Artificial Delayed Output Twisting Algorithm

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***Abstract*—For uncertain systems with relative degree two, an output feedback based twisting algorithm is proposed. This algorithm ensures practical asymptotic convergence to the origin based on the information of output and its artificially delayed version. We prove the convergence of the proposed algorithm by a continuous, weighted homogeneous and strict Lyapunov function.**

***Index Terms*—Twisting controller, stability and stabilization, time-delay, strict Lyapunov function.**

I. Introduction

FOR UNCERTAIN systems with relative degree two, *Twisting controller*, forces both, the position or its equivalent output variable as well as its derivative to zero simultaneously in finite time. However, it requires the derivative of the output [1]. For having the information of both the output and its derivative, either two sensors are used or an observer or differentiator is used in addition to one sensor. Having more number of sensors can increase the cost of the system [2]. Use of observer and differentiator also come with their own share of problems. The performance of observers are known to be inferior as compared to the physical sensors when the parameters change during the operation of the system. Differentiators are known to be highly sensitive to noise in the output channel [1].

Many systems with time delay which are represented by functional differential equation has application in different areas [3]. Thus, analysis of stability of systems with time delay becomes important [5]. In this brief we propose an artificially delayed output feedback based controller which is structurally similar to the twisting algorithm, but instead of using the information of output and its derivative, uses the information of output and an artificially delayed version of it.

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The stability analysis of systems with time delay mostly considers the linear time-delay models due to the complexities involved [4]. However, in [5] on the introduction of delay performance of the system is boosted.

For the proposed controller which requires the information about output and its delayed values, we obtain the condition on controller gains by the application of standard Lyapunov technique in presence of time-varying disturbance. The condition for the delay has been obtained by using the Lyapunov-Razumikhin approach [6], using the same Lyapunov function that is used for obtaining the controller gains. Apart from time-varying disturbance rejection, the proposed controller can also be used as an observer. Moreover, it can also mitigate the unmatched disturbance for an uncertain second order system.

The rest of this brief is organized as follows. Section II contain the notions and preliminaries used in this brief. The main result with their proof, where we have obtained the gain conditions and condition on delay till which the controller functions, have been presented in Section III. Section IV contain the applications of the proposed controller, which includes the structure of the controller for systems with relative degree two, mitigation of unmatched disturbance and observer design. For better insight an example of state estimation Van der Pole system using the proposed algorithm have been included in Section V. Finally some concluding remarks have been made in Section VI.

II. Notions

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real numbers. The sign function is defined as $\text{sign}(x) := 1$ for $x > 0$, $\text{sign}(x) := -1$ for $x < 0$ and $\text{sign}(x) := [-1, 1]$ for $x = 0$. $C_{[-\tau,0]}$ denotes the Banach space of continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ with $\|\phi\| = sup_{-\tau\leq\zeta\leq0}|\phi(\zeta)|$ where $|.|$ denotes the standard norm. Dilation matrix is given as $\Lambda_r(\lambda) = \text{diag}\{\lambda^{r_i}\}_{i=1}^n$ where $r = [r_1, \ldots, r_n]^T$. For $\phi \in C_{[-\tau,0]}$, $\|\phi\|_r$, represents the homogeneous norm which is defined as: $\|\phi\|_r = (\sum_{i=1}^n \|\phi_i\|^{\frac{\rho}{r_i}})^{\frac{1}{\rho}}$, where, $\rho \geq max_{1\leq i\leq n} r_i$, and $r_i > 0 \quad \forall i = 1 \ldots n$. B_ρ^τ denotes a sphere of radius $\rho > 0$ in $C_{[-\tau,0]}$ and is given by $B_\rho^\tau = \{\phi \in C_{[-\tau,0]} : \|\phi\|_r \leq \rho\}$.

Weighted homogeneous functions and vector field have several elegant properties, we are going to recall a definition of weighted homogeneity for delayed functions as well as vector field, which will be used in the construction of the proposed artificial delayed output twisting algorithm of this note.

Definition 1 [7]: $f : \mathbb{R}^n \to \mathbb{R}$ is said to be weighted homogeneous of degree m if $f(\Lambda_r(\lambda)x) = \lambda^m f(x)$, for any $x \in \mathbb{R}^n$ and for all $\lambda > 0$.

Definition 2 [7]: $f : C_{[-\tau,0]} \to \mathbb{R}$ is called $r-$homogeneous, where $r_i > 0 \quad \forall i = 1 \ldots n$, of degree m if $f(\Lambda_r(\lambda)\phi) = \lambda^m f(\phi)$ for any $\phi \in C_{[-\tau,0]}$ and for all $\lambda > 0$.

Definition 3 [7]: $f : C_{[-\tau,0]} \rightarrow \mathbb{R}^n$ is called $r-$homogeneous, where $r_i > 0 \quad \forall i = 1 \dots n$, of degree m if $f(\Lambda_r(\lambda)\phi) = \lambda^m \Lambda_r(\lambda) f(\phi)$ for any $\phi \in C_{[-\tau,0]}$ and for all $\lambda > 0$.

Now, we recall a result about continuous real-valued homogeneous functions ([8], Lemma 4.2), which will be used in the proof of the main Theorem of this note.

