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On the existence and decay in a new thermoelastic theory with two temperatures

José R. Fernández, Santwana Mukhopadhyay, Ramón Quintanilla and Om Namha Shivay

Abstract. In this work, we study a new two-temperatures thermoelastic model. Both thermodynamic and conductive temperatures are included, being related by means of an elliptic or parabolic equation. Then, two problems are considered assuming the dependence or not on the rate conductivity temperature. Existence of solutions for the three-dimensional setting and the exponential energy decay in the one-dimensional case are shown.

Keywords. Two-temperatures, thermoelasticity, existence of solutions, energy decay

Mathematics Subject Classification (2010). 74F05, 74A15

1. Introduction

Heat conduction for thermoelastic models are usually based on the Fourier law. That is, the heat flux vector is proportional to the gradient of temperature. But the consequences of this assumption, jointly with the classical energy equation, implies that the thermal waves propagate instantaneously. Alternative heat conduction theories (see [3, 17, 18]) have been proposed to overcome this drawback. We can recall the books [19, 38, 40], where several mathematical studies are developed to clarify the applicability of the new theories. In this sense, we refer to the theories of Lord and Shulman [25], Green and Lindsay [12],

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phase-lag theories [39], Green and Naghdi [13–15] or Moore-Gibson-Thompson [34, 35].

In 1963, Coleman and Noll [8] provided an example where the restrictions on the constitutive parameters were obtained by using the classical Clausius-Duhem inequality. Therefore, this inequality was stated often as a basic postulate (see, e.g., [7]). However, Gurtin and Williams [16] suggested later that it is not the most general form of the inequality and they proposed a generalized form of the Clausius-Duhem inequality based on two different entropy mechanisms inside a thermoelastic body. Leigh [23] also shown the fact with other examples. Hence, in such cases (for non-simple materials) the generalized form of the Clausius-Duhem inequality should be considered. Following [16], in 1968 Gurtin and several co-workers [4–6] proposed a modification of the Fourier law by considering two temperatures (thermodynamic and conductive temperatures) which are related by means of an elliptic equation. Warren and Chen [41] investigated the wave propagation under this theory, and they noted that the two temperatures are found to have representation, in form of a traveling wave, with a response which occurs instantaneously throughout the body. This fact was also observed in the classical thermoelasticity theory although the wave speed for the two-temperatures theory is larger than the wave speed for the classical one. Even, this increase in the wave velocity is independent of the two-temperatures parameter. More recently, Youssef [42] suggested to modify, in a similar way, the theories of Lord and Shulman and Green and Lindsay. This was extended later by Quintanilla [34, 35] in the case of the dual phase-lag theories and we note that these theories have deserved much attention in the last twenty years (see, for instance, [2, 26–28, 30–33]). Shivay and Mukhopadhyay [37] suggested a new two-temperatures theory for the Green and Lindsay proposition assuming that there is a temperature-rate dependence for the two temperatures. In this case, the relation between the temperatures is modified leading to new mathematical problems. A uniqueness and instability result has been obtained recently [10] but mathematical and physical studies are needed to clarify the applicability. This work is addressed in this line.

The plan of this paper is the following. In the next section we describe briefly the thermomechanical model and we state the constitutive assumptions. In Section 3, we consider the case when the theory is independent of the rate of the conductivity temperature. The existence of solutions in the three-dimensional case and the exponential decay in the one-dimensional case are obtained. The case when we take into account the rate of the conductivity temperature is considered in Section 4 and similar results are shown.

2. Basic equations and assumptions

In this section, we recall the basic equations and the assumptions that we will impose to understand the problem. When we consider the three-dimensional problem we denote by B the domain with a boundary assumed to be smooth enough to apply the divergence theorem.

We will denote by boldface the vectors and tensors. A vector $\boldsymbol{u}$ can take the form (u_i), where we denote by "i" the i-component. A sub-index following a comma means partial derivation with respect to this component.

