CHAPTER-1

Introduction

Chapter 1

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1.1 Delay Differential Equation

Isaac Newton and Gottfried Leibniz investigated differential and integral calculus in the seventeenth century; numerous problems in biology, physics and engineering have been analyzed using ordinary differential equations. In many applications, it is assumed that the systems under consideration satisfy the *principle of causality;* that is, the rate of change of the state of system is independent of the past and is determined solely by the present inputs. But one should analyze that this is only a first approximation to the true situation. A more realistic model depicts that the rate of variation in the system's state depends not only on its current value, but also on the past history of the system. When a system is governed by an equation which does not incorporate a dependence on its past history, it generally consists of either ordinary or partial differential equations. However, in many processes time delays are not negligible; such models incorporating past history generally include functional differential equations (FDEs) or delay differential equations (DDEs).

A delay differential equation (DDE) is a type of functional differential equation where the the highest order derivative of the unknown function at a certain time depends on the solution of the function at previous times. DDEs are also referred as retarded functional differential equations, hereditary differential equation, equations with aftereffect or dead-time, differential-difference equations or in control theory as time-delay systems.

Mathematically, the delay differential equations can be expressed in the form

$$x^{'}(t) = f(t, x(t), x_t), \quad t \ge t_0, \tag{1.1.1}$$

with the initial history

$$x(t) = \phi(t), \quad t_0 - \tau \le t < t_0, \tag{1.1.2}$$

where $x(t) \in R^n$ is the state of the system at time t, $x_t = \{x(t-\tau) : \tau \le t\}$ representing the state of solution in the past, τ is a time delay or lag. Instead of a simple initial condition, an initial history function $\phi(t)$ needs to be specified on the entire interval $[t_0 - \tau, t_0]$. This initial function is usually taken to be continuous which is an infinite set of values, making the DDE problem inherently infinite-dimensional. This infinite dimensional nature of DDE is apparent in the study of dynamical system.

More general delay equations might be considered: constants time delays(τ_j are positive constant), time-dependent delays ($\tau_j = \tau_j(t)$), state-dependent delays ($\tau_j = \tau_j(t, x(t))$), continuously distributed delays, and higher derivatives all occur in applications and lead to more complicated evolution equations. However, equations of the form (1.1.1) and (1.1.2) constitute a sufficiently broad class of systems arise in practice for a variety of reasons, and provide an important category of dynamical systems.

The theories of ODEs and DDEs are very much similar. One can similarly define the ideas of linear, nonlinear and homogeneous equations for DDEs as defined for ODEs. The analytical and numerical techniquies developed for ODEs have been extended for DDEs. There are important differences as well, while the phase space for an ODE is always finite dimensional; a DDE shows an infinite dimensional dynamical system. This feature results from the fact that instead of an initial value, an initial function is provided to determine the solution.

In the eighteenth century, Leonhard Euler, Joseph Lagrange and Pierre-Simon Laplace studied DDEs in connection to different geometrical issues. In an International Congress of Mathematicians held in 1908, Emile Picard underlined the criticalness of hereditary effects in display of physical frameworks.

Vito Volterra (1909, 1928) talked about the integro differential equations those model viscoelasticity. In 1931, he composed a book on the role of hereditary effects on models for the interaction of species and he was the first to study such equations efficiently.

After 1940, the subject gained much momentum due to the consideration of significant models of engineering systems and control. It is most likely genuine that most engineers were well aware of the fact that hereditary effects happen in physical systems, yet this impact was frequently ignored due to the fact that there was inadequate hypothesis to discuss such models in point of interest.

During 1950's, there was significant movement in the subject through the valuable research contributions by Myshkis (1951), Krasovskii (1959), Bellman and Cooke (1963), Halanay (1966). Their contributions give an acceptable picture of the subject up to the early 1960's.

This has been known for some time, however the theory for such systems has widely been created currently. There were some extremely interesting improvements concerning the closure of the set of exponential results of linear equations and the extension of solutions regarding these special solutions. Then again, there appeared to be little worry around a qualitative theory in the same spirit with respect to ordinary differential equations.

Likewise with all such equations describing system dynamics, stability is an essential concern. In the book on stability theory, Krasovskii and Brenner (1963) introduced the theory of Lyapunov functional highlighting the important fact that some issues in such systems are more serious and manageable to solve if one considers the motion in a function space despite the fact that the state variable is a finite-dimensional vector. New applications also continue to arise and require modifications of even the definition of the basic equations.

Basic theories describing time-delay systems established in the 1950s and 1960s; when the topics related to existence and uniqueness of solutions, continuous dependence, stability methods and numerical approximation to study the behavior of solutions, etc. where developed which are the foundation for the later analysis of time-delay systems. The sensitivity and variational equations (sensitive to the initial history) for DDEs have also attracted the interest to the researchers.

The concept of delay differential equations is abundant in nature, developed extensively and has become part of the vocabulary of researchers dealing with wide range of applications including physics, engineering, economics, chemistry, mechanics, nuclear reactors, heat flow, distributed & neural networks, microbiology, epidemiology, physiology, sociology, ecology, nonlinear optics as well as many others certainly adding volume to its own merit.

1.2 Fractional Calculus

The concept of fractional calculus may be considered an *old* and yet *novel* topic. It is an old topic since, starting from some speculations of G.W. Leibniz (1695, 1697) and L. Euler (1730), it has been developed up to nowadays. The motivation of fractional calculus lies in the question that whether the meaning of derivative of an integer order n can be extended when n is not an integer. In a letter dated September 30th, 1695, L'Hospital curiously asked to Leibniz about a particular notation he had used in his publication for the nth-derivative of the linear function $f(x)=x$. L'Hospital posed the question to Leibniz, "What if n be 1/2?". Leibniz responded that, "It will lead to a paradox." But he added prophetically, from this apparent paradox, one day useful consequences will be drawn.

Fractional order derivative has proven to be a very suitable tool for the description of memory and hereditary properties of various processes. Nowadays, the theoretical analysis and practical applications of these operators are well established, and their applications to science and engineering are being considered as an attractive topic.

Some mathematical functions and definitions (Podlubny (1999), Miller and Ross (1993), Kilbas *et al.* (2006)) which are inherently tied to fractional calculus and will commonly be required are discussed herewith for the development of the further work.

1.2.1 Gamma function

Euler's gamma function is one of the most common functions used in the fractional calculus. The Gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z)=\int_0^{\infty} e^{-t}t^{z-1}dt\,, \tag{1.2.1}$$

which converges in the right half of the complex plane $\text{Re}(z) > 0$. One of the basic properties of the Gamma function is the following reduction formula:

$$\Gamma(z+1) = z\Gamma(z). \tag{1.2.2}$$

Clearly, $\Gamma(1) = 1$, using the aforementioned property, we get

$$\Gamma(n+1) = n! \quad \text{for } n \in N^{+}.$$

Thus, the simplest notation of gamma function is simply the generalization of factorial for all real numbers.