Lemma 1: Suppose V_1 and V_2 are continuous real-valued functions $V_i : \mathbb{R}^n \rightarrow \mathbb{R}$, homogeneous with the same weights and degrees $l_1 > 0$ and $l_2 > 0$, respectively, and V_1 is positive-definite. Then for every $x \in \mathbb{R}^n$,

$$\left[\min_{\{z:V_1(z)=1\}} V_2(z)\right][V_1(x)]^{\frac{l_2}{l_1}} \le V_2(x) \le \left[\max_{\{z:V_1(z)=1\}} V_2(z)\right][V_1(x)]^{\frac{l_2}{l_1}}.$$

Lemma 2 [7]: Let $f : C_{[-\tau,0]} \rightarrow \mathbb{R}^n$ be locally bounded and r-homogeneous with degree d, then there exists $k > 0$ such that $\|f(x)\|_r < k \max_{1\le i\le n} \|x\|_r^{1+d/r_i}$, $\forall\, x \in C_{[-\tau,0]}$.

Lemma 3 [7]: Let $f : C_{[-\tau,0]} \rightarrow \mathbb{R}^n$ be r-homogeneous with degree d and uniformly continuous in B_ρ^τ for some $\rho > 0$, then for any $\eta > 0$ there exists $k > 0$ such that

$$\|F(x) - F(z)\|_r \le \max\{k \max_{1\le i\le n} \|x - z\|_r^{1+d/r_i}, \eta\}, \forall x, z \in B_\rho^\tau.$$

III. Main Results

Let us consider a system $\dot{x}_1 = u + d(t)$, $x_1 \in \mathbb{R}$, $d : \mathbb{R}_+ \rightarrow \mathbb{R}$, where $d(t)$ is perturbations/disturbances, satisfying $|\dot{d}(t)| \le d_0$. Let the proposed controller u be taken as $u := -k_1 \int_0^t \text{sign}(x_1(s))ds - k_2 \int_0^t \text{sign}(\alpha(x_1(s), x_1(s-\tau)))ds$ with the initial condition $x(s) = \phi(s), s \in [-\tau, 0]$ where k_1, k_2 are the positive gains of the controllers, designed later in the manuscript, τ denotes the specified artificial delay and function α is selected such that the artificial delayed closed loop system is weighted homogeneous [7]. In this brief we have taken, $\alpha(x_1(t), x_1(t-\tau))$ as $|x_1|^{0.5}\text{sign}(x_1(t)) - |x_1(t-\tau)|^{0.5}\text{sign}(x_1(t-\tau))$. After substitution of control u, the closed loop system can be written as

$$\dot{x}_1 = x_2, \dot{x_2} = -k_1 \text{sign}(x_1(t)) - k_2 \text{sign}(\alpha(x_1(t), x_1(t-\tau))) + \dot{d}(t). \quad (1)$$

In general, finding gain conditions based on the Lyapunov function for any class of functional differential inclusion is not a straight forward problem. Therefore, we introduce the following time-varying change of state variables of the system (1) to simplify the problem $z_1(t) := \frac{x_1(t)}{L(t)}, z_2(t) := \frac{x_2(t)}{L(t)}$, $L(t) > 0, \ \forall t \ge 0$, in the new co-ordinates, then system (1) is given by

$$\dot{z_1} = -\left(\frac{\dot{L}}{L}\right)z_1 + z_2; \ \dot{z_2} = -\left(\frac{\dot{L}}{L}\right)z_2 - \frac{k_1}{L}\text{sign}(z_1) - \frac{k_2}{L}\text{sign}(\alpha(L(t)z_1(t), L(t-\tau)z_1(t-\tau))) + \frac{\dot{d}}{L}. \quad (2)$$

Normally, an algebraic equivalence of systems (1) and (2) does not preserve the stability properties of a dynamical system. For this, it is necessary and sufficient to have topological equivalence: algebraic equivalence plus the condition $|L(t)| \le p_1$ and $|1/L(t)| \le p_2$ for all $t \ge 0$ where p_1 and p_2 are fixed constants [12]. We are also assuming that derivative of function $L(t)$ is bounded by p_3 for all $t \ge 0$. It is easy to find $L(t)$ which satisfies the above mentioned properties. For example, a logistic function $L(t) = \frac{L_0}{2-\exp(-l_0 t)}$ where L_0 and l_0 are the curve's maximum value and the logistic growth rate of the function, respectively.

The following Theorem gives the practical asymptotic stability of the closed loop system (1) with respect to B_ϵ^τ.

Theorem 1: Consider a $L(t)$ with given properties as mentioned above and assume that the gains are selected such that the following inequality is fulfilled for $k_1 \ge k_2$, $(k_1 - k_2) \ge L(\pi_1 - \frac{2^{\frac{3}{2}}}{3}\pi_2)$, $(k_1 + k_2) \le L(t)(\frac{2^{\frac{3}{2}}}{3}\pi_2 + \pi_1)$ with $\pi_1 \ge \frac{2^2 2^{\frac{5}{6}}}{3^2}\pi_2$ where $\pi_i; i = 1,\ 2$ are positive constants. Then, the origin of the system (1) is practical asymptotically stable with respect to B_ϵ^τ in spite of disturbance $\sup_t |\dot{d}(t)| \le d_0$ for sufficiently small artificial delay $0 < \tau < \tau_0$ where τ_0 is a positive constant.

Proof: The proof has been divided into three parts.

