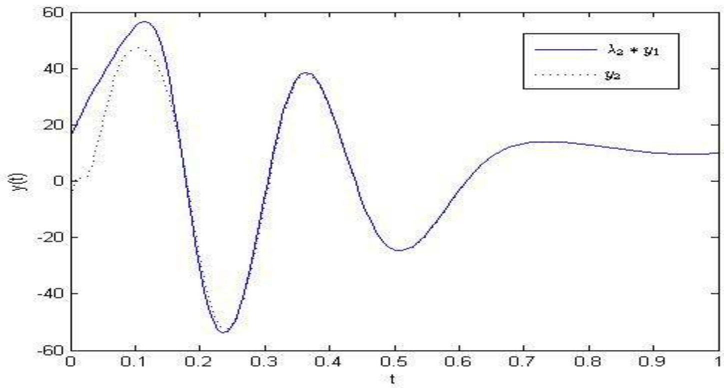
$$\begin{aligned}
e_1'(t) &= (y_2(t) - x_2(t) - \lambda_1(y_1(t) - x_1(t)))(a - \hat{a}) + (x_2(t-\tau) - \lambda_1 x_1(t-\tau))(p - \hat{p}) \\
&\quad - (1/2 + k)e_1(t) \\
e_2'(t) &= (x_2(t) - \lambda_2 x_1(t))(b - \hat{b}) + (y_2(t) - \lambda_2 y_1(t))(c - \hat{c}) - (1/2 + k)e_2(t) \\
e_3'(t) &= -(z_2(t) - \lambda_3 z_1(t))(d - \hat{d}) - (1/2 + k)e_3(t)
\end{aligned}$$

Numerical simulation and results

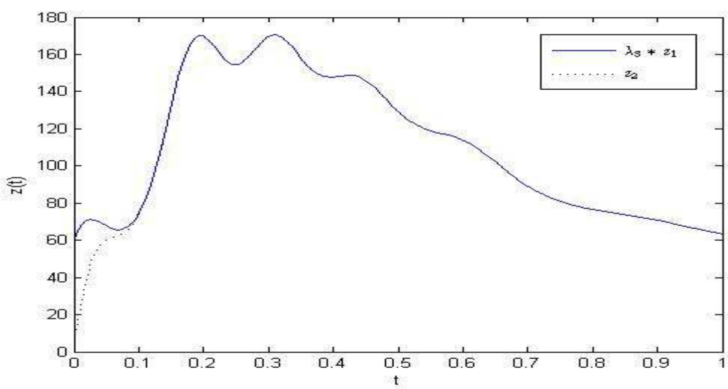
To verify and demonstrate the effectiveness of the proposed method, the simulation result for the projective synchronization between identical time-delayed Lorenz systems is discussed. For the purpose of numerical simulations, parameters are taken as in Section 4.3.1. The initial values of estimated unknown parameters are chosen as $(\hat{a},\hat{b},\hat{c},\hat{d},\hat{p})=(0,0,0,0,0)$. The initial values of state trajectories of drive and response systems are taken as $(x_1(t),y_1(t),z_1(t))=(-20,\ 8,\ 20)$ and $(x_2(t),y_2(t),z_2(t))=(10,\ -6,\ 8)$, respectively. The time delay, control input and constant scaling matrix are considered as $\tau=0.001,\ k=1$ and $\lambda=diag(4,2,3),$ respectively. The simulation results are depicted through Fig. 4.3 and Fig. 4.4. Figs. 4.3(a) – (c) show the state vectors of drive and response systems. Fig. 4.3(d) ensures that the synchronization error vector tends to zero as t becomes large, and as a result, the projective synchronization between systems (4.9) and (4.10) is achieved. Fig. 4.4 shows the variations of estimated values of unknown parameters $\hat{a},\hat{b},\hat{c},\hat{d}$ *and* $\hat{p}$ converge to the original values *a*, *b*, *c*, *d* and *p*.