The evolution equations for this theory are given by

$$\begin{aligned} \rho\ddot{u}_i &= t_{ij,j}, \\ \theta_0\rho\dot{\eta} &= q_{i,i}, \end{aligned}$$

where the constitutive equations are

$$\begin{aligned} t_{ij} &= C_{ijkl}u_{k,l} - \beta_{ij}(\theta + t_1\dot{\theta}), \\ \rho\eta &= t_2\rho c_E\theta_0^{-1}\dot{\theta} + \rho c_E\theta_0^{-1}\theta + \beta_{ij}u_{i,j}, \\ q_i &= K_{ij}\phi_{,j}, \end{aligned}$$

and the two-temperatures relation that, in the case that the theory is independent on the rate of the conductivity temperature, takes the form:

$$\phi - m(K_{ij}\phi_{,i})_{,j} = \theta + t_1\dot{\theta}.$$

In the case that the theory also takes into account the rate of the conductivity temperature, we have

$$t_1\dot{\phi} + \phi - m(K_{ij}\phi_{,i})_{,j} = \theta + t_1\dot{\theta}.$$

In the previous equations, ρ is the mass density, (u_i) is the displacement vector, (t_{ij}) is the stress tensor, η is the entropy, (q_i) is the heat flux vector, ϕ and θ are the conductivity and the thermodynamic temperatures, respectively, (C_{ijkl}) is the elasticity tensor, (K_{ij}) is the thermal conductivity tensor, (β_{ij}) is the coupling tensor, t_1 and t_2 are the two relaxation parameters, θ_0 is the reference temperature, c_E is the thermal capacity and m is a parameter typical of the two-temperatures theory.

In order to make the analysis easier, we assume that the material is homogeneous, but the analysis could be extended (straightforwardly) to the nonhomogeneous case.

When we consider the three-dimensional problem, we will obtain an existence and uniqueness theorem. To prove it we need to assume:

(i) The mass density ρ, the relaxation parameters t_1 and t_2, the thermal capacity c_E, the parameter m and the reference temperature θ_0 are strictly positive numbers.

(ii) The relaxation parameters satisfy $t_1 > t_2$.

(iii) The thermal conductivity tensor (K_{ij}) is symmetric and positive definite; that is, $K_{ij} = K_{ji}$ and there exists a positive constant C_1 such that

$$K_{ij}\xi_i\xi_j \geq C_1\xi_i\xi_i$$

for every vector (ξ_i).

(iv) The elasticity tensor (C_{ijkl}) is symmetric and positive definite; that is, $C_{ijkl} = C_{klij}$ and there exists a positive constant C_2 such that

$$C_{ijkl}\tau_{ij}\tau_{kl} \geq C_2\tau_{ij}\tau_{ij}$$

for every tensor (τ_{ij}).

Our assumptions are natural from the thermomechanical point of view. The meaning of (i) and (ii) is clear. Condition (iii) is also natural and (iv) can be interpreted with the help of the elastic stability theory.

In this paper, we also study the exponential decay of the solutions to the one-dimensional problem defined in an interval of finite length $[0, \ell]$, $\ell > 0$. In this case, the evolution equations are

$$\begin{aligned} \rho\ddot{u} &= t_x, \\ \theta_0\rho\dot{\eta} &= q_x, \end{aligned}$$

where

$$\begin{aligned} t &= \mu u_x - \beta(\theta + t_1\dot{\theta}), \\ \rho\eta &= t_2\rho c_E\theta_0^{-1}\dot{\theta} + \rho c_E\theta_0^{-1}\theta + \beta u_x, \\ q &= K\phi_x, \end{aligned}$$

and the relations between the two temperatures are

$$\phi - mK\phi_{xx} = \theta + t_1\dot{\theta}$$

or

$$t_1\dot{\phi} + \phi - mK\phi_{xx} = \theta + t_1\dot{\theta}.$$

The assumptions we propose in this case are similar to the ones imposed previously. Therefore, we assume that ρ, t_1, t_2, m, θ_0, μ, c_E and K are positive constants and that $t_1 > t_2$. However, to obtain the exponential decay of the solutions we also need to assume that $\beta \neq 0$.