1.2.2 Beta function

The Beta-function and its relation with gamma function is defined as

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx, \tag{1.2.3}$$

where $\text{Re}(z) > 0, \text{Re}(w) > 0$ and

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{1.2.4}$$

1.2.3 Mittag-Leffler function

The Mittag-Leffler function (Mittag-Leffler, 1903) $E_\alpha(z)$ defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \text{Re}(\alpha) > 0, \tag{1.2.5}$$

which is known as Mittag-Leffler function of the first kind. The Mittag-Leffler function of the second kind $E_{\alpha,\beta}(z)$ has the following form

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \text{Re}(\alpha),\ \text{Re}(\beta) > 0. \tag{1.2.6}$$

Obviously when $\beta = 1$, one can see that both functions are same i.e.,

$$E_{\alpha,1}(z) = E_{\alpha}(z). \tag{1.2.7}$$

Definition 1.2.1 A real function $f(x),\ x > 0$, is said to be in the space $C_{\mu},\ \mu \in \Re$, if there exists a real number $p > \mu$, such that $f(x) = x^{p} f_1(x)$, where $f_1(x) \in C[0,\infty)$.

Definition 1.2.2 A function $f(x),\ x > 0$ is said to be in the space $C_{\mu}^{n}\ , n \in N_0 = N \cup \{0\}$ iff $f^{(n)} \in C_{\mu}$.

In 1819, S. F. Lacroix became the first mathematician to present an article that mentioned fractional derivative (Ross (1975)). He expressed the n-th derivative of the function $f(x) = x^{m}$, where m is the positive integer as

$$\frac{d^{n}}{dx^{n}} x^{m} = \frac{m!}{(m-n)!} x^{m-n}. \tag{1.2.8}$$

Using the symbol Γ, which denotes the generalized factorial, one can see that the generalization for a power function can also be extended to real numbers as

$$\frac{d^{\alpha}}{dx^{\alpha}} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}. \tag{1.2.9}$$

This shows the visualization of fractional calculus mathematically.

1.2.4 Grünwald-Letnikove fractional integral

In 1867, A. K. Grünwald proposed perhaps the most difficult, yet in some way the most natural approach for fractional differentiation. His method was based on the generalization of the finite difference quotients for fractional derivatives, obtaining the formula (Miller (1995)),

$$ {}_aD_x^{\alpha} f(x) = \lim_{h\to\infty} h^{-\alpha} \sum_{j=0}^{\left\lceil \frac{t-a}{h} \right\rceil} (-1)^j \binom{\alpha}{j} f(t-jh). \qquad (1.2.10) $$

1.2.5 Riemann-Liouville fractional integral

The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f(x)$ is defined as

$$ J_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi)\, d\xi\,,\ \alpha > 0,\ x > 0, \qquad (1.2.11) $$

$$ J_x^0 f(x) = f(x). $$

1.2.6 Riemann-Liouville fractional derivative

The Riemann-Liouville fractional derivative operator of order $\alpha > 0$ of a function $f(x)$ is defined as

$$ \begin{aligned} D_x^{\alpha} f(x) &= D_t^n (J_x^{n-\alpha} f)(x),\ n-1 < \alpha < n\ ,\ n \in \mathrm{N} \\ &= \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau. \end{aligned} \qquad (1.2.12) $$

1.2.7 Properties of the Riemann-Liouville operator

The followings are basic properties of the Riemann-Liouville fractional integral operator J_x^{α} for $f \in C_{\mu}^n$, $\mu \ge -1$, α, $\beta > 0$ and $\gamma > -1$.

(i) $J_x^{\alpha} J_x^{\beta} f(x) = J_x^{\alpha+\beta} f(x)$,

(ii) $J_x^{\alpha} J_x^{\beta} f(x) = J_x^{\beta} J_x^{\alpha} f(x)$,

(iii) $J_x^{\alpha} x^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

1.2.8 Caputo fractional derivative

Caputo (1967) introduced the definition of fractional derivative of $f(x)$, $x>0$ as

$$\begin{aligned} {}^{c}D_x^{\alpha} f(x) &= (J_x^{n-\alpha} D_t^{n} f)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\xi)^{n-\alpha-1} f^{(n)}(\xi)\, d\xi,\ n-1<\alpha<n\ ,\ n\in \mathrm{N}. \end{aligned} \tag{1.2.13}$$

1.2.9 Properties of the Caputo fractional derivative

The properties of the Caputo fractional order derivative are given by

(i) Let $f \in C_{-1}^{n}$, $n \in N_0$, then ${}^{c}D_x^{\alpha} f(x),\ 0<\alpha\le n$ is well defined and

${}^{c}D_x^{\alpha} f(x) \in C_{-1}$.

(ii) Let $n-1\le\alpha\le n, n\in N$ and $f \in C_{\mu}^{n}$, $\mu\ge -1$, then

$$(J_x^{\alpha}\, {}^{c}D_x^{\alpha}) f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^{+}) \frac{x^k}{k!},\quad x\ge 0.$$

In this thesis Caputo definition of fractional derivatives and integrals are considered. The important reason of choosing Caputo derivatives for solving initial value problem of fractional order differential equations is that it holds for both homogeneous and non-homogeneous conditions. Another importance of Caputo definition is that the Caputo derivative of a constant is zero, whereas in the case of a finite value of the lower terminal at the Riemann- Liouville fractional derivative of constant is not equal to zero but $D_t^{\alpha} C = C \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)}$.

1.3 Dynamical System

A dynamical system is a concept in mathematics where a fixed rule describes the time dependence of a point in a geometrical space (wikipedia). To describe a dynamical system one needs to explain phase space or state space, whose coordinates describe exactly the state of some real or hypothetical system at any instant and a dynamical rule that specifies the immediate future of all state variables, given the current state of those same state variables. In mathematical language, the dynamical rule is based on a function mapping that takes as its input the state of the system at one time and gives as its output the state of the system at the next time. Dynamical systems are deterministic if for a given time interval a unique future state follows from the current state, or stochastic or random if there is a probability distribution of possible consequents. Mathematically, a dynamical system is described by an initial value problem. In this way, dynamical system can be considered to be a model describing the temporal evolution of a system. Modelling is a powerful analytical tool for understanding and predicting behavior of physical and artificial systems that changes over time and it has had a history of success.

1.4 Stability

In the theory of dynamical system, stability analysis allows us to determine the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions. This is vital in an extensive variety of applications since much behaviour observed in the real world can be described using differential equations.

Let us consider the autonomous system of ODE as

$$D_t\,x = f(x,\,y),$$
$$D_t\,y = g(x,\,y), \tag{1.4.1}$$

where x and y are the state variables of the system, and f and g are specified nonlinear functions. The concepts equilibrium point and stability are motivated by the desire to keep a dynamical system in, or at least close to, some desirable state. The equilibrium point of a dynamical system is utilized for a state of the system that does not change in the process of time, i.e., if the system is in equilibrium at time t_0, and afterward it will stay there for all times $t > t_0$. A typical starting point for the analysis of (1.4.1) is to find out the equilibrium (fixed) points and to perform a local stability analysis. The equilibrium point, (x^*, y^*), satisfies

$$f(x,y) = 0\,,\; g(x,y) = 0. \tag{1.4.2}$$

The behavior of solutions near (x^*, y^*) can be examined by linearizing (1.4.1) at (x^*, y^*). Considering $\xi = x - x^*$ and $\eta = y - y^*$ as the perturbations, the linearized system is given by

$$\begin{bmatrix} \dfrac{d\xi}{dt} \\ \dfrac{d\eta}{dt} \end{bmatrix} = J\big|_{(x^*,y^*)} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{1.4.3}$$

where $J\big|_{(x^*,y^*)}$ is the Jacobian matrix at equlibrium point. The local stability of this system can be easily determined from the eigenvalues of the Jacobian matrix.

