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Generalized-Hukuhara subgradient and its application in optimization problem with interval-valued functions

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Abstract. In this article, the concepts of *gH*-subgradient and *gH*-subdifferential of interval-valued functions are illustrated. Several important characteristics of the *gH*-subdifferential of a convex interval-valued function, e.g., closeness, boundedness, chain rule, etc. are studied. Alongside, we prove that *gH*-subdifferential of a *gH*-differentiable convex interval-valued function contains only the *gH*-gradient. It is observed that the directional *gH*-derivative of a convex interval-valued function is the maximum of all the products between *gH*-subgradients and the direction. Importantly, we prove that a convex interval-valued function is *gH*-Lipschitz continuous if it has *gH*-subgradients at each point in its domain. Furthermore, relations between efficient solutions of an optimization problem with interval-valued function and its *gH*-subgradients are derived.

Keywords. Convex programming; *gH*-subgradient; *gH*-subdifferential; interval optimization problems.

1. Introduction

In real-life decision-making processes, we often face the optimization problem with nonsmooth functions. To deal with optimization problems with nonsmooth functions, the concepts of subgradient and subdifferential inevitably arise. Due to the inexact and imprecise nature of many real-world occurrences the study of Interval-Valued Functions (IVFs) and optimization problems with IVFs, known as Interval Optimization Problems (IOPs), become substantial topics to the researchers. In this article, we illustrate the concepts of subgradient and subdifferential for IVFs and study several important characteristics of subgradient and subdifferential of IVFs. We also study the optimality conditions for nonsmooth IOPs. As intervals are the inextricable things in IVFs and IOPs, before making a survey on IVFs and IOPs, we make a short survey on the arithmetic and ordering of intervals

1.1 *Literature survey*

In the literature of IVFs, to deal with compact intervals and IVFs, Moore [1] introduced interval arithmetic. There are a few limitations (see [2], for details) of Moore's interval arithmetic; especially, Moore's interval arithmetic cannot provide the additive inverse of a nondegenerate interval (interval with unequal limits). For this inefficiency of Moore's arithmetic, Hukuhara [3] proposed 'Hukuhara difference' of intervals. Although the Hukuhara difference provides the additive inverse of any compact interval, it is not applicable between all pairs of compact intervals (see [2], for details). To overcome this difficulty, a new rule for the subtraction of intervals, i.e, 'nonstandard subtraction', was introduced by Markov [4] and named as 'generalized Hukuhara difference (*gH*-difference)' by Stefanini [5]. The *gH*-difference has the property of providing additive inverse of any compact interval.

Unlike real numbers, any linear ordering for intervals is still undeveloped. Isibuchi and Tanaka [6] suggested a few partial ordering relations of intervals. In [7], some ordering relations with the help of parametric representation of intervals are proposed. Another ordering relation is provided in [8] by a bijective function from the set of intervals to $\mathbb{R}^2$. However, all the ordering relations of [7, 8] can be derived from the ordering relations of [6]. The concept of variable ordering relation of intervals is introduced in [9].

Calculus is one of the most important tools for optimization. Therefore, alike to the real-valued and vector-

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valued functions, the development of the calculus for IVFs is essential to study extrema of IVFs. Towards this endeavor, the definition of differentiability of IVFs was initially introduced by Hukuhara [3] with the help of Hukuhara difference of intervals. However, this definition of Hukuhara differentiability is restrictive [10]. Based on *gH*-difference, the notions of *gH*-derivative, *gH*-partial derivative, *gH*-gradient, and *gH*-differentiability for IVFs are provided in [4, 11–15]. Lupulescu studied the differentiability and the integrability for the IVFs on time scales in [16] and developed the fractional calculus for IVFs in [17]. The concept of directional *gH*-derivative for IVF is depicted in [15, 18]. Ghosh *et al* [2] have introduced the concepts of *gH*-Gâteaux derivative, and *gH*-Fréchet derivative of IVFs. Recently, the idea of *gH*-Clarke derivative is proposed by Chauhan *et al* [19].