3. First problem

In this section, we consider the problem determined by the system:

$$\left.\begin{aligned} \rho\ddot{u}_i &= \Big(C_{ijkl}u_{k,l} - \beta_{ij}(\theta + t_1\dot{\theta})\Big)_{,j}, \\ \rho c_E t_2\ddot{\theta} &= -\beta_{ij}\theta_0\dot{u}_{i,j} - \rho c_E\dot{\theta} + K_{ij}\dot{\phi}_{,ij}, \\ \phi - mK_{ij}\phi_{,ij} &= \theta + t_1\dot{\theta}, \end{aligned}\right\} \text{ in } B \times (0, \infty), \tag{1}$$

the initial data, for a.e. $\boldsymbol{x} \in B$,

$$\left.\begin{array}{ll} u_i(\boldsymbol{x},0) = u_i^0(\boldsymbol{x}), & \dot{u}_i(\boldsymbol{x},0) = v_i^0(\boldsymbol{x}), \\ \theta(\boldsymbol{x},0) = \theta^0(\boldsymbol{x}), & \dot{\theta}(\boldsymbol{x},0) = \vartheta^0(\boldsymbol{x}), \end{array}\right\} \tag{2}$$

and the boundary conditions:

$$u_i(\boldsymbol{x},t) = \phi(\boldsymbol{x},t) = 0, \quad \boldsymbol{x} \in \partial B, \quad t > 0. \tag{3}$$

In this case, we study the theory where the rate of the conductivity temperature is not considered.

Remark 3.1. The map $\phi \to \phi - mK_{ij}\phi_{,ij}$ defines an isomorphism between $W^{2,2} \cap W_0^{1,2}$ and L^2. If we denote by Φ the inverse, we have that

$$\phi = \Phi(\theta + t_1\dot{\theta}) = \Phi(\theta) + t_1\Phi(\dot{\theta}).$$

3.1. Existence of solutions. We study the problem defined previously and we prove an existence theorem in the Hilbert space:

$$\mathcal{H} = \boldsymbol{W}_0^{1,2} \times \boldsymbol{L}^2 \times L^2 \times L^2.$$

Let its inner product be given as

$$< (\boldsymbol{u},\boldsymbol{v},\theta,\vartheta),(\boldsymbol{u}^*,\boldsymbol{v}^*,\theta^*,\vartheta^*) >= \frac{1}{2}\int_B \Big(\rho v_i\overline{v_i^*} + C_{ijkl}u_{i,j}\overline{u_{k,l}^*}$$
$$+\frac{\rho c_E}{\theta_0}(\theta + t_2\vartheta)\overline{(\theta^* + t_2\vartheta^*)} + \frac{t_2(t_1-t_2)}{\theta_0}\vartheta\overline{\vartheta^*}\Big)\, d\boldsymbol{x},$$

where, as usual, the bar denotes the conjugated complex. It is relevant to say that this inner product defines a norm which is equivalent to the usual one in $\mathcal{H}$.

If we denote $U = (\boldsymbol{u},\boldsymbol{v},\theta,\vartheta)$, our problem can be written as

$$\frac{dU}{dt} = \mathcal{A}U, \quad U(0) = (\boldsymbol{u}^0,\boldsymbol{v}^0,\theta^0,\vartheta^0),$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & \boldsymbol{I} & 0 & 0 \\ \boldsymbol{A} & 0 & \boldsymbol{B} & \boldsymbol{C} \\ 0 & 0 & 0 & I \\ 0 & D & F & E \end{pmatrix}$$

and

$$\boldsymbol{A}\boldsymbol{u} = \frac{1}{\rho}(C_{ijkl}u_{k,l})_{,j}, \quad \boldsymbol{B}\theta = -\frac{1}{\rho}\beta_{ij}\theta_{,j},$$
$$\boldsymbol{C}\vartheta = -\frac{t_1}{\rho}\beta_{ij}\vartheta_{,j}, \quad D\boldsymbol{v} = -\frac{1}{\rho c_E t_2}\beta_{ij}v_{i,j},$$
$$E\vartheta = -\frac{\vartheta}{t_2} + \frac{t_1}{\rho c_E t_2}K_{ij}(\Phi(\vartheta))_{,ij},$$
$$F\theta = \frac{1}{\rho c_E t_2}K_{ij}(\Phi(\theta))_{,ij}.$$

Theorem 3.2. *The operator $\mathcal{A}$ defines a contractive semigroup in the Hilbert space $\mathcal{H}$.*