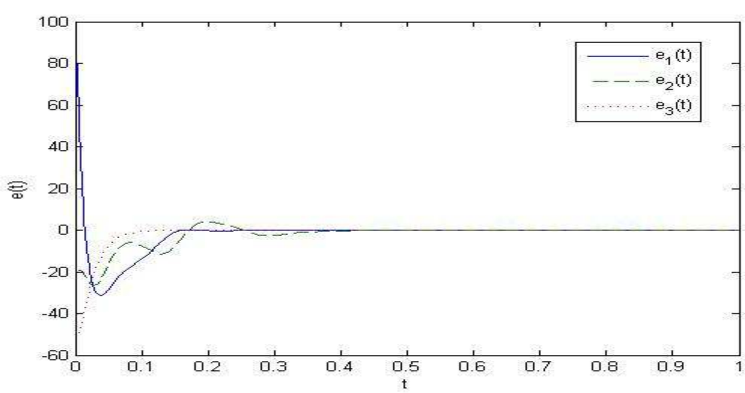
(a) State trajectories between state vectors $x_1(t)$ and $x_2(t)$.



(b) State trajectories between state vectors $y_1(t)$ and $y_2(t)$.



(d) State trajectories between state vectors $z_1(t)$ and $z_2(t)$.



(d) The evolution of errors state $(e_1(t), e_2(t), e_3(t))$.

Fig. 4.4. State trajectories of drive system (4.9) and response system (4.10) between state vectors and evolution of error vectors.

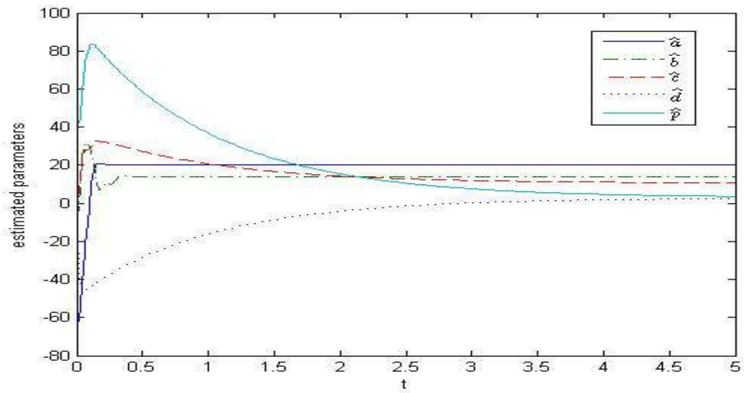


Fig. 4.5. The estimated parameter vectors of time-delayed advanced Lorenz systems.

4.5. Projective synchronization between non-identical systems

This section presents two examples to show projective synchronization between a pair of non-identical systems.

4.5.1. Projective synchronization between advanced Lorenz system and Rössler system

The drive system is described as

$$\begin{aligned} x_1'(t) &= a(y_1(t) - x_1(t)) + px_1(t-\tau) \\ y_1'(t) &= -x_1(t)z_1(t) + bx_1(t) + cy_1(t) \\ z_1'(t) &= x_1^2(t) - dz_1(t) \end{aligned} \tag{4.11}$$

and the response system as

$$\begin{aligned} x_2'(t) &= -y_2(t) - z_2(t) + a_1x_2(t-\tau_1) + a_2x_2(t-\tau_2) + u_1(t) \\ y_2'(t) &= x_2(t) + b_1y_2(t) + u_2(t) \\ z_2'(t) &= b_2 + x_2(t)z_2(t) - c_1z_2(t) + u_3(t). \end{aligned} \tag{4.12}$$

Similar to the previous section, we define state errors that yield the following error dynamics as

$$e_1'(t) = -y_2(t) - z_2(t) + a_1 x_2(t-\tau_1) + a_2 x_2(t-\tau_2) - \lambda_1(a(y_1(t) - x_1(t)) + p x_1(t-\tau)) + u_1(t)$$
$$e_2'(t) = x_2(t) + b_1 y_2(t) - \lambda_2(-x_1(t) z_1(t) + b x_1(t) + c y_1(t)) + u_2(t)$$
$$e_3'(t) = b_2 + x_2(t) z_2(t) - c_1 z_2(t) - \lambda_3(x_1^{\,2}(t) - d z_1(t)) + u_3(t).$$