Let us take an autonomous system of the form

$$D_t\,x = f(x) \tag{1.4.4}$$

Definition 1.4.1: A critical point x_e of the system (1.4.4) is said to be **stable** if, for all ε > 0 there is a δ > 0 such that every solution $x = \phi(t)$ satisfying $\|\phi(0) - x_e\| < \delta$ exists for all positive t and satisfies $\|\phi(t) - x_e\| < \varepsilon$ for all t ≥ 0.

Definition 1.4.2: A critical point x_e is said to be **asymptotically stable** if it is stable and if there exists a δ_0, with $0 < \delta_0 < \delta$, such that if a solution $x = \phi(t)$ satisfies $\|\phi(0) - x_e\| < \delta_0$ then φ(t) → x as t → ∞.

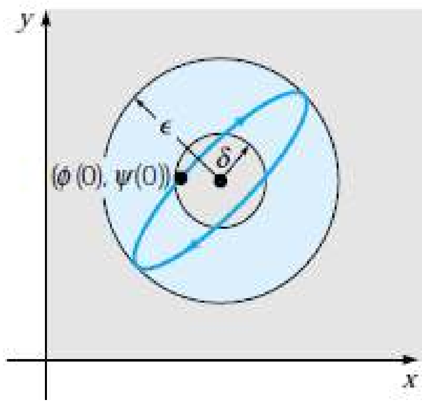

Fig. 1.1. Stability.

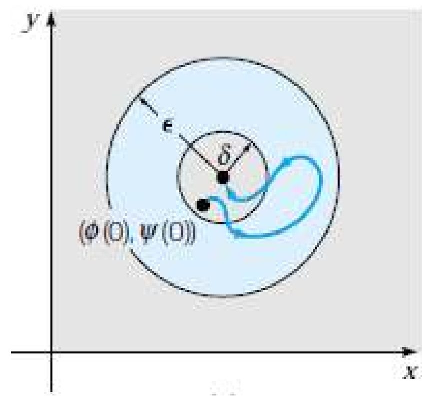

Fig. 1.2. Asymptotic stability.

A geometrical interpretation of stability, and asymptotic stability notions are given in Figs. 1.1 and 1.2, respectively. They show typical behaviours of different solution trajectories as per the type of stability they possess. This is demonstrated geometrically that all solutions that start "sufficiently close" (within the distance δ) to x_e stay "close" (within the distance ε) to x_e. However, the trajectory of the solution does not have to approach the critical point x_e as $t \to \infty$. A critical point which is not stable is said to be unstable. For the case of asymptotic stability, trajectories that start "sufficiently close" to x_e not only stay "close" but must eventually approach to x_e as $t \to \infty$.

1.4.1 Lyapunov's Direct Method

Stability Theory Early results include the work of the Russian mathematician A. M. Lyapunov, who, in 1892, gave the first precise definition of stability and developed the theory of stability of a motion or solution for a system of ordinary differential equations. The use of Lyapunov functions to prove stability has become common place and is known alternatively as the Lyapunov's direct method or Lyapunov's second method. Lyapunov's direct method involves determining a family of closed curves or closed surfaces in state space such that the general behavior of nearby trajectories of a dynamical system can be inspected. This technique is relevant for examining the global stability of nonlinear systems and for deciding trapping regions for a dissipative chaotic flow.

Let us consider an autonomous nonlinear dynamical system

$$x'(t) = f(x(t)), \qquad x(0) = x_0, \tag{1.4.5}$$

where $x(t) \in D \subseteq R^n$ is state vector, D is an open set containing the origin, and $f : D \to R^n$ is continuous. Assume that $x = 0$ be an equilibrium point of nonlinear system and consider $V : D \to R$ be a positive definite continuously differentiable function on an open neighborhood D of the origin, such that $V'(x) \le 0$ in D along the path of the system. Then $V(x)$ is called a Lyapunov function and the equilibrium point is stable in the sense of Lyapunov. Furthermore, if $V'(x) < 0$ in $D - \{0\}$, then the equilibrium point is said to be asymptotically stable.

Let $x = 0$ be an equilibrium point of a nonlinear system $x' = f(x)$ and let $V : R^n \to R$ be a positive definite continuously differentiable function, such that $V(x) \to \infty$ as $\|x\| \to \infty$ and

$$\dot{V}(x) \le 0, \quad \forall x \ne 0,$$

then equilibrium point $x = 0$ is globally asymptotically stable. Note that Lyapunov functions usually are not unique for a given system.

1.4.2 Stability of Delay Differential Equations

The study of stability of systems of differential equations which contain delays has been an active area of research in many fields of science and engineering. The question of stability of nonlinear functional differential equations is complicated by the lack of a complete Lyapunov functional structure whose existence is necessary for stability of a general nonlinear time-delay system. Lyapunov-Krasovskii stability criterion, introduced by Krasovskii (1959), can be interpreted as a natural generalization of the classical Lyapunov stability theory for ordinary systems to the case of time-delay systems (infinite dimensional systems). The choice of an appropriate Lyapunov–Krasovskii functional is the key-point for deriving of stability criteria. For this reason, he extended the complete theory of Lyapunov by using functional $V : C \to R$ where $C = C([-\tau, 0], R^n)$

Theorem 1.4.1: (Krasovskii (1963)) Suppose that $u, v, w : [0, \infty) \to [0, \infty)$ are continuous nonnegative nondecreasing functions, $u(s), v(s)$ are positive for $s > 0$, $u(0) = v(0) = 0$. If there is a continuous function $V : C \to R$, such that

$$u(|\varphi(0)|) \le V(\varphi) \le v(|\varphi|), \qquad \varphi \in C,$$

$$V'(\varphi) = \limsup_{t \to 0} \frac{1}{t}[V(x_t(.,\varphi)) - V(\varphi)] \le -w(|\varphi(0)|),$$

then equilibrium point of time-delayed system $x = 0$ is stable. If, in addition, $w(s) > 0$ for $s > 0$, then equilibrium point $x = 0$ is asymptotically stable.

1.4.3 Stability of Fractional Order Systems

This section presents the definitions for stability condition of certain class of the linear and nonlinear fractional order systems of finite dimension, which is of great interest towards the investigation of fractional order dynamical systems. It is important to note that the stability and asymptotic behavior of fractional order system are not of exponential type however it is in the form of power law $t^{-\alpha} (\alpha \in R)$, the so called long memory behavior (Matignon (1996), Petras (2009)).