Based on the existing ordering relations of intervals and calculus of IVFs, many authors developed the theories to characterize the solutions to IOPs. For instance, using the concept of Hukuhara differentiability, Wu [20] proposed KKT conditions for IOPs. In [21], Wu presented the solution concepts of IOPs with the help of bi-objective optimization. Also, Wu reported some duality conditions for IOPs in [22, 23]. Using the concept of *gH*-differentiability, Chalco-Cano *et al* [11] presented KKT conditions for IOPs. Ghosh *et al* [24] developed generalized KKT conditions to obtain the solution of the IOPs. Further, Ghosh developed a Newton method [12] and a quasi-Newton method [25] to solve IOPs. The optimality conditions for IOPs using the concepts of directional *gH*-derivative and total *gH*-derivative of interval-valued objective functions are depicted by Stefanini *et al* in [15], and using the concepts of *gH*-Gâteaux derivative of interval-valued objective functions are depicted by Ghosh *et al* in [2]. Also, Chauhan *et al* [19] proposed the optimality conditions for IOPs using *gH*-Clarke derivative of interval-valued objective function.

The authors of [26–30] proposed various optimality and duality conditions for nonsmooth IOPs by converting them into real-valued multiobjective optimization. However, in this approach, one needs the closed-form of boundary functions of the interval-valued objective and constrained functions, which is practically difficult; even for a very simple IVF $\mathbf{T}(p_1, p_2) = \frac{[-1,6]\odot p_1 \oplus [3,5]\odot p_2}{[-2,7]\odot p_1 \oplus [-4,0]\odot p_2}$ for all $(p_1, p_2) \in \mathbb{R}^2$, the closed forms of the lower boundary function $\underline{t}$ and upper boundary function $\overline{t}$ of $\mathbf{T}$ are not easy to obtain. Apart from these, based on parametric representations of the IVFs, some authors [7, 12, 25] studied IOPs and developed theories to obtain the solutions to IOPs by converting them into real-valued optimization problems. The authors of [31] proposed some optimality conditions and duality results of a nonsmooth convex IOP using the parametric representation of its interval-valued objective and constrained functions. However, the parametric process is also practically difficult since, in the parametric process, the number of variables increases with the number of intervals involved in the IVFs, and to verify any property of an IVF one has to verify it for an infinite number of its corresponding real-valued function. For instance, see definition 9 in [31]. Furthermore, using the parametric representation of IVFs and converting an IOP into a real-valued optimization problem, one can obtain only one solution to the IOP. Whereas, an IOP may have infinite solutions (see example 6) of the present article.

1.2 *Motivation and contribution*

From the literature of IVFs and IOPs, it is observed that the concepts of subgradient and subdifferentials for IVFs are yet to be introduced deeply. However, the authors of [32] proposed the concepts of subgradient and subdifferentials for *n*-cell convex fuzzy-valued functions (FVFs) and proved that the subdifferentials of convex FVFs are convex. However, other important properties of subgradient and subdifferentials of FVFs, such as closeness, boundedness, chain rule, etc. of subdifferentials are not found in [32]. As IVFs are the special case of FVFs, in this article, adopting the concept of subgradient for convex FVFs of the article [32] we define subgradient of convex IVFs (namely *gH*-subgradient). Thereafter, we illustrate the concept of subgradient for convex IVFs in terms of linear IVFs. Subsequently, we define the subdifferential of convex IVFs (namely *gH*-subdifferential) and study its various important properties. We prove that *gH*-subdifferentials of convex IVFs are closed and bounded sets. In order to prove these properties, the norm on the set of *gH*-continuous bounded linear IVFs is defined and the idea of sequences with their convergence on the set of *n*-tuple of compact intervals is described. Although the author of [33] provided the concept of subgradients for IVFs in terms of linear functions, our concept is more general (please see remark 7 of this article for details).

In this article, along with the aforementioned properties of *gH*-subdifferentials, several important characteristics of *gH*-subgradients are also studied in this article. Interestingly, it is observed that a convex IVF is *gH*-Lipschitz continuous if it has *gH*-subgradients at each point in its domain. It is reported that for a convex IVF, the directional *gH*-derivative is the maximum of the products of the *gH*-subgradients and the concerning direction. The chain rule of a convex IVF and the *gH*-subgradient of the sum of finite numbers of convex IVFs are illustrated. Also, some optimality conditions of nonsmooth convex IOP *without applying the parametric approach* are explored in this article. Most importantly, it is to mention that all the proposed definitions and the results of this article are applicable to all the IVFs regardless of whether or not

(i) their parametric representations can be found, or
(ii) the explicit form of their lower and upper boundary functions are readily available.