Part I: In this part, the validity of the results of Delay-differential equations for the system (1) has been ascertained. Considering the representation where $x_t(s) = x(t+s)$ and $-\tau \le s \le 0$, equation (1) can be written as

$$\dot{x}_{t,1}(0) = x_{t,2}(0),\ \dot{x}_{t,2}(0) \in -k_1 \text{sign}\big(x_{t,1}(0)(t)\big) - [k_2 - d_0, k_2 + d_0]\text{sign}\big(\alpha\big(x_{t,1}(0), x_{t,1}(-\tau)\big)\big), \quad (3)$$

where the autonomous functional differential inclusion (1) has state $x = [x_1 \ \ x_2]^T \in \mathbb{R}^2$ and the state function $x_t \in C_{[-\tau,0]}$ Let the right hand side of (3) be written as $\dot{x}_t(0) \in F[x(t), x(t-\tau))]$ with $x_0(\tau) = \phi(-\tau), \tau \in [0, \tau_0](\tau_0 > 0)$. The functional $F : C_{[-\tau,0]} \rightarrow \mathbb{R}^2$ is locally bounded and is discontinuous in $x_1 \cup \alpha(x_1, x_1(t-\tau))$. Moreover, the functional F is continuous except for the several points defined by $S_i = \{x_{t,1}(0) | x_{t,1}(0) = 0 \cup \alpha(x_{t,1}(0)(t), x_{t,1}(-\tau)) = 0\} \in C_{[-\tau,0]}$, where $i = 1, \dots, m$ and m denotes number of conditions for which F is discontinuous. Let $S_f = \cup_{i=1}^{i=m} S_i$ denotes all the points where the functional F is discontinuous. A functional following this property is said to be piecewise continuous [9]. The solution of such functionals can be understood in terms of generalized set-valued mapping with respect to the functional F. With the set-valued mapping, define $\mathbf{K}[F](x_t) = \text{co}\{\lim_{i\to\infty} F(x_t^i) | x_t^i \rightarrow x_t, x_t^i \notin S_f\}$, where "co" denotes the convex hull. Further, an equilibrium point of (3) is a point $0 \in C_{[-\tau,0]}$ such that $0 \in \mathbf{K}[F](0)$. Suppose that t_0 is the initial time when the trajectories of (3) hit discontinuous set S_i. It is important to mention here that no trajectory of (3) is going to stay on the discontinuous set S_i for the following cases: $(x_1(t) = 0 \cap x_2(t) \neq 0) \cup (x_1(t-\tau) = 0 \cap x_2(t) \neq 0) \cup (x_1(t) \cap x_1(t-\tau) = 0 \cap x_2(t) \neq 0)$ $\forall t \ge t_0$, except when $x_1(t) = x_2(t) = 0$ if $k_1 > k_2 + d_0$. Therefore, condition on the delay for the asymptotic stability at the origin remains the same as the [7, Lemma 4].

Part II: This part derives the conditions on gain using the direct Lyapunov function. Consider the following Lyapunov function in the new coordinates

$$V(z) = \left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{3}{2}} + \pi_2 z_1 z_2. \quad (4)$$

Applying Young's inequality to the term $\pi_2|z_1||z_2|$, it can be shown that proposed Lyapunov function (4) is bounded from below by zero.

$$\begin{aligned} V(z) &\ge \left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{3}{2}} - \pi_2|z_1||z_2| \\ &\ge \left(\pi_1|z_1|\right)^{\frac{3}{2}} + \left(\frac{1}{2}z_2^2\right)^{\frac{3}{2}} - \pi_2\left(\frac{2}{3}g^{\frac{3}{2}}|z_1|^{\frac{3}{2}} + \frac{1}{3}g^{-3}|z_2|^3\right) \\ &= \left(\pi_1^{\frac{3}{2}} - \frac{2}{3}\pi_2 g^{\frac{3}{2}}\right)|z_1|^{\frac{3}{2}} + \left(\left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{1}{3}\pi_2 g^{-3}\right)|z_2|^3, \end{aligned} \quad (5)$$

where $g > 0$. Now, $V \geq 0$, $\forall$ z if $(\pi_1^{\frac{3}{2}} - \frac{2}{3}\pi_2 g^{\frac{3}{2}}) > 0$ and $((\frac{1}{2})^{\frac{3}{2}} - \frac{1}{3}\pi_2 g^{-3}) > 0$, i.e., $2^{\frac{1}{2}}(\frac{\pi_2}{3})^{\frac{1}{3}} < g < \pi_1(\frac{3}{2\pi_2})^{\frac{2}{3}}$, which implies $(\frac{3}{2\pi_2})^{\frac{2}{3}}\pi_1 \geq 2^{\frac{1}{2}}(\frac{\pi_2}{3})^{\frac{1}{3}}$. thus, $\pi_1 \geq \frac{2^{\frac{1}{2}}2^{\frac{2}{3}}}{3}\pi_2$. Further, selecting g to be the linear combination of $2^{\frac{1}{2}}(\frac{\pi_2}{3})^{\frac{1}{3}}$ and $(\frac{3}{2\pi_2})^{\frac{2}{3}}\pi_1$ will assure that $V \geq 0$ and is a convex function. Thus $g = 2^{\frac{1}{2}}\beta(\frac{\pi_2}{3})^{\frac{1}{3}} + (1-\beta)(\frac{3}{2\pi_2})^{\frac{2}{3}}\pi_1, \quad 0 \leq \beta \leq 1$. Therefore, if we are able to establish $\dot{V} < 0$ in all argument then $V = 0$ is the global minima. Now our next aim is to show $\dot{V} < 0$,

$$\dot{V} = \left\{\frac{3}{2}\big(\pi_1|z_1| + \frac{1}{2}z_2^2\big)^{\frac{1}{2}}\pi_1 \mathrm{sign}(z_1) + \pi_2 z_2\right\}\dot{z_1} + \left\{\frac{3}{2}\Big(\pi_1|z_1| + \frac{1}{2}z_2^2\Big)^{\frac{1}{2}}z_2 + \pi_2 z_1\right\}\dot{z_2}$$