Consider the n dimensional fractional order system

$$D_t^{\alpha_i} x_i = f_i(x_1, x_2, ..., x_n), \qquad (1.4.6)$$

where $0 < \alpha_i \leq 1$, is fractional order such that $\alpha_i = k_i / m_i$, $\gcd(k_i, m_i) = 1$, $k_i, m_i \in N$, $i = 1,2,...,n.$ and let m be the least common multiple of the denominators $m_i's$ of $\alpha_i's$. $D_t^{\alpha_i}$ is the fractional order time derivative, then we have the following results:

For the case, when the autonomous system (1.4.6) is linear that is $[f_1(x), f_2(x),..., f_n(x)]^T = [a_{ij}]_{i,j=1}^n x = Ax$ where $x \in R^n$, then

- Matignon (1996) gave the qualitative result that the stability is guaranteed iff the roots of some polynomial lie outside the closed angular sector $|\arg(\lambda))| \leq \pi\alpha/2$. Thus generalizing in an amazing way the well known results for the integer case $\alpha = 1$.
  If $\alpha = \alpha_1 = \alpha_2 = ... = \alpha_n$, then the fractional order system is asymptotically stable iff $|\arg(spec(A))| > \pi\alpha/2$ and the system is stable iff $|\arg(spec(A))| \geq \pi\alpha/2$ with those critical eigenvalues which satisfies

$\left|\arg(spec(A))\right| = \pi\alpha/2$ have geometric multiplicity one. In this case the components of the state decay towards zero like $t^{-\alpha}$.

- Later Deng *et al.* (2007) observed that that if all roots of the characteristic equation have negative real parts, then the equilibrium of the incommensurate linear system with fractional order is Lyapunov globally asymptotical stable if the equilibrium exists which is almost the same as that of classical differential equations. Thus, if α_i's are different rational numbers, then the system (1.4.6) is asymptotically stable if all roots of the equation $\det(diag(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \ldots, \lambda^{m\alpha_n}) - A) = 0$ satisfy $\left|\arg(\lambda)\right| > \gamma\pi/2$, where $\gamma = 1/m$.

Again if the function f_i is nonlinear and has second continuous partial derivatives in a ball centered at an equilibrium point $x^* = (x_1, x_2, \ldots\ldots, x_n)$, that is $f_i(x_1, x_2, \ldots\ldots, x_n) = 0$, for every $i = 1, 2, \ldots, n$, then we have interesting results:

- (Ahmed *et al.*, (2007)) If $\alpha = \alpha_1 = \alpha_2 = \ldots = \alpha_n$, then the equilibrium point x^* of the system (1.4.6) is asymptotically stable iff $\left|\arg(spec(J|_{x^*}))\right| > \pi\alpha/2$, where the matrix J is Jacobian matrix of the system (1.33) which is defined by $J = \left[\frac{\partial f_i}{\partial x_j}\right]_{i,j=1}^{n}$
- In the year 2009, Petras (2009) presented a survey paper where he reviewed the methods for stability investigation of a certain class of fractional order linear and nonlinear systems and illustrated the result that if α_i's are different rational numbers, then the equilibrium point

x^* of the system (1.4.6) is asymptotically stable if all the roots of the equation

$$\det(diag(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \ldots, \lambda^{m\alpha_n}) - J|_{x^*}) = 0 \tag{1.4.7}$$

satisfy $|\arg(\lambda)| > \gamma\pi/2$, where $\gamma = 1/m$. Then the condition inequality of this condition can be written in another way as

$$\frac{\pi}{2m} - \min_i\{|\arg(\lambda_i)|\} < 0. \tag{1.4.8}$$

Thus, equilibrium point of the system (1.4.6) is asymptotically stable if the condition (1.4.8) is satisfied.

Hence, a fundamental condition for fractional order system to exhibit a chaotic attractor is

$$\frac{\pi}{2m} - \min_i\{|\arg(\lambda_i)|\} \geq 0. \tag{1.4.9}$$

The aforementioned condition is a necessary condition for the existence of chaos. It might be utilized as a powerful tool to determine the minimum order for which a given fractional order system can't observe chaotic attractor and for what value of fractional order derivative, the system may generate chaos (Tavazoei and Haeri (2007)).

These stability results play an important role during the study of chaos and synchronization between fractional order systems.

1.5 Chaos

Chaos is derived from the Greek word 'Χαος', significances a state without order or predictability. According to ancient Greek mythology, chaos is the "primeval emptiness preceding the genesis of the universe, turbulent and disordered, mixing all the elements" (William (1997)).

Chaotic dynamics may have had its early development with the work of the French mathematical physicist Henri Poincaré in the late 1800's (Poincaré (1890)). He attempted to tackle the issue of the motion of three objects in mutual gravitational attraction, the so-called three-body problem (Sun, planet and moon). Poincaré found that orbits are aperiodic, and yet not increasing infinitely (meaning deterministic) nor approaching any fixed points or limit cycles. He was able to show that complication in solving the three body problem was due to the sensitive dependence on initial conditions making long term prediction impossible. Therefore, Poincaré might be considered to be the first person to imagine "Chaos".

In 1898, Jacques Hadamard remarked general divergence of trajectories in spaces of negative curvature. He observed that all trajectories are unstable, in that, all particle trajectories separate exponentially from one another, with a positive Lyapunov exponent.

One imperative year was 1963, when the meteorologist E.N. Lorenz contributed significantly to chaotic theory while concentrating the dynamics of the weather and described a simple mathematical model for forecasting the weather behaviour using computer simulations (Lorenz (1963)). Lorenz's model was the first numerical model to analyze chaos in a non-linear dynamical system. His discovery planted the seed for the new hypothesis of chaos science. He also observed that the trajectory of the system being evolving with time in a complex and non-repeating pattern, oscillated in an irregular manner, but always remaining in a bounded region. Lorenz explained that whenever he started his simulations from two slightly different initial conditions, the ensuing result soon got to be entirely unexpected. He additionally noticed that the solution settled down in a fascinating butterfly shaped set of points, which caused the concept of this high level sensitive dependence on initial conditions to become popularly known as the “butterfly effect”. Finally, he concluded that the earth’s weather is a chaotic and therefore, a long-range prediction is an impossible task. Lorenz’s work had little impact until the 1970’s and later it turned into a boon for chaos.

In 1971, David Ruelle and Floris Takens described a phenomenon in an alternative mathematical explanation of the turbulence in fluid dynamics based on the existence of so-called "strange attractors" (Ruelle and Takens (1971)).

In 1975, Li and Yorke showed the sustained aperiodic and unpredictable behaviors arising in deterministic nonlinear maps. Their research article (Li and Yorke (1975)) demonstrated the term chaos for the various phenomena that demonstrated aperiodicity along with sensitive dependence on initial conditions.

What Lorenz accomplished for climate, Robert May (May (1976)) did for ecology. His work exhibited the logistic map as a plausible population model with a period-doubling cascade of bifurcations and chaotic trajectories.

Some important works were carried out by the physicist Mitchell Feigenbaum, who discovered order in disorder and have rekindled interest towards low dimensional discrete dynamical systems. David Ruelle, Floris Takens and S. E. Newhouse have played an important role in the investigation of deterministic chaos in hydrodynamic systems. One of the foremost contributors to this area of research was Benoit Mandelbrot. Using a home computer, Mandelbrot (1982) spearheaded the mathematics of fractals. His fractals (the geometry of fractional dimensions) served to depict the actions of chaos, rather than explain it.

Currently, chaos theory grabbed the attention of the researchers and contributes to a significant amount of the ongoing research concerning numerous fields, such as electronic systems, message encryption, Brownian motion, change of the weather, evolution of the solar system, behavior of the stock markets, fluid dynamics, biological processes in the living organisms, control of chemical reactions, fluctuation of the astronomical orbit, information theory, etc. Before going for further study in the domain of chaos, let us make an attempt to characterize it.