1.3 *Delineation*

The article is demonstrated as follows. The next section deals with prerequisites of interval analysis and calculus of IVFs. The notions of *gH*-subgradients and *gH*-subdifferentials of IVFs with their several important characteristics are illustrated in section 3. It is shown that the *gH*-subdifferential of a convex IVF is closed and bounded. It is proved that a *gH*-differentiable convex IVF has only one *gH*-subgradient. It is also observed that the directional *gH*-derivative of a convex IVF in each direction is the maximum of all the products of *gH*-subgradients and the direction. Further in section 3, it is shown that a convex IVF is *gH*-Lipschitz continuous if it has *gH*-subgradients at each point in its domain. Apart from these, the chain rule of a convex IVF and the *gH*-subgradient of the sum of finite numbers of convex IVFs are illustrated. The relations between efficient solutions of an IOP with *gH*-subgradients of its objective function are derived in section 4. Finally, the last section is concerned with a few future directions for this study.

2. Preliminaries and terminologies

In this section we discuss a few basic notions on intervals. Thereafter, we describe the convexity and calculus of IVFs. The ideas and notations that we describe in this section are used throughout the paper. We denote

- $\mathbb{R}$ as the set of real numbers
- $\mathbb{R}_+$ as the set of positive real numbers
- $\mathbb{R}^n$ as the Euclidean space
- $I(\mathbb{R})$ as the set of all compact intervals
- $\mathcal{S}$ as a nonempty subset of $\mathbb{R}^n$
- $\mathcal{X}$ as a nonempty linear subset of $\mathbb{R}^n$
- $\widehat{\mathcal{X}}$ as the set of all *gH*-continuous linear IVF on $\mathcal{X}$

2.1 *Interval arithmetic, dominance relation and sequence of intervals*

We represent an interval $\mathbf{A} \in I(\mathbb{R})$ in the following way

$$\mathbf{A} = [\underline{a}, \overline{a}].$$

Also, we represent a singleton set $\{x\}$ of $\mathbb{R}$ by the interval $\mathbf{X} = [\underline{x}, \overline{x}]$ with $\underline{x} = x = \overline{x}$. As for example, $\mathbf{0} = \{0\} = [0, 0]$.

In this article, along with the Moore's interval addition ($\oplus$), substraction ($\ominus$), multiplication ($\odot$), and division ($\oslash$) [1]:

$$\begin{aligned}
\mathbf{U} \oplus \mathbf{V} =& [\underline{u} + \underline{v}, \overline{u} + \overline{v}], \mathbf{U} \ominus \mathbf{V} = [\underline{u} - \overline{v}, \overline{u} - \underline{v}], \\
\mathbf{U} \odot \mathbf{V} =& [\min\{\underline{u}\underline{v}, \underline{u}\overline{v}, \overline{u}\underline{v}, \overline{u}\overline{v}\}, \max\{\underline{u}\underline{v}, \underline{u}\overline{v}, \overline{u}\underline{v}, \overline{u}\overline{v}\}], \\
\mathbf{U} \oslash \mathbf{V} =& [\min\{\underline{u}/\underline{v}, \underline{u}/\overline{v}, \overline{u}/\underline{v}, \overline{u}/\overline{v}\}, \max\{\underline{u}/\underline{v}, \underline{u}/\overline{v}, \overline{u}/\underline{v}, \overline{u}/\overline{v}\}], \\
& \text{provided } 0 \notin \mathbf{V},
\end{aligned}$$

we use *gH*-difference ($\ominus_{gH}$) of intervals because $\mathbf{U} \ominus \mathbf{U} \neq \mathbf{0}$ for a nondegenerate interval $\mathbf{U}$. The *gH*-difference [4, 5] of the interval $\mathbf{V}$ from the interval $\mathbf{U}$ is defined by

$$\mathbf{U} \ominus_{gH} \mathbf{V} = [\min\{\underline{u} - \underline{v}, \overline{u} - \overline{v}\}, \max\{\underline{u} - \underline{v}, \overline{u} - \overline{v}\}]$$