$\dot{V}$ can be rewritten as,

$$\begin{aligned}
\dot{V} &= -W_1(z)\Big(\frac{\dot{L}}{L}\Big) + W_2(z)\Big(\frac{\dot{d}}{L}\Big) - W_3^*(z), \text{ where,}\\
W_1(z) &= \frac{3}{2}\Big(\pi_1|z_1| + \frac{1}{2}z_2^2\Big)^{\frac{1}{2}}\Big(\pi_1|z_1| + z_2^2\Big) + 2\pi_2 z_1 z_2\\
W_2(z) &= \frac{3}{2}\Big(\pi_1|z_1| + \frac{1}{2}z_2^2\Big)^{\frac{1}{2}}z_2 + \pi_2 z_1\\
W_3^*(z) &= \Big(k_1\frac{\pi_2}{L} + k_2\frac{\pi_2}{L}\mathrm{sign}\Big(z_1\chi\Big)\Big)|z_1|\\
&\quad + \frac{3}{2}\Big(\pi_1|z_1| + \frac{1}{2}z_2^2\Big)^{\frac{1}{2}}\Big(\Big(\frac{k_1}{L} - \pi_1\Big)\mathrm{sign}\Big(z_1 z_2\Big)\Big)|z_2|\\
&\quad + \frac{3}{2}\Big(\pi_1|z_1| + \frac{1}{2}z_2^2\Big)^{\frac{1}{2}}\Big(\frac{k_2}{L}\mathrm{sign}\Big(z_2\chi\Big)\Big)|z_2| - \pi_2 z_2^2,\\
&\quad \text{with, } \chi = \alpha\Big(L\Big(t\Big)z_1\Big(t\Big), L\Big(t-\tau\Big)z_1\Big(t-\tau\Big)\Big).
\end{aligned} \tag{6}$$

We are going to show that $W_3^*(z)$ would dominate over $W_2(z)$. Since, $(\pi_1|z_1| + \frac{1}{2}z_2^2)^{\frac{1}{2}} \geq (\frac{1}{2})^{\frac{1}{2}}|z_2|$ (all positive arguments), implies, $\frac{3}{2}(\pi_1|z_1| + \frac{1}{2}z_2^2)^{\frac{1}{2}}|z_2| \geq \frac{3}{2}(\frac{1}{2})^{\frac{1}{2}}z_2^2$, therefore,

$$\pi_2(2)^{\frac{1}{2}}(\pi_1|z_1| + \frac{1}{2}z_2^2)^{\frac{1}{2}}|z_2| \geq -\pi_2 z_2^2 \text{ and } W_3^* \leq W_3', \text{ where,}$$

$$\begin{aligned}
W_3'(z) &= \Big(k_1\frac{\pi_2}{L} + k_2\frac{\pi_2}{L}\mathrm{sign}(z_1\chi)\Big)|z_1| + \frac{3}{2}\Big(\pi_1|z_1| + \frac{1}{2}z_2^2\Big)^{\frac{1}{2}}\\
&\quad\times\left(\frac{2^{\frac{3}{2}}}{3}\pi_2 + \Big(\frac{k_1}{L} - \pi_1\Big)\mathrm{sign}(z_1 z_2) + \frac{k_2}{L}\mathrm{sign}(z_2\chi)\right)|z_2|.
\end{aligned} \tag{7}$$

For $W_3' > 0\ \forall\ z$, both the coefficients of equation (7) should be independently greater than zero, that is,

$$\begin{aligned}
&k_1\frac{\pi_2}{L} + k_2\frac{\pi_2}{L}\mathrm{sign}(z_1\chi) \geq 0,\\
&\frac{2^{\frac{3}{2}}}{3}\pi_2 + \Big(\frac{k_1}{L} - \pi_1\Big)\mathrm{sign}(z_1 z_2) + \frac{k_2}{L}\mathrm{sign}(z_2\chi) \geq 0.
\end{aligned} \tag{8}$$

These two inequalities are satisfied if

$$\begin{aligned}
&k_1 + k_2 \geq 0;\ \ k_1 - k_2 \geq 0,\\
&\frac{2^{\frac{3}{2}}}{3}\pi_2 + \frac{k_1}{L} - \pi_1 \pm \frac{k_2}{L} \geq 0,\ \ \frac{2^{\frac{3}{2}}}{3}\pi_2 - \frac{k_1}{L} + \pi_1 \pm \frac{k_2}{L} \geq 0,
\end{aligned} \tag{9}$$

which can be rewritten as

$$k_1 \geq k_2,\ \ (k_1 + k_2) \leq L\left(\frac{2^{\frac{3}{2}}}{3}\pi_2 + \pi_1\right),$$

$$(k_1 - k_2) \geq L\left(\pi_1 - \frac{2^{\frac{3}{2}}}{3}\pi_2\right). \tag{10}$$

Now, $W_3'(z) \geq W_3^f(z) \triangleq A|z_1| + B(\pi_1|z_1| + \frac{1}{2}z_2^2)^{\frac{1}{2}}|z_2|$, since, $A = \min_z[k_1\frac{\pi_2}{L} + k_2\frac{\pi_2}{L}\mathrm{sign}(z_1\chi)] \geq 0$ $B = \min_z[\frac{2^{\frac{3}{2}}}{3}\pi_2 + (\frac{k_1}{L} - \pi_1)\mathrm{sign}(z_1 z_2) + \frac{k_2}{L}\mathrm{sign}(z_2\chi)] \geq 0$, Thus, $W_3^f(z)$ is a continuous and homogeneous positive definite function. According to Lemma 1, it follows that $\forall\ z \in \mathbb{R}^2$, $W_2(z) \leq \gamma W_3^f(z)$ is satisfied, with $\gamma = \max_{\{z: W_3^f(z)=1\}} > 0$, because both $W_2(z)$ and $W_3^f(z)$ are continuous and homogeneous with same weights and degree. Finally,