The controllers are calculated as follows

$$u_1(t) = y_2(t) + z_2(t) + \lambda_1(y_1(t) - x_1(t))\hat{a} + x_1(t-\tau)\hat{p} - x_2(t-\tau_1)\hat{a}_1 - x_2(t-\tau_2)\hat{a}_2 \\ \quad - (1/2 + k) e_1(t)$$
$$u_2(t) = -x_2(t) - \lambda_2 x_1(t) z_1(t) + \lambda_2 x_1(t)\hat{b} + \lambda_2 y_1(t)\hat{c} - y_2(t)\hat{b}_1 - (1/2 + k) e_2(t)$$
$$u_3(t) = -x_2(t) z_2(t) + \lambda_3 x_1^{\,2}(t) - \lambda_3 z_1(t)\hat{d} - \hat{b}_2 + z_2(t)\hat{c}_1 - (1/2 + k) e_3(t)$$

and the estimated parameters are taken as

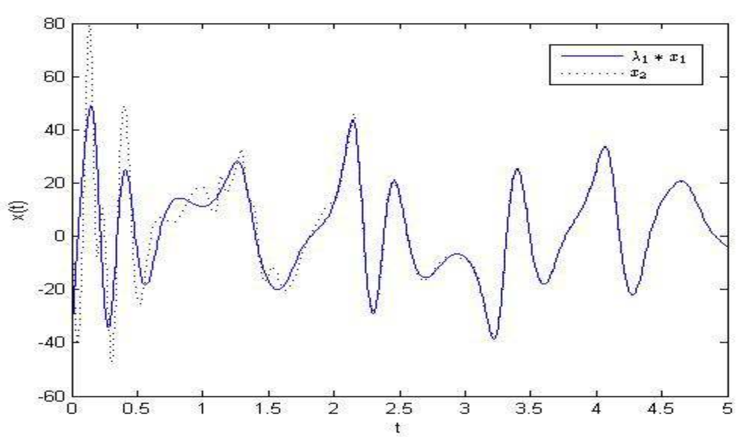
$$\hat{a}' = -\lambda_1(y_1(t) - x_1(t)) e_1(t) - (\hat{a} - a)$$
$$\hat{b}' = -\lambda_2 x_1(t) e_2(t) - (\hat{b} - b)$$
$$\hat{c}' = -\lambda_2 y_1(t) e_2(t) - (\hat{c} - c)$$
$$\hat{d}' = \lambda_3 z_1(t) e_3(t) - (\hat{d} - d)$$
$$\hat{p}' = -\lambda_1 x_1(t-\tau) e_1(t) - (\hat{p} - p)$$
$$\hat{a}_1' = x_2(t-\tau_1) e_1(t) - (\hat{a}_1 - a_1)$$
$$\hat{a}_2' = x_2(t-\tau_2) e_1(t) - (\hat{a}_2 - a_2)$$
$$\hat{b}_1' = y_2(t) e_2(t) - (\hat{b}_1 - b_1)$$
$$\hat{b}_2' = e_3(t) - (\hat{b}_2 - b_2)$$
$$\hat{c}_1' = -z_2(t) e_3(t) - (\hat{c}_1 - c_1),$$

which helps to obtain the error dynamical system as follows to achieve the projective synchronization.

