CHAPTER-2

Anti-Synchronization between Identical and Non-Identical Fractional-Order Chaotic Systems Using Active Control Method

Chapter 2

Anti-synchronization between identical and non-identical fractional-order chaotic systems using active control method

2.1 Introduction

The applications of dynamical systems have nowadays spread to a wide spectrum of disciplines including science, engineering, biology, sociology, etc. The introduction of fractional calculus in nonlinear dynamical systems has rendered a new dimension to the existing problems. Fractional differential equations have garnered a lot of attention and appreciation recently due to their ability to provide a more accurate description of different nonlinear phenomena. The advantage of using fractional order is to allow greater flexibilities in the model. The applications of fractional calculus are growing rapidly. During last few years, the applications can be found in physics and engineering, visco-elastic systems, dielectric polarization, electrode–electrolyte polarization, electromagnetic waves, quantitative finance, quantum evolution of complex system, the control of fractional-order dynamic systems, etc. (Hifer (2001), Podlubny (1999), Koeller (1984), Sun *et al.* (1984), Ichise *et al.* (1971), Heaviside (1971), Laskin (2000), Kunsezov *et al.* (1999), Hartley and Lorenzo (2002)). Analysis of fractional-order dynamical systems involving Riemann-Liouville as well as Caputo derivatives have been found in the scientific contribution of Daftardar-Gejji and Babakhani (2004).

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The field of chaos has grabbed the attention of the researchers and this contributes to a significant amount of the ongoing research these days.

Stability is a fundamental concept of dynamical system which is extended to fractional-order system later on. The main purpose of developing stability theory is to examine the dynamic response of a system's disturbances as time approaches to infinity. The problem of stability is very essential and crucial for fractional-order dynamical system, due to complexity of the relations, which have been investigated and studied by many researchers working in mathematics and engineering for last few decades. Some stability results in fractional-order systems are found in Matignon (1996), Deng *et al.* (2007), Ahmed *et al.* (2007).

Synchronization of chaotic systems is very active topic in nonlinear sciences, which has already been studied in various fields of science and engineering for last few decades. This is the phenomenon that may occur when two or more chaotic oscillators are coupled. The most familiar synchronization phenomenon is the complete synchronization associated with the vanishing of the differences of states of the synchronized systems. The pioneering work of Pecore and Corrall (1990) introduced a method about synchronization between the drive (master) and response (slave) systems of two identical or non-identical systems with different initial conditions, which have attracted a great deal of interest in various fields due to its important applications in ecological system, physical system, chemical system, modeling brain activity, system identification, pattern recognition phenomena and secure communications (Blasius *et al.* (1999), Lakshmanan and Murali (1996), Han *et al.* (1995), Cuomo and Oppenheim (1993), Murali and Lakshmanan (2003)) etc. Balasubramaniam and Vembarasan (2012) have shown the asymptotic stability of error system while solving the synchronization problem for two identical recurrent neural networks through novel output feedback controller using Lyapunov stability theory. The same authors in their article (Vembarasan and Balasubramaniam (2013)) have studied the synchronization between two identical Rikitake system using T-S fuzzy control techniques, which is one of the powerful techniques for synchronizing the complex nonlinear systems. Theesar *et al.* (2012) proposed the synchronization between identical Lur'e systems using sampled data controller introduced by Yang and Chua (1997, 1998). Recently, different types of synchronization between chaotic systems with time delay are considered by researchers and the evidence can be found in the scientific contributions of Sahaverdiev and Shore (2002), Sahaverdiev *et al.* (2002), Zhan *et al*. (2003) and Senthilkumar *et al*. (2006). In 2012, Theesar *et al.* (2012) used time-delayed feedback control during adaptive synchronization between identical Lorenz-Stenflo systems where noise perturbation is considered in the response system.

In the recent years various synchronization schemes, such as linear and nonlinear feedback synchronization, time delay feedback approach, adaptive control, active control, back-stepping design method, sliding mode control etc. (Huang *et al*. (2004), Park and Kwon (2005), Chen and Lu (2002), Al-sawalha and Noorani (2009), Yassen (2005), Wu and Lü (2003), Fang (2013), Yau (2004)), have been successfully applied to chaos synchronization. The concept of synchronization has been extended to the scope such as generalized synchronization, complete synchronization, lag synchronization, phase synchronization, anti-phase synchronization etc. (Yang and Duan (1998), Yu and Liu (2003), Rosenblum *et al.* (1997), Erjaee and Taghvafard (2011), Liu (2006), Liu *et al*. (2000)). Anti synchronization is a phenomenon to use the output of the drive system to control the response system so that the output of the second one has the same amplitude but opposite in sign to the first one. Therefore, the sum of two outputs of drive and response systems are expected to converge to zero when anti-synchronization occurs.