$$\begin{aligned}
W_1(z) &= \frac{3}{2}\Big(\pi_1|z_1| + \frac{1}{2}z_2^2\Big)^{\frac{1}{2}}\Big(\pi_1|z_1| + z_2^2\Big) + 2\pi_2 z_1 z_2\\
&\geq \frac{3}{2}(\pi_1|z_1|)^{\frac{1}{2}}\pi_1|z_1| + \frac{3}{2}\Big(\frac{1}{2}z_2^2\Big)^{\frac{1}{2}}z_2^2 - 2\pi_2\Big(\frac{2}{3}g^{\frac{3}{2}}|z_1|^{\frac{3}{2}} + \frac{1}{3}g^{-3}|z_2|^3\Big)\\
&= \Big(\frac{3}{2}\pi_1^{\frac{3}{2}} - \frac{4}{3}\pi_2 g^{\frac{3}{2}}\Big)|z_1|^{\frac{3}{2}} + \Big(\frac{3}{2^{\frac{3}{2}}} - \frac{2\pi_2}{3}g^{-3}\Big)|z_2|^3
\end{aligned} \tag{11}$$

Thus, $W_1(z)$ is positive-definite if $\frac{2^{\frac{5}{6}}}{3^{\frac{2}{3}}}\pi_2^{\frac{1}{3}} < g < \frac{3^{\frac{4}{3}}}{2^2}\frac{\pi_1}{\pi_2^{\frac{2}{3}}}$ and such a value of g exists if $\pi_1 > \frac{2^{\frac{5}{6}}2^2}{3^2}\pi_2$. It can be noted that $\pi_1 > \frac{2^{\frac{5}{6}}2^2}{3^2}\pi_2$ also fulfills $\pi_1 \geq \frac{2^{\frac{1}{2}}2^{\frac{2}{3}}}{3}\pi_2$ required for $V \geq 0$. Thus, we have obtained the condition for gains.

Part III: In this part, the condition on delay is calculated. It is based on the fundamental result developed in [7].

Let $x_0 \in B_\rho^\tau$ for some $\rho > 0$ and $\tau > 0$, then

$$x(t, x_0) = x_0(0) + \int_0^t F[x(s, x_0), x(s-\tau, x_0)]ds \tag{12}$$

denotes the solution of dynamics represented by (1) on the interval [0, t] $\forall$ t ≥ 0, Using the relation between standard norm for homogeneous norm, $\underline{\sigma}_r(|x|_r) \leq |x| \leq \overline{\sigma}_r(|x|_r)$, and $\underline{\rho}_r(\|\phi\|_r) \leq \|\phi\| \leq \overline{\rho}_r(\|\phi\|_r)$, where $\overline{\rho}_r, \overline{\sigma}_r, \underline{\rho}_r, \underline{\sigma}_r$ are radially unbounded and attains a value zero at 0, we get, $\|x_0\| \leq 2\overline{\rho}_r(\rho)$. For all $\|\phi\| \leq 2\overline{\rho}_r(\rho)$ according to Lemma 2, there exists $k > 0$ for almost everywhere such that $|F[\phi(0), \phi(-\tau)]|_r \leq k\max_{1\leq i\leq n}\|\phi\|_r^{1+q/r_i}$, from Lemma 3

$$\begin{aligned}
&|F[\phi(0), \phi(-\tau)]| \leq \overline{\sigma}_r(|F[\phi(0), \phi(-\tau)]|_r)\\
&\quad \leq \overline{\sigma}_r(k\max_{1\leq j\leq n}\|\phi\|_r^{1+q/r_i}) \leq \overline{\sigma}_r(k\max_{1\leq j\leq n}[\underline{\rho}_r^{-1}(\|\phi\|)]^{1+q/r_i}) = \iota_k(\|\phi\|)
\end{aligned}$$

where $\iota_k(s) = \overline{\sigma}_r(k\max_{1\leq j\leq n}[\underline{\rho}_r^{-1}(s)]^{1+q/r_i})$ is a function of class κ_∞ for $q > -\min_{1\leq i\leq n} r_i$. Now, let's first consider the case when $t \in (0, \tau]$, selecting $0 < \tau < \frac{\overline{\rho}_r(\rho)}{\iota_k(2\overline{\rho}_r(\rho))}$, ensures that $\|x(t, x_0\| \leq 2\overline{\rho}_r(\rho)$ for $t \in (0, \tau]$. Now lets suppose $\|x_s\| < 2\overline{\rho}_r(\rho)$ for all $s \in (0, t')$for some $t' \leq \tau$ and $\|x_s\| \geq 2\overline{\rho}_r(\rho)$ for $s \in [t', \tau]$, then from (12),

$$\begin{aligned}
\Big|x\Big(t', x_0\Big)\Big| &\leq |x_0(0)| + \int_0^{t'}|F[x(s, x_0), x(s-\tau, x_0)]|ds\\
&\leq |x_0(0)| + \int_0^{t'}\iota_k(\|x_s\|)ds \leq |x_0(0)| + t'\ \underset{0\leq s<t'}{sup}\ \iota_k(\|x_s\|)\\
&< \overline{\rho}_r(\rho) + t'\iota_k(2\overline{\rho}_r(\rho)) < 2\overline{\rho}_r(\rho)
\end{aligned} \tag{13}$$

This is a contradiction, thus $\|x_s\| \leq 2\overline{\rho}_r(\rho)$ for $s \in (0, \tau]$. Hence for any $\rho > 0$ there exists $\tau > 0$ such that when the initial point lies within B^{τ}_{ρ} then the trajectory $x_t \in B^{\tau}_{\rho'}$, for $t \in (0, \tau]$, where $\rho' = \underline{\rho}_r^{-1}[2\overline{\rho}_r(\rho)]$.