1.5.1 Definition of chaos

Despite the fact there is no unified, universally accepted, rigorous definition of chaos in the current scientific literature however a commonly used definition confines the fundamental nature of chaos, which everyone will agree with the following quotation as mentioned by Strogatz (1994).

“Chaos is aperiodic long-term behaviour, in a deterministic system that exhibits sensitive dependence on initial conditions.”

The three properties of chaos mentioned in the definition might be explained as follows:

"**Aperiodic long-term behaviour**" means that there are trajectories which do not settle down to fixed points, periodic orbits, or quasi-periodic orbits as time becomes large.

"**Deterministic**" means that the system has no random or noisy inputs or parameters. The irregular behaviour arises from the system's nonlinearity, rather than from noisy driving forces.

"**Sensitive dependence on initial conditions**" means that nearby trajectories separate exponentially fast, i.e., the system has a positive Lyapunov exponent.

Nowadays, chaos theory has four fundamental questions, which are chaos synchronization, chaos control, chaotification, and ultimate boundedness. Incredible advancement and interest have been attained in these areas. This thesis deals with the synchronization of time delayed chaotic systems and the synchronization between fractional order chaotic systems.

1.6 Chaos Synchronization

Fujisaka and Yamada (1983, 1983a, 1984) paved the way with their pioneering studies on chaos synchronization, but it was not until 1990 when Pecora and Carroll (1990) introduced their method of chaotic synchronization and suggested application to secure communications that the subject received considerable attention within the scientific community. Pecora and Carroll wrote that

"*Chaotic systems would seem to be dynamical systems that defy synchronization. Two identical autonomous chaotic systems started at nearly the same initial points-in phase space have trajectories which quickly become uncorrelated, even though each maps out the same attractor in phase space. It is thus practical impossibility to construct, identical, chaotic, synchronized system in laboratory.*"

It might seem that the synchronization of chaotic systems is difficult to achieve due to their extremely sensitive dependence on initial conditions. The synchronization scenario has been of long standing interest and studied extensively. Synchronization of chaotic systems is defined as a process wherein two (or many) chaotic systems (either identical or nonidentical) adjust a given property of their motion to a common behavior, due to coupling or forcing (Boccaletti *et al.* (2002)).

This contribution of the present thesis reveals the significant influence of time delay and fractional order time derivative on chaos synchronization and has suggested a new approach to get the synchronization of chaotic systems. The methodologies applied in this thesis include both theoretical analysis and numerical simulation.

1.6.1 Types of chaotic synchronization

Inspired by the seminal works of Fujisaka and Yamada (1983, 1983a, 1984) and of Pecora and Carroll (1990) on synchronization of chaotic systems, various types of synchronization scenario have been investigated, viz. complete synchronization (Pecora and Carroll (1990)), phase synchronization (Rosenblum *et al.* (1996)), anti-phase synchronization (Zhang and Sun (2004)), hybrid synchronization (Xie and Chen (2002), Sudheer and Sabir (2009)), lag synchronization (Rosenblum *et al.* (1997), Boccaletti *et al.* (2002)), generalized synchronization (Rulkov *et al.* (1995), Yang and Duan (1998)), projective synchronization (Mainieri and Rehacek (1999), Li and D. Xu (2004)), Function projective synchronization (Chen and Li (2007)), etc. These different types can be grouped into the following categories.

1.6.1.1 Complete synchronization

The easiest type of synchronization to detect is complete synchronization; it appears as the equality of the state variables while evolving in time. In complete synchronization the chaotic trajectories of the coupled systems remain in step with each other in the course of time. This is observed in coupled chaotic systems with identical elements (i.e., each component having the same dynamics and parameter set) and is also referred as conventional synchronization or identical synchronization. This kind of synchronization was first described by Pecora and Carroll (1990).

Mathematically, two continuous-time chaotic systems

$$x'(t) = f(x(t)) \tag{1.6.1}$$

and

$$y'(t) = f(y(t)) + \mu(x(t), y(t)), \tag{1.6.2}$$

where $x, y \in R^n$ are the state vectors, $f : R^n \to R^n$ is continuous nonlinear vector functions and $\mu(x, y)$ is the vector controller, are said to be completely synchronized if $\lim_{t\to\infty} \|y(t) - x(t)\| = 0$, for any combination of initial states x(0) and y(0).

1.6.1.2 Anti- synchronization

Anti-synchronization between two chaotic systems is occurred when the respective states of chaotic systems have the same magnitude but opposite in sign. It may takes place in both the identical and non-identical chaotic systems. Mathematically, the anti-synchronization of two chaotic systems

$$x'(t) = f(x(t)) \tag{1.6.3}$$

and

$$y'(t) = g(y(t)) + \mu(x(t), y(t)), \tag{1.6.4}$$

where $x, y \in R^n$ are the state vectors, $f, g : R^n \to R^n$ are continuous nonlinear vector functions and $\mu(x, y)$ is the vector controller, is achieved when $\lim_{t\to\infty} \|y(t) + x(t)\| = 0$, for any combination of initial states $x(0)$ and $y(0)$.

1.6.1.3 Hybrid Synchronization

Hybrid synchronization is an attractive case where one part of the system is anti- synchronized while the other part is completely synchronized so that complete synchronization and anti-synchronization co-exist in the system.

1.6.1.4 Phase synchronization

This scenario of the synchronization occurs when the coupled chaotic systems keep their phase difference bounded by a constant while their amplitudes remain uncorrelated (Rosenblum *et al.* (1996), Rosa *et al.* (1998)). This phenomenon is mostly achieved in coupled non identical systems. In case of phase synchronization, if $\varphi_1(t)$ and $\varphi_2(t)$ denote the phases of the two coupled chaotic systems, synchronization of the phase is described by the relation $n\varphi_1(t) = m\varphi_2(t)$, with m and n whole numbers.

1.6.1.5 Projective synchronization

Projective synchronization was proposed by Mainieri and Rehacek (1999) in partially linear systems, where they showed that the responses of two identical systems synchronized up to a constant scaling factor.

Considering the drive and response systems in the form of

$$x'(t) = f(x(t)) \tag{1.6.5}$$

and

$$y'(t) = g(y(t)) + u(x(t), y(t)), \tag{1.6.6}$$

the error system is defined as

$$e(t) = y(t) - \lambda x(t), \tag{1.6.7}$$

where λ is a constant.

The systems (1.6.5) and (1.6.6) are said to be projective synchronized, if there exists a constant λ such that $\lim_{t\to\infty}\|e(t)\|=0$.

Complete synchronization and anti-phase synchronization can be regarded as special cases of projective synchronization characterized by $\lambda=1$ and $\lambda=-1$, respectively. Xu *et al.* (2001, 2002) introduced several control schemes based on Lyapunov stability theory to manipulate the scaling factor onto a required value, and derived a general condition for projective synchronization.