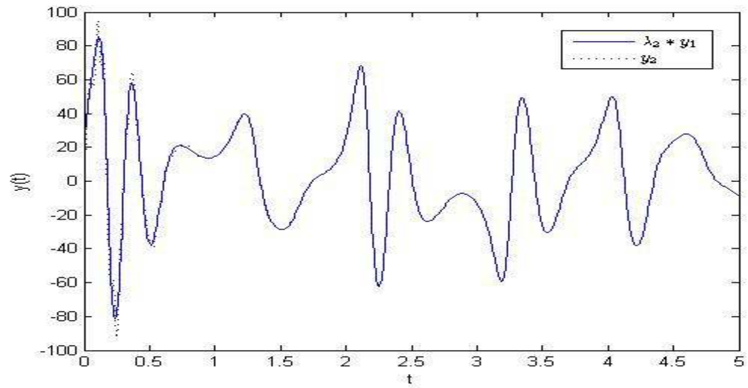
$$e_1'(t) = -\lambda_1(y_1(t) - x_1(t)(a - \hat{a}) - \lambda_1 x_1(t-\tau)(p - \hat{p}) + x_2(t-\tau_1)(a_1 - \hat{a}_1) \\ \quad + x_2(t-\tau_2)(a_2 - \hat{a}_2) - (1/2 + k) e_1(t)$$
$$e_2'(t) = -\lambda_2 x_1(t)(b - \hat{b}) - \lambda_2 y_1(t)(c - \hat{c}) + y_2(t)(b_1 - \hat{b}_1) - (1/2 + k) e_2(t)$$
$$e_3'(t) = \lambda_3 z_1(t)(d - \hat{d}) + (b_2 - \hat{b}_2) - z_2(t)(c_1 - \hat{c}_1) - (1/2 + k) e_3(t).$$

Numerical simulation and results

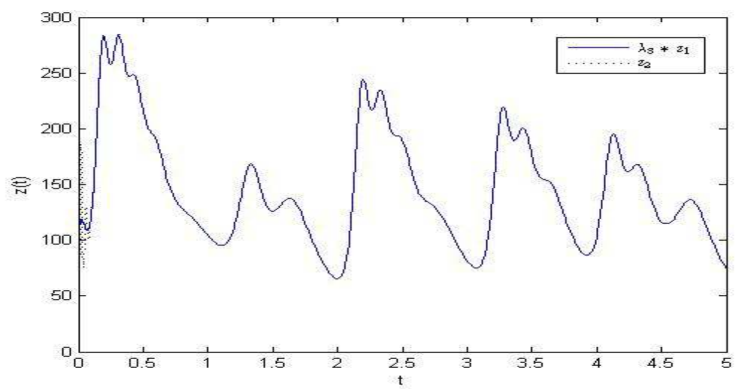
In the present section, the numerical simulation for the projective synchronization of time-delayed Lorenz system and multiple delay Rössler system is studied. The time delay and parameters are chosen same as given in sections 4.3.1 and 4.3.2. The initial values of estimated unknown parameter vectors are $(\hat{a},\hat{b},\hat{c},\hat{d},\hat{p})=(0,0,0,0,0)$ and $(\hat{a}_1,\hat{a}_2,\hat{b}_1,\hat{b}_2,\hat{c}_1)=(0,0,0,0,0)$. The constant scaling matrix and control input are chosen as $\lambda = diag(4,2,3)$ and $k=1$ respectively. Numerical simulations show that the proposed controller provides an adaptive projective synchronization between drive and response systems as shown in Figs. 4.6(a) – (c), and thus, synchronization error vector asymptotically converge to zero which are displayed through Fig. 4.6(d). Figs. 4.7(a) and 4.7 (b) show that the estimated values of unknown parameter vectors $(\hat{a},\hat{b},\hat{c},\hat{d},\hat{p})$ and $(\hat{a}_1,\hat{a}_2,\hat{b}_1,\hat{b}_2,\hat{c}_1)$ converge to the original vectors (a,b,c,d,p) and (a_1,a_2,b_1,b_2,c_1) for both the drive and response systems.