Proof. First, we have, for every $U \in \mathcal{D}(\mathcal{A})$,

$$\begin{aligned} Re < \mathcal{A}U, U >= -\frac{1}{2}\int_B K_{ij}\Phi(\theta + t_1\vartheta)_{,i}\overline{\Phi(\theta + t_1\vartheta)_{,j}}\, d\boldsymbol{x} \\ -\frac{t_1 - t_2}{2\theta_0}\int_B |\vartheta|^2\, d\boldsymbol{x} - \frac{1}{2}\int_B m|K_{ij}\Phi(\theta + t_1\vartheta)_{,ij}|^2\, d\boldsymbol{x} \le 0. \end{aligned} \tag{4}$$

Secondly, the domain of the operator $\mathcal{A}$ is $(\boldsymbol{u}, \boldsymbol{v}, \theta, \vartheta) \in \mathcal{H}$ such that

$$\boldsymbol{v} \in \boldsymbol{W}_0^{1,2}, \quad \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\theta + \boldsymbol{C}\vartheta \in \boldsymbol{L}^2.$$

Clearly, it is a dense subspace of the Hilbert space $\mathcal{H}$.

Now, we will see that zero belongs to the resolvent of the operator $\mathcal{A}$. Let us consider $F = (\boldsymbol{f}_1, \boldsymbol{f}_2, f_3, f_4) \in \mathcal{H}$. We need to solve the system:

$$\begin{aligned} &\boldsymbol{v} = \boldsymbol{f}_1, \quad \vartheta = f_3, \quad \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\theta + \boldsymbol{C}\vartheta = \boldsymbol{f}_2, \\ &D\boldsymbol{v} + F\theta + E\vartheta = f_4. \end{aligned}$$

We can write

$$\begin{aligned} &\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\theta = \boldsymbol{f}_2 - \boldsymbol{C}f_3, \\ &F\theta = f_4 - D\boldsymbol{f}_1 - Ef_3. \end{aligned}$$

It is clear that we can solve the second equation and the solution θ belongs to L^2. Then, we can also solve the equation

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}_2 - \boldsymbol{C}f_3 - B\theta.$$

If we use the Lax-Milgram lemma, we conclude the existence of $\boldsymbol{u} \in \boldsymbol{W}_0^{1,2}$ solving this equation. In fact, we can also show that

$$\|(\boldsymbol{u}, \boldsymbol{v}, \theta, \vartheta)\| \le C\|F\|,$$

for a certain positive constant C.

Therefore, we have proved the existence of solutions (see [1, 11, 24, 29, 36]).

In view of the previous theorem we can conclude the following.

Theorem 3.3. *Let $(\boldsymbol{u}^0, \boldsymbol{v}^0, \theta^0, \vartheta^0) \in \mathcal{D}(\mathcal{A})$. Then, there exists a unique solution to the problem determined by (1)-(3).*

Remark 3.4. It is well known that we cannot expect exponential decay of the solutions in the classical thermoelastic case when the dimension of the domain is greater than one [20–22]. As the coupling terms are very similar for the different thermoelastic theories, we suspect that a similar behavior also happens for other thermoelastic theories, including the ones considered in this paper. In fact, the examples proposed by Dafermos [9] of undamped isothermal solutions can be also extended (word by word) to our case and, therefore, we should restrict our attention to the one-dimensional case if we want to obtain the exponential decay of the solutions. This will be the aim of the next subsection.

3.2. Exponential decay. In this subsection, we restrict our attention to the one-dimensional problem in order to obtain the exponential decay of solutions. Therefore, we study the problem in a spatial finite interval $[0, \ell]$ for $\ell > 0$.

We recall that our system is written as

$$\left.\begin{array}{l} \rho\ddot{u} = \mu u_{xx} - \beta(\theta_x + t_1\vartheta_x), \\ \rho c_E t_2\ddot{\theta} = -\theta_0\beta v_x - \rho c_E\vartheta + K\Phi(\theta + t_1\vartheta)_{xx}, \\ \phi - mK\phi_{xx} = \theta + t_1\dot{\theta}. \end{array}\right\} \quad \text{in} \quad (0, \ell) \times (0, \infty).$$

We have the following.