1.6.1.6 Modified Projective synchronization

Modified projective synchronization was proposed by Li (2007), where the chaotic systems could be synchronized to a constant scaling matrix. In the error system (1.13), if we replace λ by a constant scaling matrix A then the error system (1.13) becomes

$$e(t)=y(t)-A\,x(t)\,, \tag{1.6.8}$$

where $A=diag(a_1,a_2,...,a_n)$ is the scaling constant matrix such that $\lambda_i's$ are constant scaling factors $\forall i\in N$. Then the systems (1.6.5) and (1.6.6) are said to be in modified projective synchronization, if there exists a constant scaling matrix A such that $\lim_{t\to\infty}\|e(t)\|=0$. By choosing the scaling factors in the scaling matrix, one can flex the scales of the different states independently.

1.6.1.7 Generalized synchronization

Coupled chaotic systems are said to exhibit Generalized synchronization if there exists some functional relation between systems, i.e., $y(t)=\varphi(x(t))$, means the states of the two interacting systems are functionally synchronized. This type of synchronization occurs mainly when the coupled chaotic systems are different, although it is also been investigaed between identical chaotic systems. Projective synchronization is a particular case of generalized synchronization where one-to-one mapping function is a simple linear function $\varphi(x(t)) = \lambda x(t)$.

1.6.1.8 Function Projective Synchronization

Function Projective Synchronization generalizes projective synchronization, in which drive and response systems are synchronized up to a scaling function $\lambda(t)$, but not a constant. Firstly, Chen and Li (2007) had introduced function projective synchronization.

Considering the drive and response systems as

$$x'(t) = f(x(t)) \tag{1.6.9}$$

and

$$y'(t) = g(y(t)) + u(x(t), y(t)), \tag{1.6.10}$$

where $x, y \in R^n$ are the state vectors, $f, g : R^n \to R^n$ are continuous nonlinear vector functions and $u(x, y)$ is the controller function. The error system is written as

$$e(t) = y(t) - \lambda(t)x(t), \tag{1.6.11}$$

where $\lambda(t)$ is the continuously differentiable function with $\lambda(t) \neq 0 \ \forall \ t$.

The systems (1.6.9) and (1.6.10) are said to be function projective synchronized, if there exists a scaling function $\lambda(t)$ such that $\lim_{t\to\infty}\|e(t)\| = 0$.

The freedom of choosing the scaling function in FPS has advantage and can additionally enhance the security of communication.

1.6.1.9 Modified Function Projective Synchronization

Modified Function Projective Synchronization is more general than modified projective synchronization and FPS. In this method the responses of the synchronized dynamical states synchronize up to a desired scaling function matrix.

In the error system (1.6.11), if we replace $\lambda(t)$ by a function scaling matrix $A(t)$, then the error system becomes

$$e(t) = y(t) - A(t)\,x(t)\,, \tag{1.6.12}$$

where $A(t) = diag(a_1(t), a_2(t), ..., a_n(t))$ is the function scaling matrix such that $a_i(t) \neq 0, i = 1,2,...,n$ are the continuously differentiable function for all t. Then the systems (1.6.9) and (1.6.10) are said to be in modified function projective synchronized, if there exists a function scaling matrix $A(t)$ such that $\lim_{t\to\infty}\|e(t)\| = 0$. It is obvious that compared with FPS, the MFPS can provide more security in communication.

1.6.1.10 Anticipating and Lag synchronization

In these cases, the synchronized state is characterized by a time interval τ such that the dynamical variables of the chaotic systems are related by $y(t) = x(t+\tau)$, which means that the dynamics of one of the systems follows or anticipates the dynamics of the other. These types of synchronization may occur in time-delayed chaotic systems, coupled in a drive-response configuration. In case of Anticipating synchronization, , the response anticipates the dynamics of the drive. In lag Synchronization, $\tau < 0$, appears as the asymptotic boundedness of the difference between the output of one system at time t and the output of the other shifted in time of a lag time.

In particular, if the time delay may become zero, i.e., $\tau = 0$, the anticipating synchronization and lag synchronization are further simplified to complete synchronization.

1.6.2 Methodology of chaos synchronization

In this sub-section, a non exhaustive rapid overview of methods for obtaining the synchronization is discussed in detail. The most effective and widely studied approach is discovered by L. M. Pecora and T. L. Carroll where they synchronized two identical chaotic systems with different initial conditions (Pecora and Carroll (1990, 1991)). Active control, adaptive control, feedback control and tracking control methods have been extensively studied in recent literature for obtaining various types of synchronization.

1.6.2.1 Active control method

In 1997, E. W. Bai and K. E. Lonngren proposed the active control method. The proposed scheme has received considerable attention during the last few decades. The method of synchronizing two identical chaotic systems through active control method can be illustrated by using following algorithm.

Let us consider the chaotic system in the form of

$$Dx(t) = A\,x(t) + f(x(t)), \tag{1.6.13}$$

where $x(t) \in R^n$, is the state vector, A is constant matrix and $f : R^n \to R^n$ defines the non-linear function. For investigating the chaos synchronization, the drive and response systems are represented as

$$D\,x_1(t) = A\,x_1(t) + f(x_1(t)), \tag{1.6.14}$$

$$Dx_2(t) = A\,x_2(t) + f(x_2(t)) + \mu(t), \tag{1.6.15}$$

where $\mu(t) \in R^n$ is the active control functions. If we define the error state as $e(t) = x_2(t) - x_1(t)$, the error dynamical system becomes

$$De(t) = A\,e(t) + f(x_2(t)) - f(x_1(t)) + \mu(t). \tag{1.6.16}$$

Design the active controller function $\mu(t)$ as

$$\mu(t) = f(x_1(t)) - f(x_2(t)) + V(t),$$

where $V(t)$ is the linear control function, as a function of $e(t)$. There are many possible choices for the linear control function $V(t)$. We choose, $V(t) = Me(t)$, where M is a constant matrix, the elements of the matrix M are properly chosen in such a way that the error system will have all eigen values with negative real parts. Finally the error dynamical system reduces to

$$De(t) = (A + M)\,e(t). \tag{1.6.17}$$

This choise leads to the error system converges to zero as time t tends to infinity and thus the synchronization of two chaotic systems is achieved i.e., $\lim_{t\to\infty}\|e(t)\| = 0.$ This method can also be applied for the synchronization of non-identical chaotic systems.

1.6.2.2 Adaptive control method

Most of the analyses in synchronization involve two identical or non-identical chaotic systems under the hypotheses that all the parameters of the drive and response systems are known a priori. But in practical situations, there exists partially or even fully uncertain parameters in either/both drive system and/or response system that may destroy the synchronization and even break it. The conventional control approaches are not applicable in such case, as the desired synchronization would be destroyed by these uncertainties. So it is necessary to design adaptive controller and parameter update laws of unknown parameters for synchronization of coupled chaotic systems. The definition of adaptive synchronization implies that no direct information about the system parameters is available for designing the controller functions and update laws (Zang *et al.* (2006), Chen *et al.* (2002a), Sun (2013)).

Consider the drive system as

$$x' = f(x) + F(x)\alpha \tag{1.6.18}$$

and the response system as

$$y' = g(y) + G(y)\beta + \mu(t), \tag{1.6.19}$$

where $x, y \in R^n$ are the state vectors of drive and response systems respectively, $\alpha, \beta \in R^m$ are the unknown parameter vectors, $f(x), g(y) \in R^n$ and $F(x), G(y) \in R^{nxm}$ are nonlinear functions, the elements $F_{ij}(x)$ in matrix $F(x)$ and $G_{ij}(y)$ in matrix $G(y)$ are satisfying $F_{ij}(x) \in L_\infty, \forall x \in R^n$ and $G_{ij}(y) \in L_\infty, \forall y \in R^n$ respectively and $\mu(t) \in R^n$ is the controller to be determined.