Mathematically, the anti-synchronization of two systems is achieved when $\lim_{t\to\infty} \|x_1(t)+x_2(t)\| = 0$, where $x_1(t)$ and $x_2(t)$ are the state vectors of the drive and response systems respectively. In 2005, Hu *et al.* proposed an adaptive control method for anti- synchronization of uncertain Chua's chaotic system. In 2004, Zhang and Sun (2004) investigated the complete synchronization and anti-synchronization of chaotic systems based on a suitable separation of systems using the Lyapunov stability theory and Matrix measure. Mossa *et al.* (2009) studied anti-synchronization of two identical and different hyperchaotic systems using active control method. Recently, Al-Sawalha and Noorani (2012) have given the concept of the reduced-order anti-synchronization of uncertain chaotic systems. But anti-synchronization between two fractional-order chaotic systems is less in number. The study of anti-synchronization between two standard order chaotic systems has invoked the interest in the author to investigate the time required for anti-synchronization between fractional-order chaotic systems.

Active control method proposed by Bai and Lonngren (1997) is simple and easy to implement in practical applications of synchronization of coupling of a pair of systems and has received huge attention during last few years (Agrawal *et al*. (2012, 2013a, 2013)). Anti-synchronization between identical and non-identical chaotic systems using active control method in both standard order and fractional-order systems are already been studied (Wang and Shi (2009), Al-sawalha *et al*. (2001), Bhalekar and Gejji (2011)).

Keeping in view, the challenges for detecting transformation of dynamical variables between the identical or non-identical systems during synchronization and anti-synchronization, tremendous applications of fractional calculus in various areas of science & engineering and effectiveness of the Active Control Method, the author is motivated to make an attempt to do a coupling of identical and non-identical fractional-order chaotic systems to receive different types of information from the systems due to its memory effect and greater flexibilities.

In this chapter, the anti-synchronization between two identical and non-identical fractional-order chaotic systems using active control method have been studied. Using the Adams-Boshforth-Moulton method (Diethelm *et al*. (2004), Diethelm and Ford (2004)), computer simulations are carried out for different order fractional time derivatives and are displayed graphically to demonstrate the efficiency of the proposed approach. The author is optimist that the present chapter will be a useful contribution to the scientific literature on the methods of control for nonlinear dynamical systems.