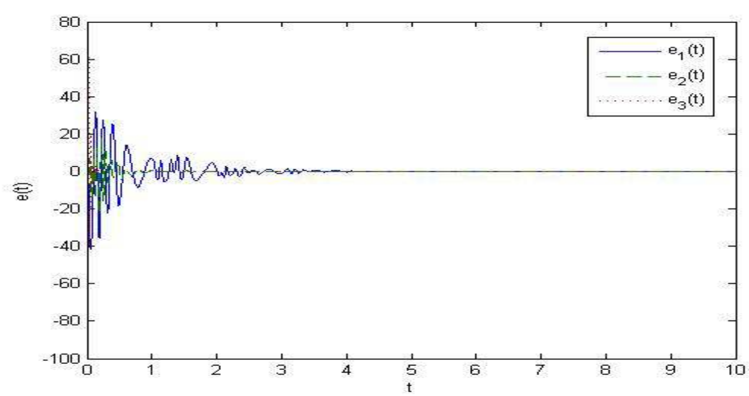
(a) State trajectories between state vectors $x_1(t)$ and $x_2(t)$.



(b) State trajectories between state vectors $y_1(t)$ and $y_2(t)$.

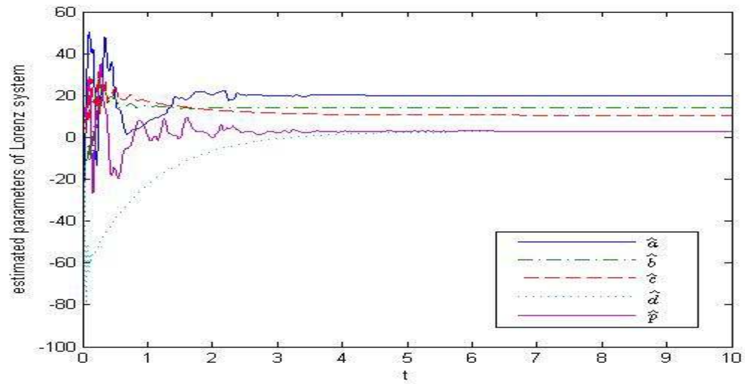


(c) State trajectories between state vectors $z_1(t)$ and $z_2(t)$.

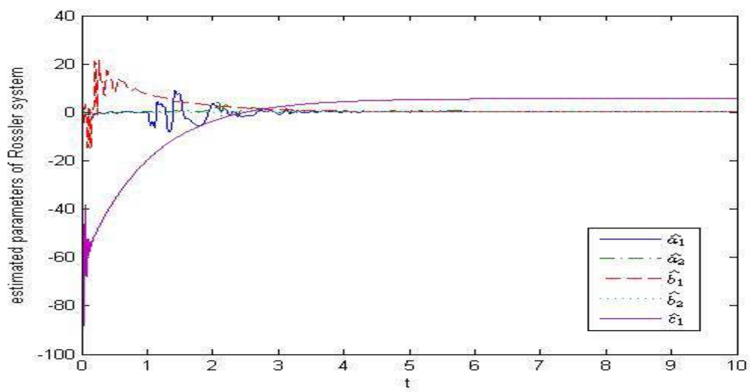


(d) The evolution of the errors $(e_1(t), e_2(t), e_3(t))$.

Fig. 4.6. State trajectories of drive system (4.11) and response system (4.12) between state vectors and evolution of error vectors.



(a) The estimated parameter vectors of advanced Lorenz system.



(b) The estimated parameter vectors of multiple delay Rössler system.

Fig. 4.7. State trajectories of estimated parameter vectors drive system (4.11) and response system (4.12).

4.5.2 Projective synchronization between advanced Lorenz system and time-delay feedback Chua's oscillator

In order to observe the projective synchronization behaviour between advanced Lorenz systems and Chua system, the drive and response systems are defined as

$$x_1'(t) = a(y_1(t) - x_1(t)) + px_1(t-\tau)$$
$$y_1'(t) = -x_1(t)z_1(t) + bx_1(t) + cy_1(t) \tag{4.13}$$
$$z_1'(t) = x_1^2(t) - dz_1(t)$$

and

$$x_2'(t) = \alpha(y_2(t) - x_2(t) - f(x_2(t))) + u_1(t)$$
$$y_2'(t) = x_2(t) - y_2(t) + z_2(t) + u_2(t) \tag{4.14}$$
$$z_2'(t) = -\beta y_2(t) - \gamma z_2(t) - \mu\varepsilon \sin(\sigma x_2(t-\tau)) + u_3(t),$$