Let, $e = y - x$ be the synchronization error vector. Then the error dynamical system can be written as

$$e' = g(y) + G(y)\beta - f(x) - F(x)\alpha + \mu(t). \tag{1.6.20}$$

If nonlinear controller is designed as (Zang *et al.* (2006))

$$\mu(t) = f(x) + F(x)\hat{\alpha} - g(y) - G(y)\hat{\beta} - ke \tag{1.6.21}$$

and adaptive law of parameters are taken as

$$\begin{cases} \hat{\alpha}' = -[F(x)]^T e, \\ \hat{\beta}' = [G(y)]^T e, \end{cases} \tag{1.6.22}$$

where k is a positive constant, $\hat{\alpha}$ and $\hat{\beta}$ are estimated values of the unknown parameters α and β, respectively.

Consider a Lyapunov function as

$$V = \frac{1}{2}[e^T e + \overline{\alpha}^T \overline{\alpha} + \overline{\beta}^T \overline{\beta}],$$

where $\overline{\alpha} = (\alpha - \hat{\alpha})$ and $\overline{\beta} = (\beta - \hat{\beta})$ are the estimation errors of the parameters α and β. The time derivative of V along the the trajectories of (1.6.20) is

$$V = e^T (G(y)(\beta - \hat{\beta}) - F(x)(\alpha - \hat{\alpha}) - ke) + \overline{\alpha}'^T \overline{\alpha} + \overline{\beta}'^T \overline{\beta}.$$

Applying the parameter update laws rule, we get

$$V = -ke^T e,$$

where $V \in R^n$ is positive definite function and $V' \in R^n$ is negative definite function. Therefore, according to Lyapunov stability theorem, the error system is globally and asymptotically stable which means that the synchronization of coupled chaotic systems is achieved and it is also seen that the parameters' estimation errors $\overline{\alpha}$ and $\overline{\beta}$ decay to zero as time goes to infinity.

1.6.2.3 Tracking control method

Tracking control method is mainly applied for the synchronization of chaotic systems with different order. The fractional order chaotic drive and response systems are defined as

$$D_t^{\alpha} x = F(x) \tag{1.6.23}$$

and

$$D_t^{\beta} x = G(y) + \phi(y, x), \tag{1.6.24}$$

where $0 < \alpha, \beta < 1$ are fractional order time derivatives, $x, y \in R^n$ are n-dimensional state vectors of the systems (1.6.23) and (1.6.24) respectively. $F, G : R^n \to R^n$ are two continuous nonlinear vector functions, $\phi(x, y) : R^{n \times n} \to R^n$ is a controller function which have to be determined. For the systems (1.6.23) and (1.6.24), if there exists a control function $\phi(x, y)$ such that

$$\lim_{t \to \infty} \|e\| = \lim_{t \to \infty} \|y - K(x)x\| = 0, \tag{1.6.25}$$

where $e = (e_1, e_2, \ldots. e_n)^T \in R^n$ is an error state vector, $K(x) = diag(k_1(x), k_2(x), \ldots., k_n(x)),\quad i = 1, 2, \ldots., n$ is a continuous scaling function, such synchronization is called FPS. Based on the idea of tracking control, in order to achieve the equation (1.6.25), it is assumed that the function

$$\phi(x, y) = D_t^{\beta}(K(x)x) - G(K(x)x) + N(x, y)\, e \tag{1.6.26}$$

is a suitable control function, where $N(x, y) \in R^{n \times n}$. Using equation (1.6.26), the error system is obtained as

$$D_t^{\beta} e = (M(x, y) + N(x, y))\, e, \tag{1.6.27}$$

where $M(x, y)e = G(y) - G(K(x)x) \in R^{n \times n}$ is a polynomial matrix. Now with proper choice of $N(x, y)$, if all the eigenvalues of matrix $M(x, y) + N(x, y)$ have negative real part, then $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$, where $\lambda_i, i = 1, 2, \ldots n,$ are the eigenvalues of the matrix $M(x, y) + N(x, y)$. Thus the error dynamical system (1.6.27) is locally asymptotically stable and as a consequence FPS between the systems (1.6.23) and (1.6.24) is achieved (Zhou and Zhu (2011), Hegazi *et al*. (2013)).

1.7 Numerical Methods

1.7.1 Runge-Kutta method for Delay Differential Equations

In 2001, L.F. Shampine and S. Thompson (Shampine and Thompson (2001)) developed a program, dde23, to solve delay differential equations (DDEs) with constant delays in MATLAB. The aim was to make it as easy as possible to solve effectively a large class of DDEs. The method was proposed based on the Runge–Kutta triple BS(2,3) used in ode23, which was nicely explained how explicit Runge-Kutta triples can be extended and used to solve DDEs.

Let us consider a system of the nonlinear delay differential equation as

$$y'(t)=f(t,y(t),y(t-\tau_1),y(t-\tau_2),...y(t-\tau_k)), \quad a \le t \le b, \tag{1.7.1}$$

with the initial history

$$y(t)=S(t), \quad t \le a, \tag{1.7.2}$$

where τ_j $(j=1,2,...,k)$ are constant delays such that $\tau=\min(\tau_1,...,\tau_k)>0$.

In order to motivate the construction of numerical strategy for DDEs let us discuss the explicit Runge-Kutta triples to solve the general ordinary differential equation as

$$y'(t)=f(t,y(t)), \quad a \le t \le b, \tag{1.7.3}$$

with the initial condition $y(a)$.

Assuming an approximation as $y_n = y(t_n)$, let us proceed to obtain the approximation at $t_{n+1} = t_n + h_n$. A triple of s stages involves three formulas. For $i = 1,2,\ldots,s,$ the stages $f_{ni} = f(t_{ni}, y_{ni})$ are defined in terms of $t_{ni} = t_n + c_i h_n$ and

$$y_{ni} = y_n + h_n \sum_{j=1}^{i-1} a_{ij} f_{nj} \quad .$$

Considering $\Phi(t_n, y_n)$ as an increment function, the approximation used to advance the integration is

$$\begin{aligned} y_{n+1} &= y_n + h_n \sum_{i=1}^{s} b_i f_{ni} \\ &= y_n + h_n \Phi(t_n, y_n). \end{aligned}$$

The solution satisfies this formula with a residual called the local truncation error, lte_n,

$$y(t_{n+1}) = y(t_n) + h_n \Phi(t_n, y_n) + lte_n,$$

which provides an error $O(h_n^{p+1})$ for sufficiently smooth *f* and $y(t)$. Selecting the step size triple gives rise to another formula as

$$\begin{aligned} y^*_{n+1} &= y_n + h_n \sum_{i=1}^{s} b^*_i f_{ni} \\ &= y_n + h_n \Phi^*(t_n, y_n). \end{aligned}$$