Theorem 3.5. *$\mathcal{A}$ generates a semigroup exponentially stable, that is, there exist two positive constants M and ω such that*

$$\|U(t)\| \leq Me^{-\omega t}\|U(0)\|.$$

Proof. To obtain the exponential decay, it is sufficient (see [24]) to show that the imaginary axis is contained in the resolvent of the operator and that

$$\overline{\lim_{\lambda\to\infty}}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{5}$$

Let us prove that the imaginary axis is contained in the resolvent. We assume that this is false, then there exists a sequence $\lambda_n \to \lambda \neq 0$ and a sequence of $(u_n, v_n, \theta_n, \vartheta_n) \in \mathcal{D}(\mathcal{A})$, with unit norm, such that

$$\begin{array}{l} i\lambda_n u_n - v_n \to 0 \quad \text{in} \quad W^{1,2}, \\ i\lambda_n\rho v_n - \mu u_{n,xx} + \beta(\theta_{n,x} + t_1\vartheta_{n,x}) \to 0 \quad \text{in} \quad L^2, \\ i\lambda_n\theta_n - \vartheta_n \to 0 \quad \text{in} \quad L^2, \\ i\rho c_E t_2\lambda_n\vartheta_n + \theta_0\beta v_{n,x} + \rho c_E\vartheta_n - K(\Phi(\theta_n + t_1\vartheta_n))_{xx} \to 0 \quad \text{in} \quad L^2. \end{array}$$

The dissipation inequality (4) implies that

$$\vartheta_n \to 0 \quad \text{in} \quad L^2, \quad \Phi(\theta_n + t_1\vartheta_n) \to 0 \quad \text{in} \quad W^{2,2}.$$

This last condition allows us to conclude that $\theta_n + t_1\vartheta_n \to 0$ in L^2. Then, it follows that $\theta_n \to 0$ in L^2 and, from the last convergence, we also obtain that $\dfrac{v_{n,x}}{\lambda_n} \to 0$ in L^2, which is equivalent to $u_{n,x} \to 0$ in L^2. From here, we also find that $v_n \to 0$ in L^2. This contradicts the assumption and we conclude that the imaginary axis is contained in the resolvent.

The asymptotic condition (5) can be obtained following a similar argument.

Therefore, our theorem is proved.

4. Second problem

In this section, we consider the problem determined by the system:

$$\left.\begin{array}{l}\rho\ddot{u}_i=\left(C_{ijkl}u_{k,l}-\beta_{ij}(\theta+t_1\dot{\theta})\right)_{,j},\\ \rho c_E t_2\ddot{\theta}=-\beta_{ij}\theta_0\dot{u}_{i,j}-\rho c_E\dot{\theta}+K_{ij}\phi_{,ij},\\ t_1\dot{\phi}+\phi-mK_{ij}\phi_{,ij}=\theta+t_1\dot{\theta},\end{array}\right\}\quad \text{in } B\times(0,\infty), \tag{6}$$

the initial data, for a.e. $\boldsymbol{x}\in B$,

$$\left.\begin{array}{l}u_i(\boldsymbol{x},0)=u_i^0(\boldsymbol{x}),\quad \dot{u}_i(\boldsymbol{x},0)=v_i^0(\boldsymbol{x}),\quad \theta(\boldsymbol{x},0)=\theta^0(\boldsymbol{x}),\\ \dot{\theta}(\boldsymbol{x},0)=\vartheta^0(\boldsymbol{x}),\quad \phi(\boldsymbol{x},0)=\phi^0(\boldsymbol{x}),\end{array}\right\} \tag{7}$$

and the boundary conditions:

$$u_i(\boldsymbol{x},t)=\phi(\boldsymbol{x},t)=0,\quad \boldsymbol{x}\in\partial B,\quad t>0. \tag{8}$$

Now, we have taken into account the dependence on the rate of the conductivity temperature.