For a weighted Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$, of degree $v > -q$ the following quantities are defined

$$a = -\sup_{\zeta \in \mathbb{S}_r} \frac{\partial V}{\partial \zeta} F(\zeta, \zeta) > 0,\ 0 < b = \sup_{|\zeta|_r \le 1} |\frac{\partial V(\zeta)}{\partial \zeta}| < +\infty$$
$$c_1 |x|_r^v \le V(x) \le c_2 |x|_r^v \quad \forall x \in \mathbb{R}^n. \tag{14}$$

Let us also define $\lambda_1 = |\phi(0)|_r$ and $\lambda_2 = \|\phi\|_r$, where $\phi(0) = \Lambda_r(\lambda_1)\zeta$ for some $\zeta \in \mathbb{S}_r$ and $\phi = \Lambda_r(\lambda_2)\psi$ for some $\psi \in S_r \subset C_{[-\tau,0]}$ and $|\psi(0)|_r \le |\zeta|_r = 1$. Thus, the derivative in equation (14) calculated for system (3) is given as, $\frac{\partial V(\phi(0))}{\partial \phi(0)} F[\phi(0), \phi(-\tau)] = \frac{\partial V(\phi(0))}{\partial \phi(0)} F[\phi(0), \phi(0)] + \frac{\partial V(\phi(0))}{\partial \phi(0)} \{F[\phi(0), \phi(-\tau)] - F[\phi(0), \phi(0)]\} = \lambda_1^{q+v} \frac{\partial V}{\partial \zeta} F[\zeta, \zeta] + \lambda_2^{q+v} \frac{\partial V(\psi(0))}{\partial \psi(0)} \{F[\psi(0), \psi(-\tau)] - F[\psi(0), \psi(0)]\}$. By Lemma 3 for all $\psi \in S_r$ and for some $\eta > 0$, there exists $G > 0$ such that,

$$|F[\psi(0), \psi(-\tau)] - F[\psi(0), \psi(0)]|_r \le \max\{G \max_{1 \le i \le n} |\psi(0) - \psi(-\tau)|_r^{1+q/r_i}, \eta\}. \tag{15}$$

Now, we consider the case of $t \ge \tau$, for the trajectory F, there exists $M > 0$ such that $|\phi(0) - \phi(-\tau)| \le M\tau$, where, $M = sup_{|z|_r \le \rho', |y|_r \le \rho'} |F[z, y]|$; $\|\phi\|_r |\psi(0) - \psi(-\tau)|_r = |\phi(0) - \phi(-\tau)|_r \le \underline{\sigma}_r^{-1}(M\tau)$. So, equation (15) becomes, $|F[\psi(0), \psi(-\tau)] - F[\psi(0), \psi(0)]| \le \overline{\sigma}_r \circ \max\{G \max_{1 \le i \le n} \|\phi\|_r^{-1-q/r_i} \underline{\sigma}_r^{-1}(M\tau)^{1+q/r_i}, \eta\}$. Finally, on solution for $t \ge \tau$, for $\mu_1(\tau) = G \max_{1 \le i \le n} \underline{\sigma}_r^{-1}(M\tau)^{1+q/r_i}$ and

$$\mu_2(\lambda_2) = \begin{cases} \lambda_2^{-1-q/\min_{1 \le i \le n} r_i} & if \quad \lambda_2 \ge 1 \\ \lambda_2^{-1-q/\max_{1 \le i \le n} r_i} & if \quad \lambda_2 < 1 \end{cases}$$

$|F[\psi(0), \psi(-\tau)] - F[\psi(0), \psi(0)]| \le \overline{\sigma}_r \circ \max\{\mu_1(\tau)\mu_2(\lambda_2), \eta\}$.

For applying the Lyapunov-Razumikhin method to prove asymptotic stability, we assume some $\gamma > 1$, such that, $c_1 \, sup_{\theta \in [-\tau,0]} |\phi(\theta)|_r^v \le sup_{\theta \in [-\tau,0]} V[\phi(\theta)] < \gamma V[\phi(0)] \le \gamma c_2 |\phi(0)|_r^v$, which implies,

$$\sup_{\theta \in [-\tau,0]} |\phi(\theta)|_r \le (\gamma c_1^{-1} c_2)^{\frac{1}{v}} |\phi(0)|_r = (\gamma c_1^{-1} c_2)^{\frac{1}{v}} \lambda_1$$
$$\text{so, } \lambda_2 = \|\phi\|_r = \left(\sum_{i=1}^n \sup_{\theta \in [-\tau,0]} |\phi_i(\theta)|^{\frac{\rho}{r_i}}\right)^{\frac{1}{\rho}}$$
$$\le \left(\sum_{i=1}^n \sup_{\theta \in [-\tau,0]} |\phi(\theta)|_r^{\rho}\right)^{\frac{1}{\rho}} = n^{\frac{1}{\rho}} \sup_{\theta \in [-\tau,0]} |\phi(\theta)|_r \le R\lambda_1$$

where $R = n^{\frac{1}{\rho}} (\gamma c_1^{-1} c_2)^{\frac{1}{v}}$.