where

$$f(x_2(t)) = bx_2(t) + \frac{1}{2}(a-b)(|x_2(t)+1| - |x_2(t)-1|).$$

The corresponding error dynamical system is obtained as

$$e_1'(t) = \alpha(y_2(t) - x_2(t) - f(x_2(t))) - \lambda_1(a(y_1(t) - x_1(t)) + px_1(t-\tau)) + u_1(t)$$
$$e_2'(t) = x_2(t) - y_2(t) + z_2(t) - \lambda_2(-x_1(t)z_1(t) + bx_1(t) + cy_1(t)) + u_2(t)$$
$$e_3'(t) = -\beta y_2(t) - \gamma z_2(t) - \mu\varepsilon \sin(\sigma x_2(t-\tau)) - \lambda_3(x_1^2(t) - dz_1(t)) + u_3(t).$$

Applying Theorem 1, we obtain the synchronization controller as

$$u_1(t) = \lambda_1(y_1(t) - x_1(t))\hat{a} + \lambda_1 x_1(t-\tau)\hat{p} - (y_2(t) - x_2(t) - f(x_2(t)))\hat{\alpha} - (1/2+k)e_1(t)$$
$$u_2(t) = -\lambda_2 x_1(t)z_1(t) + \lambda_2 x_1(t)\hat{b} + \lambda_2 y_1(t)\hat{c} - (x_2(t) - y_2(t) + z_2(t)) - (1/2+k)e_2(t)$$
$$u_3(t) = \lambda_3 x_1^2(t) - \lambda_3 z_1(t)\hat{d} + y_2(t)\hat{\beta} + z_2(t)\hat{\gamma} + \varepsilon \sin(\sigma x_2(t-\tau))\hat{\mu} - (1/2+k)e_3(t)$$

and the adaptive parameters as

$$\hat{a}' = -\lambda_1(y_1(t) - x_1(t))e_1(t) - (\hat{a} - a)$$
$$\hat{b}' = -\lambda_2 x_1(t)e_2(t) - (\hat{b} - b)$$
$$\hat{c}' = -\lambda_2 y_1(t)e_2(t) - (\hat{c} - c)$$
$$\hat{d}' = \lambda_3 z_1(t)e_3(t) - (\hat{d} - d)$$
$$\hat{p}' = -\lambda_1 x_1(t-\tau)e_1(t) - (\hat{p} - p)$$

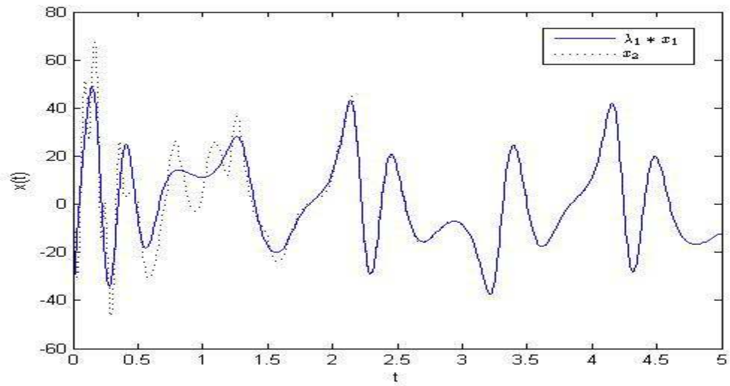
$$\hat{\alpha}' = (y_2(t) - x_2(t) - f(x_2(t)))e_1(t) - (\hat{\alpha} - \alpha)$$
$$\hat{\beta}' = -y_2(t)e_3(t) - (\hat{\beta} - \beta)$$
$$\hat{\gamma}' = -z_2(t)e_3(t) - (\hat{\gamma} - \gamma)$$
$$\hat{\mu}' = -\varepsilon \sin(px_2(t-\tau))e_3(t) - (\hat{\mu} - \mu).$$