4.1. Existence of solutions. We will prove an existence and uniqueness result to this problem in the Hilbert space $\mathcal{H}_1$, which is now defined as

$$\mathcal{H}_1=\boldsymbol{W}_0^{1,2}\times\boldsymbol{L}^2\times L^2\times L^2\times W_0^{1,2},$$

and let its inner product be given by

$$< (\boldsymbol{u},\boldsymbol{v},\theta,\vartheta,\phi),(\boldsymbol{u}^*,\boldsymbol{v}^*,\theta^*,\vartheta^*,\phi^*) >=\frac{1}{2}\int_B\Big(\rho v_i\overline{v_i^*}+C_{ijkl}u_{i,j}\overline{u_{k,l}^*}$$
$$+\frac{\rho c_E}{\theta_0}(\theta+t_2\vartheta)\overline{(\theta^*+t_2\vartheta^*)}+\frac{t_2(t_1-t_2)}{\theta_0}\vartheta\overline{\vartheta^*}+t_1K_{ij}\phi_{,i}\overline{\phi_{,j}^*}\Big)\,d\boldsymbol{x}.$$

Again, this inner product is equivalent to the usual one in $\mathcal{H}_1$.

If we denote $U=(\boldsymbol{u},\boldsymbol{v},\theta,\vartheta,\phi)$, our problem can be written as

$$\frac{dU}{dt}=\mathcal{A}_1U,\quad U(0)=(\boldsymbol{u}^0,\boldsymbol{v}^0,\theta^0,\vartheta^0,\phi^0),$$

where

$$\mathcal{A}_1=\begin{pmatrix}0&\boldsymbol{I}&0&0&0\\ \boldsymbol{A}&0&\boldsymbol{B}&\boldsymbol{C}&0\\ 0&0&0&I&0\\ 0&D&0&E&F\\ 0&0&G&H&J\end{pmatrix}$$

and

$$\begin{aligned}
&\boldsymbol{A}\boldsymbol{u} = \frac{1}{\rho}(C_{ijkl}u_{k,l})_{,j}, \quad \boldsymbol{B}\theta = -\frac{1}{\rho}\beta_{ij}\theta_{,j},\\
&\boldsymbol{C}\vartheta = -\frac{t_1}{\rho}\beta_{ij}\vartheta_{,j}, \quad D\boldsymbol{v} = -\frac{1}{\rho c_E t_2}\beta_{ij}v_{i,j},\\
&E\vartheta = -\frac{\vartheta}{t_2}, \quad F\phi = \frac{1}{\rho c_E t_2}K_{ij}\phi_{,ij},\\
&G\theta = \frac{\theta}{t_1}, \quad H\vartheta = \vartheta, \quad J\phi = \frac{1}{t_1}(mK_{ij}\phi_{,ij} - \phi).
\end{aligned}$$

Theorem 4.1. *The operator $\mathcal{A}_1$ defines a contractive semigroup in $\mathcal{H}_1$.*

Proof. First, we have, for every $U \in \mathcal{D}(\mathcal{A}_1)$,

$$\begin{aligned}
Re < \mathcal{A}_1 U, U >= &-\frac{t_1 - t_2}{2\theta_0}\int_B |\vartheta|^2\, d\boldsymbol{x} - \frac{1}{2}\int_B K_{ij}\phi_{,i}\overline{\phi_{,j}}\, d\boldsymbol{x}\\
&-\frac{m}{2}\int_B |K_{ij}\phi_{,ij}|^2\, d\boldsymbol{x} \le 0.
\end{aligned} \tag{9}$$

Secondly, the domain of the operator $\mathcal{A}_1$ is $(\boldsymbol{u}, \boldsymbol{v}, \theta, \vartheta, \phi) \in \mathcal{H}_1$ such that $\boldsymbol{v} \in \boldsymbol{W}_0^{1,2}$, $\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\theta + \boldsymbol{C}\vartheta \in L^2$ and $\phi \in W_0^{1,2} \cap W^{2,2}$. It is a dense subspace of $\mathcal{H}_1$.