Therefore,

$$\frac{\partial V(\phi(0))}{\partial \phi(0)} F[\phi(0), \phi(-\tau)]$$
$$\le b\overline{\sigma}_r \circ \max\{\mu_1(\tau)\mu_2(\lambda_2), \eta\}\lambda_2^{q+v} - aR^{-q-v}\lambda_2^{q+v}$$
$$= \|\phi\|_r^{q+v}\big(b\overline{\sigma}_r \circ \max\{\mu_1(\tau)\mu_2(\|\phi\|_r), \eta\} - aR^{-q-v}\big).$$

Now select $\eta = \frac{\overline{\sigma}_r^{-1}(aR^{-q-v} - \epsilon)}{b}$ and $\rho > \epsilon > 0$ then for some $\epsilon \in (0, aR^{-q-v})$ and for all $\tau \in (0, \tau_0]$ where,

$$\tau_0 = min\left\{\frac{\overline{\rho}_r(\rho)}{\iota_k 2\overline{\rho}_r(\rho)}, \frac{1}{\mu_2(\epsilon)}\mu_1^{-1}\Big(\frac{\overline{\sigma}_r^{-1}\big(aR^{-q-v} - \epsilon\big)}{b}\Big)\right\}, \tag{16}$$

it can be written that $max_{\theta \in [-\tau,0]} V[\phi(\theta)] < \gamma V[\phi(0)]$, which implies, $\frac{\partial V(\phi(0))}{\partial \phi(0)} F[\phi(0), \phi(-\tau)] \le -\epsilon |\phi(0)|_r^{q+v}$ for all solution $\phi \in B^{\tau}_{\rho'} \setminus B^{\tau}_{\epsilon}$ of (3) for $t \ge \tau$, i.e., for $t \ge \tau$ the trajectory of (3) are decreasing to B^{τ}_{ϵ}. Hence, we have proved the asymptotic stability for $t \ge \tau$, while stability for $t \le \tau$ can be seen from (13). Hence, this gives asymptotic stability with respect to B^{τ}_{ϵ} in B^{τ}_{ρ} for any delay $0 < \tau \le \tau_0$ where τ_0 given by (16). This completes the proof.

Calculation of τ_0: From the Lyapunov function (4) and given conditions in (14), one can calculate the value of parameters $a = 58.5, b = 3.535, c_1 = 0.744$ and $c_2 = 2.834$ by choosing $\pi_1 = \pi_2 = 1$, $L(t) = \frac{2}{2-\exp(-t)}$, $k_1 = 4$, $k_2 = 3$, $d_0 = 1$ and the initial condition of states as $0.5 \le |x(0)|_r \le 5$. In the closed loop system (1),for $\alpha(x_1(t), x_1(t-\tau)) = |x_1|^{0.5}\text{sign}(x_1(t)) - |x_1(t-\tau)|^{0.5}\text{sign}(x_1(t-\tau))$, $q = -1$ and for the choosen Lyapunov function, $v = 3$. The other parameters for the calculation of maximum value of delay τ_0 are obtained as $\underline{\sigma}_r(s) = 2^{-1/2}s$, $\overline{\sigma}_r(s) = s$, $\underline{\rho}_r^{-1}(s) = (\sum_{i=1}^n s^{\frac{4}{r_i}})^{\frac{1}{4}}$, $\overline{\rho}_r(s) = (\sum_{i=1}^n s^{2r_i})^{\frac{1}{2}}$, $\rho' = \underline{\rho}_r^{-1}[2\overline{\rho}_r(\rho)] = 9.431$ for $r = [2 \quad 1]$, $M = 12.367$, $R = 2.784$ (for $\gamma = 2$ and $\rho = 2$). Therefore, for $\epsilon = 5$, the delay condition can be obtained from equation (16) as $\tau_0 = \min\{1.456, 2.612\} = 1.456$. ■

IV. Applications

In this section, we demonstrate that the proposed artificial delayed output twisting algorithm works as a controller as well as an observer for uncertain second order systems. Full state feedback based twisting algorithm [1] is only able to mitigate the matched uncertainty/disturbance. However, we are going to show that artificial delayed output feedback twisting controller is also able to mitigate the unmatched disturbance which is an extra benefit of the proposed algorithm over the full state information based twisting controller. Another benefit comes into picture in terms of observer design, whereas the classical twisting [1] works only like a controller.

A. Controller Design for Second Order System

For the system, $\ddot{x} = u + d(t)$, with relative degree 2 with respect to output x, consider the controller $u := -k_1\text{sign}(x(t)) - k_2\text{sign}(\alpha(x(t), x(t-\tau)))$, where $d(t)$ is the bounded matched disturbance. The system under consideration can be transformed into

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = -k_1\text{sign}(x_1(t)) - k_2\text{sign}(\alpha(x_1(t), x_1(t-\tau))) + d(t) \tag{17}$$

taking $x_1 = x$ and $x_2 = \dot{x}$. It can be noted that the control signal would be discontinuous in this case.

Remark 1: System (17) is structurally similar to the twisting controller given in [1]. Since the controller used in system (17) requires the information of present state and its delayed version, the proposed controller is named as “Artificial Delayed Output Twisting algorithm”.

B. Mitigation of Unmatched Disturbance

Consider the following system, $\dot{x}_1 = x_2 + d_1(t); \dot{x}_2 = u + d_2(t)$, where u, $d_1(t)$ and $d_2(t)$ are control, unmatched and matched perturbations respectively. Assume that $d_1(t)$ is Lipschitz continuous and $d_2(t)$ is bounded. Suppose that first; we want to design full state feedback based twisting controller by converting unmatched disturbance into matched disturbance as follows: $\dot{x}_1 = p(t); \dot{p}(t) = u + \dot{d}_1(t) + d_2(t)$, where $p(t) := x_2(t)+d_1(t)$. Now, one can design classical twisting [1] as $u := -k_1\text{sign}(x_1) - k_2\text{sign}(p)$ with $k_1 > k_2 + |\dot{d}_1(t) + d_2(t)|$. However, this controller is not practically feasible because it requires the explicit information of disturbance $d_1(t)$. Next, we are going to suggest a solution which is based on the proposed delayed output twisting algorithm.