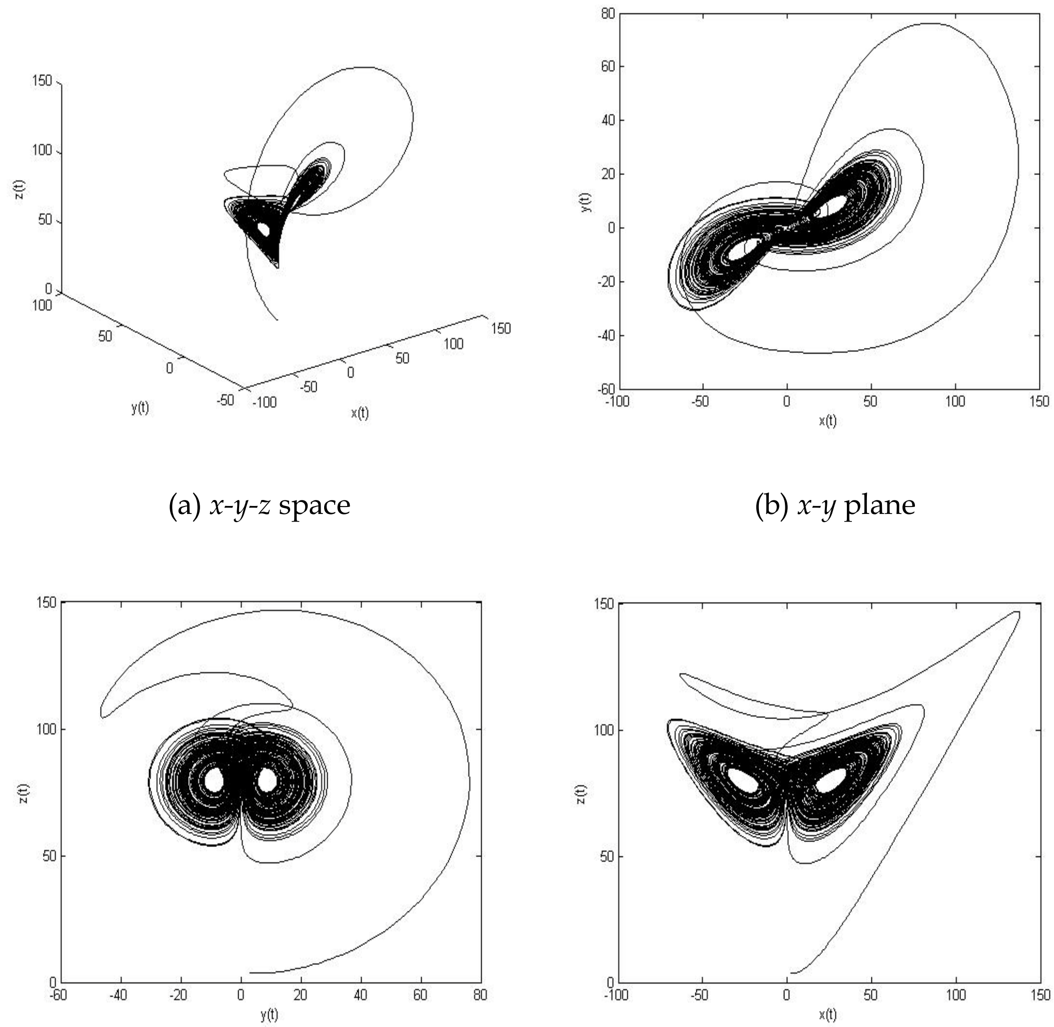
2.2 Systems' descriptions

2.2.1 Fractional-Order Qi System

The Fractional-order Qi chaotic system (Wu and Yang (2010)) is described by

$$\begin{aligned} D_t^{\alpha} x &= p(y-x) + yz \\ D_t^{\alpha} y &= rx - y - xz \\ D_t^{\alpha} z &= -qz + xy, \end{aligned} \tag{2.1}$$

where α $(0<\alpha<1)$ is the fractional-order time derivative and for the parameter values $p=35,\ q=8/3,\ r=80$ the system yields chaotic trajectory. The system has large chaotic region when the parameters are varied. The lowest value of α for which the system remains chaotic is 0.915. The chaotic attractors in the *x-y-z* space and the *x-y, y-z, z-x* planes are depicted through Figs. 2.1 (a) – (d) respectively for $\alpha = 0.93$.



(a) x-y-z space (b) x-y plane

(c) y-z plane (d) x-z plane

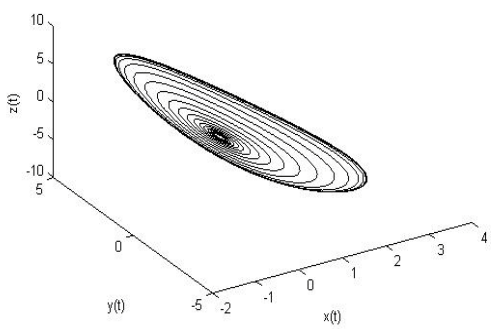
Fig. 2.1. Projections of phase portraits of Qi attractor for $\alpha = 0.93$ and $(p,q,r)=(35,8/3,80)$.

2.2.2 Fractional-Order Genesio–Tesi System

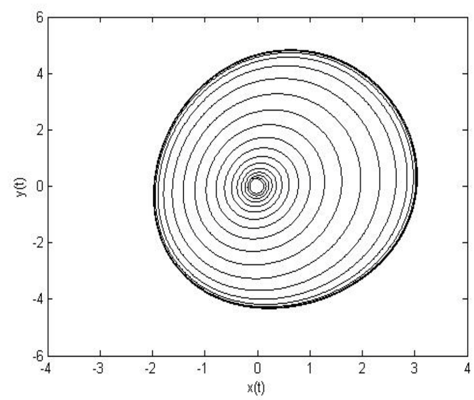
The fractional-order Genesio–Tesi chaotic system (Faieghi and Delavari (2011)) is governed by

$$
\begin{aligned}
&D_t^{\alpha} x = y \\
&D_t^{\alpha} y = z \\
&D_t^{\alpha} z = -a\,x - b\,y - c\,z + m\,x^2 .
\end{aligned}
\tag{2.2}
$$

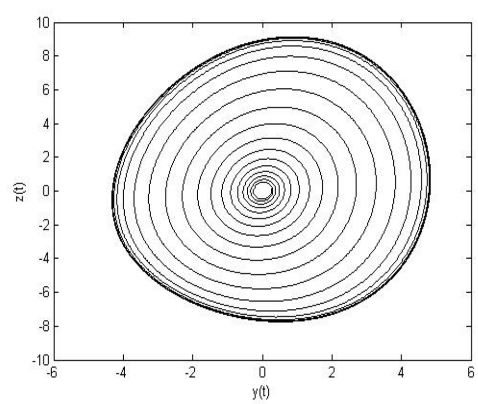
When $a = 6,\ b = 2.92,\ c = 1.2$ and $m = 1$, the system yields chaotic trajectory. The lowest value of α for which the system remains chaotic is 0.93. The chaotic attractors in the *x-y-z* space and *x-y, y-z, z-x* planes are shown through Figs. 2.2(a) – (d) respectively for $\alpha = 0.96$.



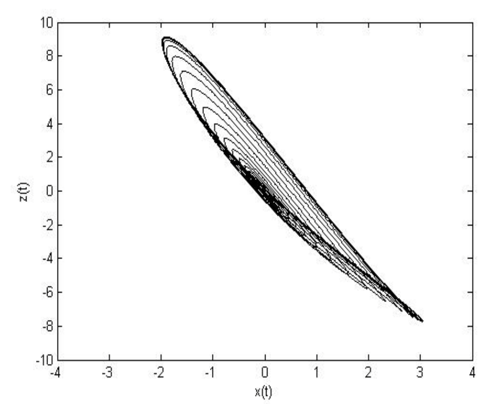
(a) *x-y-z* space



(b) *x-y* plane



(c) *y-z* plane



(d) *x-z* plane

Fig. 2.2. Projections of phase portraits of Genesio-Tesi attractor for $\alpha = 0.96$ at $(a,b,c,m) = (6, 2.92, 1.2, 1)$.