Thus, the error dynamic is reduced to

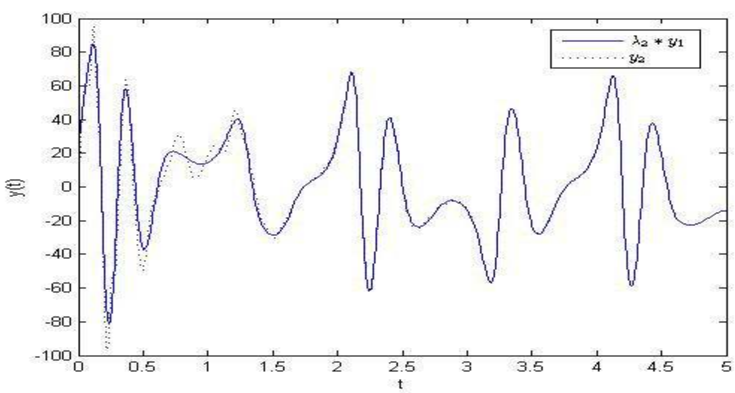
$$\begin{aligned} e_1'(t) &= (y_2(t) - x_2(t) - f(x_2(t)))(\alpha - \hat{\alpha}) - \lambda_1((y_1(t) - x_1(t))(a - \hat{a}) - x_1(t-\tau)(p - \hat{p}) \\ &\quad - (1/2 + k)e_1(t) \\ e_2'(t) &= -\lambda_2 x_1(t)(b - \hat{b}) - \lambda_2 y_1(t)(c - \hat{c}) - (1/2 + k)e_2(t) \\ e_3'(t) &= -y_2(t)(\beta - \hat{\beta}) - z_2(t)(\gamma - \hat{\gamma}) - \varepsilon \sin(\sigma x_2(t-\tau))(\mu - \hat{\mu}) + \lambda_3 z_1(t)(d - \hat{d}) \\ &\quad - (1/2 + k)e_3(t). \end{aligned}$$

Numerical simulation and results

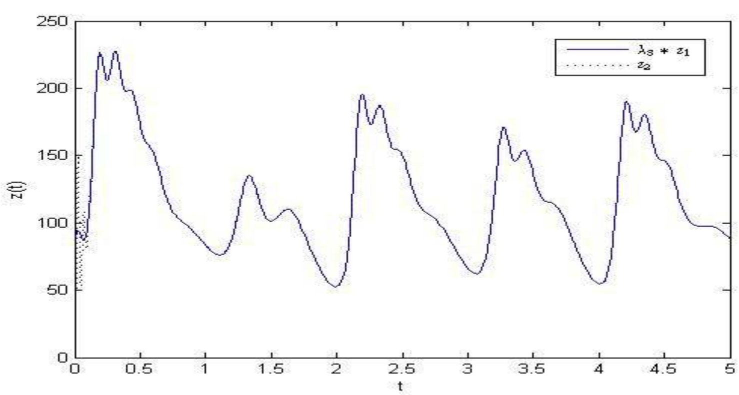
During the simulations, the parameters and initial conditions of the time-delayed Lorenz systems and time-delayed feedback Chua's Oscillator are taken as before. Here the synchronization behaviour is measured for the values of control input and constant scaling matrix as $k = 1$ and $\lambda = diag(4,2,3)$ respectively. The initial values of estimated parameter vectors of both systems are considered as $(\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{p}) = (0,0,0,0,0)$ and $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\mu}) = (0,0,0,0)$. The state vectors of drive and response systems are depicted through Fig. 4.8(a) – (c), and the error dynamics is shown through Fig. 4.8(d). The estimated values of unknown parameter vectors $(\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{p})$ and $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\mu})$ tend to the original values (a,b,c,d,p) and $(\alpha, \beta, \gamma, \mu)$, respectively, which are shown through Fig. 4.9(a) and 4.9(b). This ensures that the error states asymptotically converge to zero as $t \to \infty$ and as a consequence, adaptive projective synchronization between the systems (4.13) and (4.14) is obtained.