The solution satisfies this equation with a local truncation error lte^*_n that is $O(h_n^p)$. The third formula is given by

$$\begin{aligned} y_{n+\sigma} &= y_n + h_n \sum_{i=1}^{s} b_i(\sigma) f_{ni} \\ &= y_n + h_n \Phi(t_n, y_n, \sigma), \end{aligned}$$

where coefficients $b_i(\sigma)$ are polynomials in σ, so this represents a polynomial approximation to $y(t_n+\sigma h_n)$ for $0 \le \sigma \le 1$. The third formula is described as a continuous extension of the first because it yields the value y_n when $\sigma = 0$ and y_{n+1} when $\sigma = 1$ and assume that the order of the continuous extension is same as that of the first formula. These assumptions hold for the BS(2,3) triple. For such triples we regard the formula used to advance the integration as just the special case $\sigma = 1$ of the continuous extension. The local truncation error of the continuous extension is defined by

$$y(t_n+\sigma h_n) = y(t_n) + h_n \Phi(t_n, y(t_n), \sigma) + lte_n(\sigma).$$

Assume that for smooth f and $y(t)$, there exists a constant C_1 such that $\|lte_n(\sigma)\| \le C_1 h_n^{p+1}$ for $0 \le \sigma \le 1$.

Now the main problem is to establish approximation to the delayed term $y(t-\tau)$ which consists of two cases $h_n \le \tau$ and $h_n > \tau_j$ for some j and assume an approximation as $y(t) = S(t)$ is accessible for all $x \le x_n$. If $h_n \le \tau$, then all $t_{ni} - \tau_i \le t_n$ and $f_{ni} = f(t_{ni}, y_{ni}, S(t_{ni}-\tau_1), \ldots, S(t_{ni}-\tau_k))$ is an explicit form of the stage and thus the formulas are explicit. After taking the step to x_n, we use the continuous extension to characterize $S(t)$ on $[t_n, t_{n+1}]$ as $S(t_n+\sigma h_n) = y_{n+\sigma}$.

For the second case, the implicit formulas may be evaluated that arise when the step size is bigger than τ that is $h_n > \tau_j$ for some j, the history term $S(t)$ is evaluated in the span of the current step and the formulas are defined implicitly. Defining $S(t)$ for $x \le x_n$, when reaching x_n and extend its definition somehow to $(t_n, t_n + h_n]$ and represent the resulting function as $S^0(t)$. The simple iteration starts with the approximate solution $S^{(m)}(t)$. The following iterations are computed with the explicit formula as

$$S^{(m+1)}(t_n + \sigma h_n) = y(t_n) + h_n \Phi(t_n, y(t_n), \sigma : S^{(m)}(t)).$$

1.7.2 A Predictor-Corrector Approach for Fractional-Order Differential Equations

The investigation of an algorithm for the numerical solution of nonlinear fractional-order differential equations, equipped with suitable initial conditions can be found in the research articles of Diethelm (Diethelm *et al.* (2004), Diethelm and Ford (2004)). The scheme is the generalization of classical one-step Adams–Bashforth–Moulton scheme for first order equations. The algorithm may be used even for nonlinear problems, and it may also be extended to multi-term equations which involve more than one differential operator.

Let us consider the following nonlinear fractional order differential equation

$$D_t^{\alpha} y(t) = f(t, y(t)), \quad 0 \le t \le T, \tag{1.7.4}$$

with the initial conditions

$$y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, \ldots\ldots., m-1, \tag{1.7.5}$$

where $m = \lceil \alpha \rceil$ is the smallest integer $\ge \alpha$ i.e., $\alpha \in (m-1, m]$ and the differential operator is the Caputo derivative.

The initial value problem (1.7.4)-(1.7.5) is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{\lceil \alpha \rceil - 1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds. \tag{1.7.6}$$

Before using the Adams-Bashforth-Moulton algorithm to solve the fractional-order differential equations, let us furnish a brief idea idea of the classical one-step Adams-Bashforth-Moulton algorithm for the following ODE:

$$Dy(t)=f(t,y(t)), \quad y(0)=y_0 \tag{1.7.7}$$

Consider the uniform grid $\{t_j = jh,: j=0,1,...,N\}$ with some integer N for the interval $[0,T]$, where $h=T/N$ be the step size. Approximating $y_j = y(t_j)$, $j=1,2,...,n,$ the wish to compute an approximation y_{n+1} may be computed through the equation

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s))ds. \tag{1.7.8}$$

Since the integral in the aforementioned equation can be approximated by two-point trapezoidal quadrature formula,

$$\int_{t_n}^{t_{n+1}} f(s, y(s))ds \approx \frac{h}{2}(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))). \tag{1.7.9}$$

Thus

$$y_{n+1} = y_n + \frac{h}{2}(f(t_n, y_n) + f(t_{n+1}, y_{n+1})), \tag{1.7.10}$$

which is an implicit one-step Adams–Moulton method and cannot be solved directly. Therefore, one should adopt another numerical method to approximate y_{n+1} in the right-hand side preliminarily in order to obtain a better approximation. The preliminary approximation y_{n+1}^p, the so-called predictor can be obtained by forward Euler or one-step Adams–Bashforth method as

$$y_{n+1}^{p} = y_n + hf(t_n, y_n). \tag{1.7.11}$$

Therefore, the one-step Adams-Bashforth-Moulton method for ODEs is

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1}^{p})]. \tag{1.7.12}$$

The equations (1.7.11) and (1.7.12) are known as predictor and corrector respectively. It is well known (Hairer *et al.* (1993)) that this method is convergent of order 2, i.e.,

$$\max_{j-0,1,\ldots,N} \left| y(t_j) - y_j \right| = O(h^2).$$

Now we use an algorithm that generalizes the Adams method to solve the fractional-order differential equations. To construct the required algorithm, Diethelm *et al.* (2004), used the product trapezoidal quadrature formula to replace the integral of eq. (1.59), where nodes $t_j \ (j = 0,1,\ldots,n+1)$ are used with respect to the weight function $(t_{n+1} - .)^{\alpha+1}$. Thus one can write the integral part of eq. (1.7.6) as

$$\int_0^{t_{n+1}} (t_{n+1} - u)^{\alpha-1} g(u) du = \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n+1} a_{j,n+1} g(t_j),$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & if \quad j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & if \quad 0 \leqslant j \leqslant n, \\ 1, & if \quad j = n+1, \end{cases} \tag{1.7.13}$$

the equation (1.7.6) reduces to

$$y_h(t_{n+1}) = \sum_{k=0}^{\lceil \alpha \rceil - 1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_h, y_h(t_j)) + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, y_h^p(t_{n+1})), \tag{1.7.14}$$

where predicted value $y_h^p(t_{n+1})$ is determined by the fractional Adams–Bashforth method

$$y_h^p(t_{n+1}) = \sum_{k=0}^{[\alpha]-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j)), \tag{1.7.15}$$

with

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}((n+1-j)^\alpha - (n-j)^\alpha). \tag{1.7.16}$$

Thus the equations (1.7.14) and (1.7.15) with the weights $a_{j,n+1}$ and $b_{j,n+1}$ describes the fractional Adams-Bashforth-Moulton method.

Error in this method is

$$\max_{j=0,1,\ldots,N} \left| y(t_j) - y_h(t_j) \right| = O(h^p), \tag{1.7.17}$$

where $p = \min(2, 1+\alpha)$.