Let us take a sliding surface, $S := x_2 + k_1 \int_0^t \text{sign}(x_1(s))ds + k_2 \int_0^t \text{sign}(\alpha(x_1(s), x_1(s-\tau)))ds$, such that,

$$\dot{S} = u + d_2 + k_1\text{sign}(x_1(t)) + k_2\text{sign}(\alpha(x_1(t), x_1(t-\tau))). \quad (18)$$

Let us take the controller $u := -k_1\text{sign}(x_1(t)) - k_2\text{sign}(\alpha(x_1(t), x_1(t-\tau))) - K\text{sign}(S)$, so (18) becomes $\dot{S} = -K\text{sign}(S) + d_2$, which is finite time stable when $K > |d_2|$ and hence, S converges to 0. Thus, we get, $x_2 = -k_1 \int_0^t \text{sign}(x_1(s))ds - k_2 \int_0^t \text{sign}(\alpha(x_1(s), x_1(s-\tau)))ds$, which can be re-written as $\dot{x}_1 = z, \dot{z} = -k_1\text{sign}(x_1(t)) - k_2\text{sign}(\alpha(x_1(t), x_1(t-\tau))) + \dot{d}_1(t)$, which is similar to that of (1). By Theorem 1, the above system is asymptotically stable which in turn proves that the proposed controller mitigates unmatched disturbance.

C. Observer Design

Let us consider a system $\dot{x}_1 = x_2, \dot{x}_2 = u + d(t)$, which is mimicked as $\dot{\hat{x}}_1 := \hat{x}_2\ \dot{\hat{x}}_2 := u + L_2$, where, L_2, u and $d(t)$ are correction term, control input and bounded perturbation respectively. The estimation errors are $e_1 := \hat{x}_1 - x_1$, $e_2 := \hat{x}_2 - x_2$. The closed loop system in terms of observer's estimation errors $e^T = [e_1 \quad e_2]$ is $\dot{e}_1 = e_2, \dot{e}_2 = L_2 - d(t)$. On choosing $L_2 := -k_1\text{sign}(e_1(t)) - k_2\text{sign}(\alpha(e_1(t), e_1(t-\tau)))$, the error dynamics become structurally similar to system (17) and hence, asymptotically converges to zero by Theorem 1. Thus, the states are estimated.

V. Example

The proposed algorithm as an observer is illustrated by using following forced van der pole equation: $\dot{x}_1 = \epsilon h(x_1) + x_2$, $\dot{x}_2 = -x_1 + F(x, t)$ where $h(x_1) = x_1 - x_1^3/3$, $\epsilon > 0$ and $\|F(x, t)\| \le \delta$ is uncertain and bounded forcing term. For measured x_1 value, the proposed artificial delay based observer given as: $\dot{\hat{x}}_1 := \epsilon h(x_1) + \hat{x}_2, \dot{\hat{x}}_2 := -x_1 + C$, can be used to estimate x_2. The estimation errors are $e_1 := \hat{x}_1 - x_1, e_2 := \hat{x}_2 - x_2$, then the estimation error dynamics are given as:

$$\dot{e}_1 = e_2, \quad \dot{e}_2 = C - F(x, t). \quad (19)$$

The system (19) becomes structurally similar to proposed system (17) for $C := -k_1\text{sign}(e_1(t)) - k_2\text{sign}(\alpha(e_1(t), e_1(t-\tau)))$, to the delayed static output feedback control system with hyper-exponential convergence [10] for $C := -(k_3 + k_4)|x_1|^\beta\text{sign}(x_1(t)) + k_4|x_1(t-\tau)|^\beta\text{sign}(x_1(t-\tau))$ and to the delayed static output feedback control system [11] for $C := -k_5x_1 + k_6x_1(t-\tau)$. For simulations, $F(x, t) = \sin(2t)$, $\epsilon = 1$, $k_1 = 2, k_2 = 1$, $\tau_1 = 0.01$, $k_3 = 0.25$, $k_4 = 0.1$, $\beta = 0.8$, $\tau_2 = 0.3$, $k_5 = 0.35$, $k_6 = 0.1$ and $\tau_3 = 0.3$, $x_1(0) = 2$ and $x_2(0) = 1$ are taken. The Figure 1 shows that the proposed algorithm based estimator is able to estimate x_2 with bounded error whereas the other delayed output feedback based estimators result in large error in estimation.

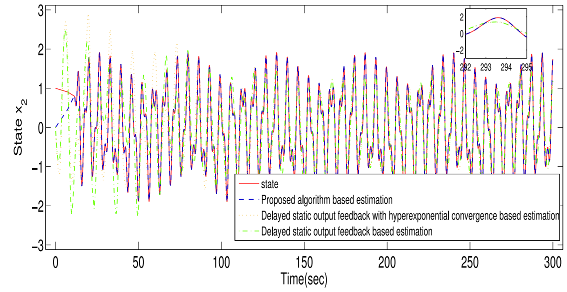

Fig. 1. Evolution of x_2 and its estimation for van der pole system using proposed observer, hyperexponential convergence delayed static output feedback based observer and delayed static output feedback based observer.

VI. Conclusion

In this brief, a controller structurally similar to the twisting controller has been proposed which, for a system with relative degree two, needs the information about one of the states and its artificially delayed version. The condition on the controller gains are obtained for stable operation. The maximum delay upto which the controller remains stable has been obtained by the application of Lyapunov-Razumikhin approach. An example of state estimation of Van der Pole system using the proposed control technique has been presented and compared with two recently proposed delayed output feedback based control techniques.

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