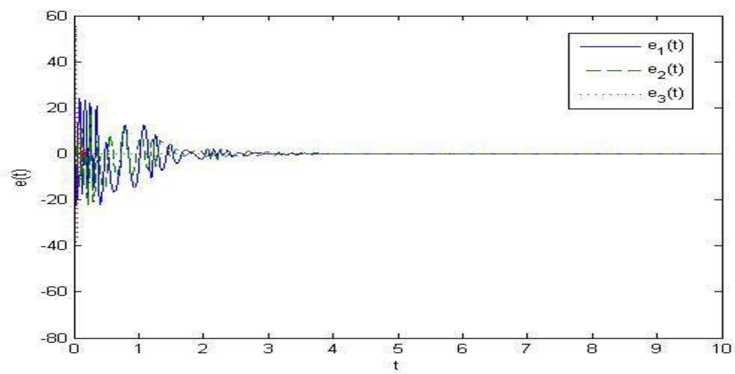
(a) State trajectories between state vectors $x_1(t)$ and $x_2(t)$.



(b) State trajectories between state vectors $y_1(t)$ and $y_2(t)$.

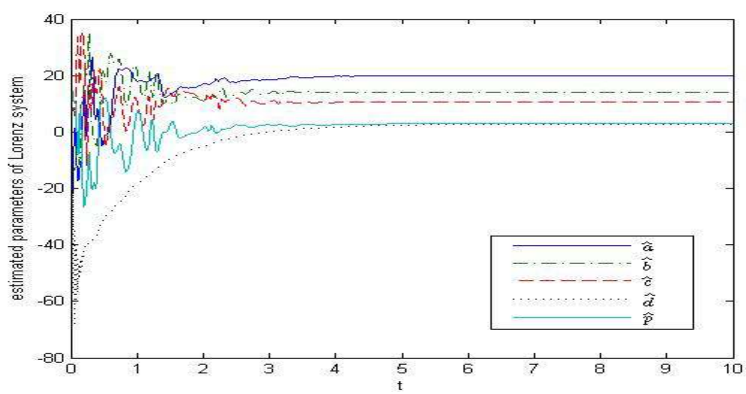


(c) State trajectories between state vectors $z_1(t)$ and $z_2(t)$.

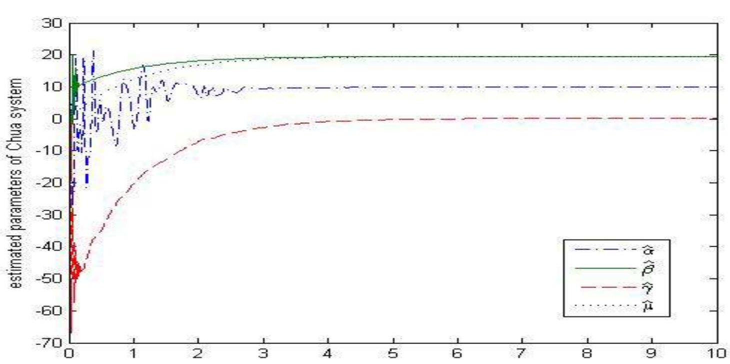


(d) The evolution of the errors $(e_1(t), e_2(t), e_3(t))$.

Fig. 4.8. State trajectories of drive system (4.13) and response system (4.14) between state vectors and evolution of error vectors.



(a) The estimated parameter vectors of advanced Lorenz system.



(b) The estimated parameter vectors of time-delayed Chua's oscillator.

Fig. 4.9. State trajectories of estimated parameter vectors of drive system (4.13) and response system (4.14).

4.6 Conclusion

The present chapter achieves three important goals. The first one is the successful investigation of synchronization scenario of time-delayed chaotic systems via adaptive control approach. The second one is finding necessary condition for adaptive projective synchronization of identical/non-identical chaotic systems with time delay. The third one is the proper design of adaptive synchronization controller and adaptive laws of parameters that are developed using Lyapunov–Krasovskii stability theory so that the components of error system and parameter estimation error decay towards zero as time becomes large for achieving global and asymptotic adaptive projective synchronization, which are demonstrated through graphical presentations.