2.3 Anti-synchronization of two identical fractional-order Qi systems

In this section, the fractional-order chaotic Qi system is considered as the drive system as

$$\begin{aligned} D_t^{\alpha} x_1 &= p(y_1 - x_1) + y_1 z_1 \\ D_t^{\alpha} y_1 &= r x_1 - y_1 - x_1 z_1 \\ D_t^{\alpha} z_1 &= -q z_1 + x_1 y_1, \end{aligned} \tag{2.3}$$

and also as the response system as

$$\begin{aligned} D_t^{\alpha} x_2 &= p(y_2 - x_2) + y_2 z_2 + \mu_1(t) \\ D_t^{\alpha} y_2 &= r x_2 - y_2 - x_2 z_2 + \mu_2(t) \\ D_t^{\alpha} z_2 &= -q z_2 + x_2 y_2 + \mu_3(t), \end{aligned} \tag{2.4}$$

where $\mu(t) = [\mu_1(t),\ \mu_2(t),\ \mu_3(t)]^T$ is the controller to be designed. To investigate the anti-synchronization of the systems (2.3) and (2.4), we define the error states as

$$e_1 = x_2 + x_1,\ e_2 = y_2 + y_1,\ e_3 = z_2 + z_1.$$

The corresponding error dynamic system obtained by adding equations (2.3) and (2.4) is

$$\begin{aligned} D_t^{\alpha} e_1 &= p(e_2 - e_1) + y_2 z_2 + y_1 z_1 + \mu_1(t) \\ D_t^{\alpha} e_2 &= r e_1 - e_2 - x_2 z_2 - x_1 z_1 + \mu_2(t) \\ D_t^{\alpha} e_3 &= -q e_3 + x_2 y_2 + x_1 y_1 + \mu_3(t). \end{aligned} \tag{2.5}$$

Choosing the control functions as

$$\begin{aligned} \mu_1(t) &= -y_2 z_2 - y_1 z_1 + V_1(t) \\ \mu_2(t) &= x_2 z_2 + x_1 z_1 + V_2(t) \\ \mu_3(t) &= -x_2 y_2 - x_1 y_1 + V_3(t), \end{aligned} \tag{2.6}$$

the error system is found as

$$
\begin{aligned}
D_t^{\alpha} e_1 &= p(e_2 - e_1) + V_1(t) \\
D_t^{\alpha} e_2 &= r\, e_1 - e_2 + V_2(t) \\
D_t^{\alpha} e_3 &= -\, q\, e_3 + V_3(t),
\end{aligned}
\tag{2.7}
$$

where $V_1(t), V_2(t), V_3(t)$ are the linear control inputs chosen such that the system (2.7) becomes stable. Next we consider

$$
\begin{bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{bmatrix} = M \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},
$$

where M is a 3×3 constant matrix. In order to make the closed loop system stable, the matrix M should be selected in such a way that eigenvalues λ_i of the error dynamical system satisfy the condition $|\arg(\lambda_i)| > \pi\alpha/2, \quad i = 1,2,3.$ Consider the following choice of the matrix M as

$$
M = \begin{pmatrix} p-1 & -p & 0 \\ -r & 0 & 0 \\ 0 & 0 & q-1 \end{pmatrix},
$$