Now, we see that zero belongs to the resolvent of this operator $\mathcal{A}_1$. We consider $F = (\boldsymbol{f}_1, \boldsymbol{f}_2, f_3, f_4, f_5) \in \mathcal{H}_1$. We need to solve the system:

$$\begin{aligned}
&\boldsymbol{v} = \boldsymbol{f}_1, \quad \vartheta = f_3, \quad \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\theta + \boldsymbol{C}\vartheta = \boldsymbol{f}_2,\\
&D\boldsymbol{v} + E\vartheta + F\phi = f_4, \quad G\theta + H\vartheta + J\phi = f_5.
\end{aligned}$$

We have

$$\begin{aligned}
&\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\theta = \boldsymbol{f}_2 - \boldsymbol{C}f_3, \quad F\phi = f_4 - D\boldsymbol{f}_1 - Ef_3,\\
&G\theta + J\phi = f_5 - Hf_3.
\end{aligned}$$

From the second equation we obtain $\phi \in W^{2,2} \cap W_0^{1,2}$. Therefore, we can solve $G\theta = f_5 - Hf_3 - J\phi \in L^2$, and then we can also obtain the solution $\boldsymbol{u} \in \boldsymbol{W}_0^{1,2} \cap \boldsymbol{W}^{2,2}$. It is easy to see that $\|(\boldsymbol{u}, \boldsymbol{v}, \theta, \vartheta, \phi)\| \le C\|F\|$ for a certain positive constant C. So we have proved the existence of solutions.

As in the previous section, from the above theorem we can conclude the following.

Theorem 4.2. *Let $(\boldsymbol{u}^0, \boldsymbol{v}^0, \theta^0, \vartheta^0, \phi^0) \in \mathcal{D}(\mathcal{A}_1)$. Then, there exists a unique solution to the problem determined by (6)-(8).*

4.2. Exponential decay. In this subsection, we prove the exponential decay for the one-dimensional problem when we define again our problem in a spatial finite interval $(0, \ell)$ for $\ell > 0$. Our system becomes now

$$\left.\begin{aligned}
&\rho\ddot{u} = \mu u_{xx} - \beta(\theta_x + t_1\vartheta_x),\\
&\rho c_E t_2\ddot{\theta} = -\theta_0\beta v_x - \rho c_E\vartheta + K\phi_{xx},\\
&t_1\dot{\phi} = \theta + t_1\vartheta - \phi + mK\phi_{xx}
\end{aligned}\right\} \text{ in } (0,\ell)\times(0,\infty).$$

Theorem 4.3. $\mathcal{A}_1$ *generates a semigroup exponentially stable.*

Proof. We are going to follow similar arguments to the ones proposed in the previous section. Let us assume that the imaginary axis is not contained in the resolvent of the operator $\mathcal{A}_1$. Then, there exist two sequences $\lambda_n \to \lambda \neq 0$ and $(u_n, v_n, \theta_n, \vartheta_n, \phi_n) \in \mathcal{D}(\mathcal{A}_1)$, with unit norm, such that

$$\begin{array}{l} i\lambda_n u_n - v_n \to 0 \quad \text{in} \quad W_0^{1,2}, \\ i\lambda_n \rho v_n - \mu u_{n,xx} + \beta(\theta_{n,x} + t_1\vartheta_{n,x}) \to 0 \quad \text{in} \quad L^2, \\ i\lambda_n \theta_n - \vartheta_n \to 0 \quad \text{in} \quad L^2, \\ i\rho c_E t_2 \lambda_n \vartheta_n + \theta_0 \beta v_{n,x} + \rho c_E \vartheta_n - K\phi_{n,xx} \to 0 \quad \text{in} \quad L^2, \\ it_1\lambda\phi_n - \theta_n - t_1\vartheta_n + \phi_n - mK\phi_{n,xx} \to 0 \quad \text{in} \quad W^{1,2}. \end{array}$$

The dissipation inequality (9) implies that $\vartheta_n \to 0$ in L^2 and $\phi_n \to 0$ in $W^{2,2}$. From the last convergence we see that

$$it_1 < \lambda_n\phi_n, \theta_n > -\|\theta_n\|^2 \to 0,$$

and, since

$$< \lambda_n\phi_n, \theta_n >=< \phi_n, \lambda_n\theta_n >\sim< \phi_n, \frac{\vartheta_n}{i} >\to 0,$$

we find that $\theta_n \to 0$ in L^2. Again, as in the previous section we conclude that $\dfrac{v_{n,x}}{\lambda_n} \to 0$ in L^2, and we can finish the proof in a similar way because the imaginary axis is contained in the resolvent.

The asymptotic condition (5) can be proved after repetition of this argument.

Therefore, the theorem is proved.

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