Remark 1 [34] The addition of intervals are commutative and associative, and

$$\mathbf{U} \ominus \mathbf{V} = \mathbf{U} \oplus (-1) \odot \mathbf{V}.$$

The algebraic operations on the product space $I(\mathbb{R})^n = I(\mathbb{R}) \times I(\mathbb{R}) \times \cdots \times I(\mathbb{R})$ (n times) are defined as follows.

Definition 1 (*Algebraic operations on* $I(\mathbb{R})^n$[34]). Let $\widehat{\mathbf{U}} = (\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_n)$ and $\widehat{\mathbf{V}} = (\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_n)$ be two elements of $I(\mathbb{R})^n$. An algebraic operation $\star$ between $\widehat{\mathbf{U}}$ and $\widehat{\mathbf{V}}$, denoted by $\widehat{\mathbf{U}} \star \widehat{\mathbf{V}}$, is defined by

$$\widehat{\mathbf{U}} \star \widehat{\mathbf{V}} = (\mathbf{U}_1 \star \mathbf{V}_1, \mathbf{U}_2 \star \mathbf{V}_2, \cdots, \mathbf{U}_n \star \mathbf{V}_n),$$

where $\star \in \{\oplus, \ominus, \ominus_{gH}\}$.

The authors of [6] defined the ordering relations of intervals of the following types '$\leq_{LR}$', '$\leq_{CW}$', and '$\leq_{LC}$'. In this article, we only use the '$\leq_{LR}$' ordering relation and simply denote it by '$\preceq$'. Also, it is to mention that in view of the ordering relation '$\preceq$', the dominance relations of intervals are as follows.

Definition 2 (*Dominance relations on intervals* [34]). For any two intervals $\mathbf{U}, \mathbf{V} \in I(\mathbb{R})$

(i) if $\underline{u} \leq \underline{v}$ and $\overline{u} \leq \overline{v}$, then we say that $\mathbf{V}$ is dominated by $\mathbf{U}$ and write $\mathbf{U} \preceq \mathbf{V}$;
(ii) if either $\underline{u} \leq \underline{v}$ and $\overline{u} < \overline{v}$ or $\underline{u} < \underline{v}$ and $\overline{u} \leq \overline{v}$, then we say that $\mathbf{V}$ is strictly dominated by $\mathbf{U}$ and write $\mathbf{U} \prec \mathbf{V}$;
(iii) if $\mathbf{V}$ is not dominated by $\mathbf{U}$, then we write $\mathbf{U} \npreceq \mathbf{V}$; if $\mathbf{V}$ is not strictly dominated by $\mathbf{U}$, then we write $\mathbf{U} \nprec \mathbf{V}$;
(iv) if $\mathbf{U} \npreceq \mathbf{V}$ and $\mathbf{V} \npreceq \mathbf{U}$, then we say that none of $\mathbf{U}$ and $\mathbf{V}$ dominates the other, or $\mathbf{U}$ and $\mathbf{V}$ are not comparable.

Now we illustrate the concept of sequence in $I(\mathbb{R})^n$ and study its convergence. To do so, we need the concepts of norms on $I(\mathbb{R})$ and $I(\mathbb{R})^n$.

Definition 3 (*Norm on* $I(\mathbb{R})$ [1]). The function $\|\cdot\|_{I(\mathbb{R})} : I(\mathbb{R}) \to \mathbb{R}_+$, defined by

$$\|\mathbf{U}\|_{I(\mathbb{R})} = \max\{|\underline{u}|, |\overline{u}|\} \forall \mathbf{U} = [\underline{u}, \overline{u}] \in I(\mathbb{R}),$$

is a norm on $I(\mathbb{R})$.