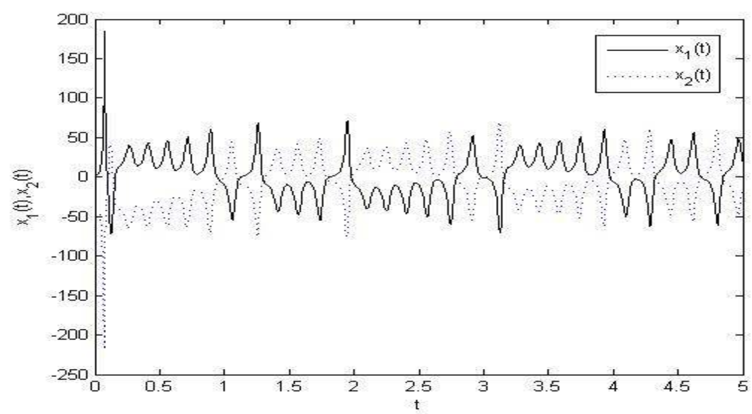
the error system is changed to

$$
D_t^{\alpha} e_i = -e_i \ , \ i = 1,2,3. \tag{2.8}
$$

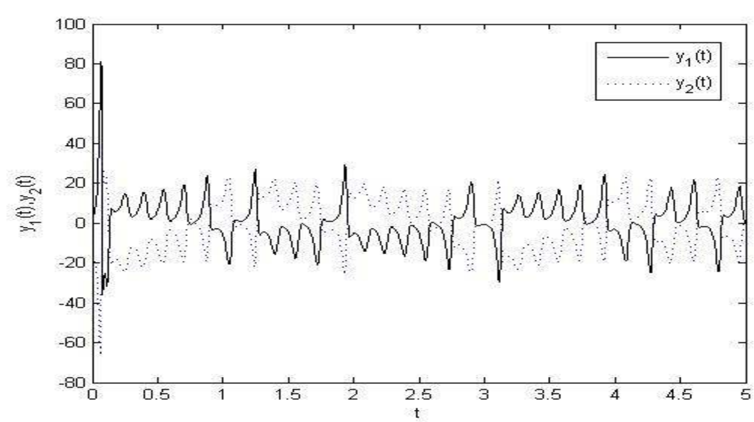
Here all the three eigen values of the system (2.8) are -1, which satisfy $|\arg(\lambda_i)| > \alpha\pi/2, \ (i = 1,2,3)$ for $0 < \alpha < 1$. Thus the error system converges to zero as $t \to \infty$ and therefore, anti-synchronization between the systems (2.3) and (2.4) is achieved.

2.3.1 Numerical simulation and results

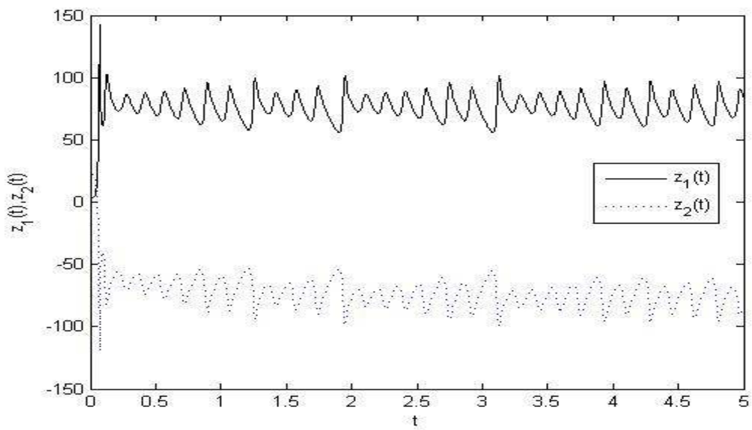
For the numerical simulation, the parameters of the fractional-order Qi-system are taken as $p = 35,\ q = 8/3,\ r = 80$. The Adams-Bashforth-Moulton method is used to solve the systems with time step size 0.005 for the fractional-order $\alpha = 0.92$. The initial states of the drive and the response systems are taken as $(2,\ 3, 4)$ and $(-50,\ -17,\ 27)$ respectively. Thus, the initial errors are $(-48,\ -14,\ 31)$. Figs. 2.3(a) – 2.3 (c) show the anti-synchronization between state vectors and Fig. 2.3(d) represents the convergent of error system towards zero as time becomes large.



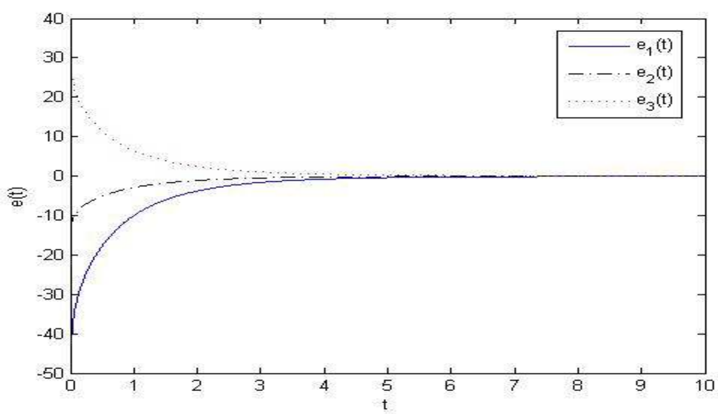
(a) State trajectories between x_1 and x_2.



(b) State trajectories between y_1 and y_2.



(c) State trajectories between z_1 and z_2.



(d) The evolution of error state $(e_1(t), e_2(t), e_3(t))$.

Fig. 2.3. State trajectories of drive system (2.3) and response system (2.4) between state vectors and evolution of error vectors for the fractional-order $\alpha = 0.92$.

2.4 Anti-synchronization of two identical fractional-order Genesio-Tesi systems

In this section, the fractional-order Genesio-Tesi systems is considered as the drive system as

$$
\begin{aligned}
D_t^{\alpha} x_1 &= y_1 \\
D_t^{\alpha} y_1 &= z_1 \\
D_t^{\alpha} z_1 &= -a\,x_1 - b\,y_1 - c\,z_1 + m\,x_1^2
\end{aligned}
\tag{2.9}
$$

and also as the response system as

$$
\begin{aligned}
D_t^{\alpha} x_2 &= y_2 + \mu_1(t) \\
D_t^{\alpha} y_2 &= z_2 + \mu_2(t) \\
D_t^{\alpha} z_2 &= -a\,x_2 - b\,y_2 - c\,z_2 + m\,x_2^2 + \mu_3(t),
\end{aligned}
\tag{2.10}
$$

where $\mu(t) = [\mu_1(t),\ \mu_2(t),\ \mu_3(t)]^T$ is the controller to be designed. To investigate the anti-synchronization of systems (2.9) and (2.10), we define the error states as $e_1 = x_2 + x_1$, $e_2 = y_2 + y_1$, $e_3 = z_2 + z_1$. The corresponding error dynamics becomes

$$
\begin{aligned}
D_t^{\alpha} e_1 &= e_2 + \mu_1(t) \\
D_t^{\alpha} e_2 &= e_3 + \mu_2(t) \\
D_t^{\alpha} e_3 &= -a\,e_1 - b\,e_2 - c\,e_3 + m\,x_2^2 + m\,x_1^2 + \mu_3(t).
\end{aligned}
\tag{2.11}
$$

Choosing the control functions as

$$
\begin{aligned}
\mu_1(t) &= V_1(t) \\
\mu_2(t) &= V_2(t) \\
\mu_3(t) &= -m\,(x_2^2 + x_1^2) + V_3(t),
\end{aligned}
\tag{2.12}
$$

we find

$$
\begin{aligned}
D_t^{\alpha} e_1 &= e_2 + V_1(t) \\
D_t^{\alpha} y_2 &= e_3 + V_2(t) \\
D_t^{\alpha} z_2 &= -a\,e_1 - b\,e_2 - c\,e_3 + V_3(t).
\end{aligned}
\tag{2.13}
$$

Representing the control inputs $V_1(t), V_2(t), V_3(t)$ as

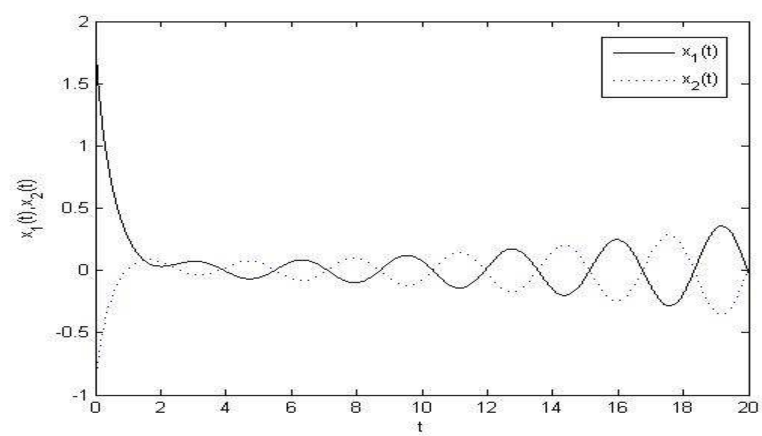
$$\begin{bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{bmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ a & b & c-1 \end{pmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

the error system is reduced to $D_t^{\alpha} e_i = -e_i$, $i = 1,2,3$.

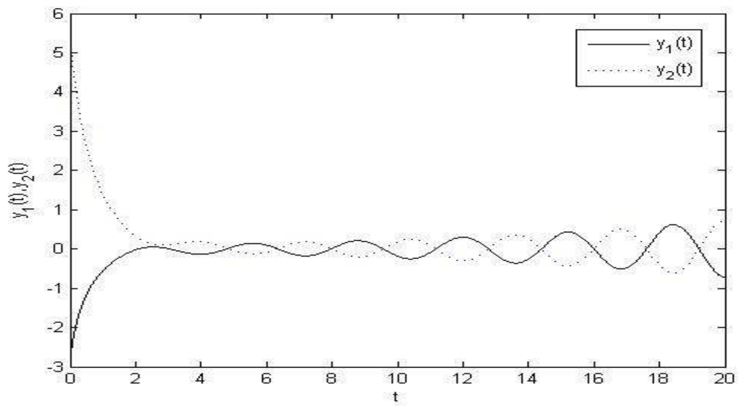
Proceeding as the previous case, since all the eigen values are -1, it may be concluded that the above error system is stable which clearly demonstrates that the anti-synchronization between the systems (2.9) and (2.10) is achieved.

2.4.1 Numerical simulation and results

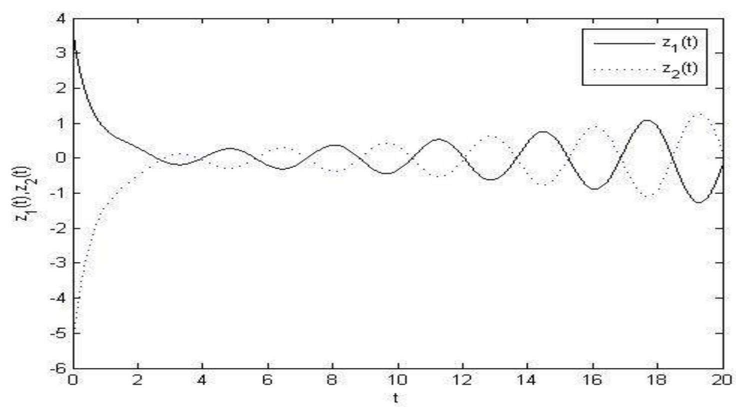
During the simulation to demonstrate anti-synchronization behavior of the two identical fractional-order Genesio system, the parameters are considered as $a = 6,\ b = 2.92,\ \ c = 1.2$ and $m = 1$, the time step size is taken as 0.005. The initial states of the drive and the response systems are taken as $(2, -3, 4)$ and $(-1, 6, -6)$ so that the initial errors are $(1, 3, -2)$. Figs. 2.4(a) – (c) and 2.4 (d) respectively depict the state vectors and the error state for the fractional-order derivative $\alpha = 0.96$.



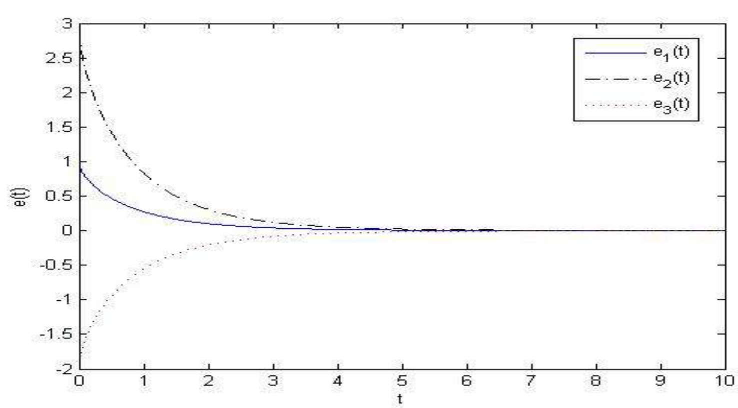
(b) State trajectories between x_1 and x_2.



(a) State trajectories between y_1 and y_2.



(c) State trajectories between z_1 and z_2.



(d) The evolution of error state $(e_1(t), e_2(t), e_3(t))$.

Fig. 2.4. State trajectories of systems (2.9) and (2.10) between state vectors and evolution of error vectors for the fractional-order $\alpha = 0.96$.

2.5 Anti-synchronization between fractional-order Genesio-Tesi and Qi system

In this section, the anti-synchronization behavior between two different fractional-order systems viz., Genesio-Tesi and Qi systems is studied. It is assumed that Genesio-Tesi system drives the Qi system. Therefore, the fractional-order Genesio-Tesi system is defined as a drive system as given in (2.9) and fractional-order Qi system as a response system described in (2.4). For investigation of the anti-synchronization of the systems (2.9) and (2.4), defining the error states as $e_1 = x_2 + x_1$, $e_2 = y_2 + y_1$, $e_3 = z_2 + z_1$, we get the error system as

$$\begin{aligned} D_t^{\alpha} e_1 &= p(e_2 - e_1) + V_1(t) \\ D_t^{\alpha} e_2 &= r e_1 - e_2 + V_2(t) \\ D_t^{\alpha} e_3 &= -q e_3 + V_3(t), \end{aligned} \tag{2.14}$$

where the linear control inputs $V_1(t), V_2(t), V_3(t)$ are related to the control functions as

$$\begin{aligned} \mu_1(t) &= p(y_1 - x_1) - y_2 z_2 - y_1 + V_1(t) \\ \mu_2(t) &= r x_1 + x_2 z_2 - y_1 - z_1 + V_2(t) \\ \mu_3(t) &= -q z_1 - x_2 y_2 + a x_1 + b y_1 + c z_1 - m x_1^2 + V_3(t). \end{aligned} \tag{2.15}$$

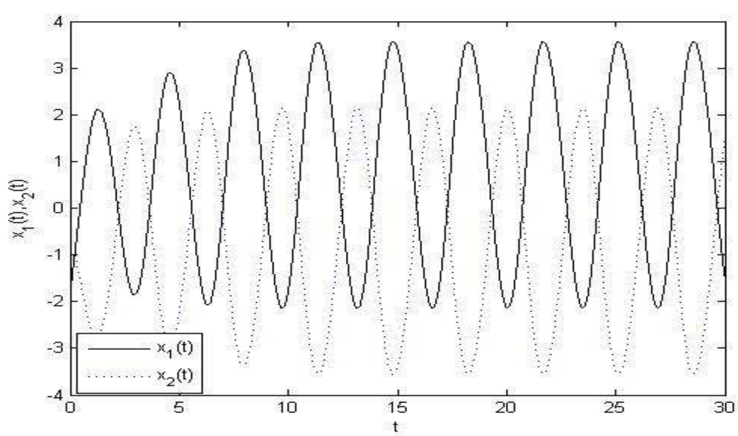
We consider

$$\begin{bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{bmatrix} = \begin{pmatrix} p-1 & -p & 0 \\ -r & 0 & 0 \\ 0 & 0 & q-1 \end{pmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

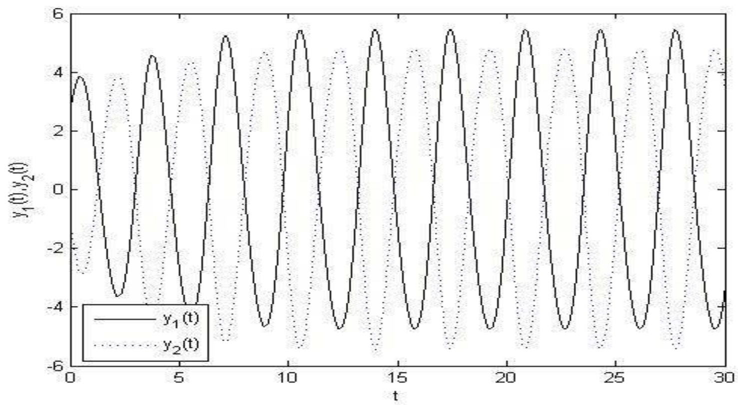
so that the error system reduces to a stable system $D_t^{\alpha} e_i = -e_i$. Thus the anti-synchronization between the systems (2.9) and (2.4) is achieved.

2.5.1 Numerical simulation and results

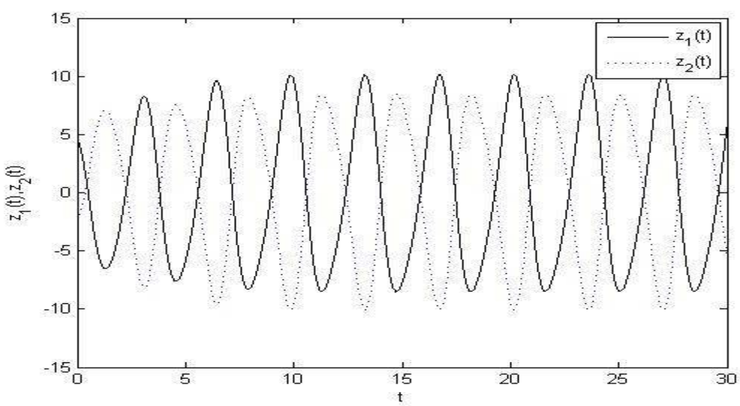
To demonstrate the effectiveness of the proposed method and the simulation results for the anti-synchronization between fractional-order Genesio and Qi systems, the parameters of Genesio and Qi systems are chosen as $a = 6,\ b = 2.92,\ c = 1.2$ and $m = 1$, and $p = 35,\ q = 8/3,\ r = 80$ respectively, so that the fractional-order systems exhibit chaotic behaviors. Time step size is taken as 0.005. The initial states of the drive and the response systems are taken as $(-2, 3, 5)$ and $(-1, -1, -2)$ respectively. Thus, the initial errors are $(-3, 2, 3)$. Figs. 2.5(a) – (c) represent the variations of state vectors and 2.5(d) represents the error system converges to zero as time becomes large for the order of the derivative $\alpha = 0.96$.



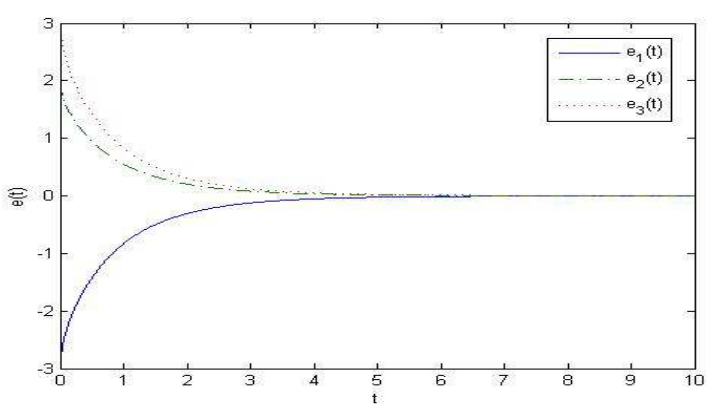
(a) State trajectories between x_1 and x_2



(b) State trajectories between y_1 and y_2.



(c) State trajectories between z_1 and z_2.



(d) State trajectories of error system.

Fig. 2.5. Plots of state trajectories of systems (2.9) and (2.4) between state vectors and evolution of error vectors for the fractional-order $\alpha = 0.96$.

2.6 Conclusion

In the present chapter, the anti-synchronizations between identical and non-identical fractional-orders chaotic systems using active control method based on fractional-order stability theory have been investigated. The author has succeeded in achieving three important objectives. First one is the exhibitions of dynamic nature of chaotic systems for fractional-order time derivative. Second one, using stability analysis, suitable conditions for anti-synchronization of fractional-order chaotic systems through linear controller input parameters on the respective systems have been established. The most important part of the analysis is the proper design of control function so that error states decay to zero for large time which helps to find the time required for anti-synchronization between the fractional-order chaotic systems. It is worth noting that the active control method clearly exhibits its simplicity, suitability, effectiveness and reliability during applications and implementation in anti-synchronizing the fractional-order chaotic systems, the outcome will surely be appreciated by the researchers working in the area of dynamical system. It is also believed that the method can be extended to the various existing chaotic systems for anti-synchronization, which may have applications in different fields of engineering including secure communication, encryption, control process etc. in fractional-order systems when those posses memory and much more sophisticated dynamics compared to their integral counterpart.