Definition 4 (*Norm on* $I(\mathbb{R})^n$). For $\widehat{\mathbf{U}} = (\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_n) \in I(\mathbb{R})^n$, the function $\|\cdot\|_{I(\mathbb{R})^n} : I(\mathbb{R}) \to \mathbb{R}_+$, defined by

$$\|\widehat{\mathbf{U}}\|_{I(\mathbb{R})^n} = \sqrt{\sum_{i=1}^{n} \|\mathbf{U}_i\|^2_{I(\mathbb{R})}},$$

is a norm on $I(\mathbb{R})^n$. To prove that the function $\|\cdot\|_{I(\mathbb{R})^n}$ satisfies all the properties of a norm please see Appendix I

In the rest of the article, we use the symbols '$\|\cdot\|$', '$\|\cdot\|_{I(\mathbb{R})}$', and '$\|\cdot\|_{I(\mathbb{R})^n}$' to denote the usual Euclidean norm on $\mathbb{R}^n$, the norms on $I(\mathbb{R})$, and the norms on $I(\mathbb{R})^n$, respectively.

Definition 5 (*Sequence in* $I(\mathbb{R})^n$). A function $\widehat{\mathbf{G}} : \mathbb{N} \to I(\mathbb{R})^n$ is called sequence in $I(\mathbb{R})^n$.

Definition 6 (*Bounded sequence in* $I(\mathbb{R})^n$). A sequence $\left\{\widehat{\mathbf{G}}_k\right\}$ in $I(\mathbb{R})^n$ is said to be bounded from below (above) if there exists an $\widehat{\mathbf{U}} \in I(\mathbb{R})^n$ (a $\widehat{\mathbf{V}} \in I(\mathbb{R})^n$) such that

$$\widehat{\mathbf{U}} \preceq \widehat{\mathbf{G}}_k \forall n \in \mathbb{N} (\widehat{\mathbf{G}}_k \preceq \widehat{\mathbf{V}} \forall n \in \mathbb{N}),$$

where for any two elements $\widehat{\mathbf{B}} = (\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_n)$ and $\widehat{\mathbf{C}} = (\mathbf{C}_1, \mathbf{C}_2, \cdots, \mathbf{C}_n)$ in $I(\mathbb{R})^n$,

$$\widehat{\mathbf{B}} \preceq \widehat{\mathbf{C}} \iff \mathbf{B}_i \preceq \mathbf{C}_i \text{ for all } i = 1, 2, \cdots, n.$$

A sequence $\left\{\widehat{\mathbf{G}}_k\right\}$ that is both bounded below and above is called a bounded sequence.

Definition 7 (*Convergence in* $I(\mathbb{R})^n$). A sequence $\left\{\widehat{\mathbf{G}}_k\right\}$ in $I(\mathbb{R})^n$ with the property:

$$\|\widehat{\mathbf{G}}_k \ominus_{gH} \widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} \to 0 \text{ as } k \to \infty,$$

where $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$, is said to be convergent sequence.

Remark 2 It is noteworthy that if a sequence $\left\{\widehat{\mathbf{G}}_k\right\}$ in $I(\mathbb{R})^n$, where $\widehat{\mathbf{G}}_k = (\mathbf{G}_{k1}, \mathbf{G}_{k2}, \cdots, \mathbf{G}_{kn})$, converges to $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \cdots, \mathbf{G}_n) \in I(\mathbb{R})^n$, then according to Definition 1 and Definition 4, corresponding each sequence $\{\mathbf{G}_{ki}\}$ in $I(\mathbb{R})$ converges to $\mathbf{G}_i \in I(\mathbb{R})$ for all $i = 1, 2, \cdots, n$. Also, due to Definition 3, the sequences $\left\{\underline{g_{ki}}\right\}$ and $\{\overline{g_{ki}}\}$ in $\mathbb{R}$ converge to $\underline{g_i}$ and $\overline{g_i}$, respectively, for all i.

2.2 *Convexity and calculus of IVFs*

An IVF is defined by the function $\mathbf{T} : \mathcal{S} \to I(\mathbb{R})$. For each argument point $p \in \mathcal{S}$, the value of $\mathbf{T}$ is presented by

$$\mathbf{T}(p) = [\underline{t}(p),\ \overline{t}(p)],$$

where $\underline{t}$ and $\overline{t}$ are real-valued functions on $\mathcal{S}$ such that $\underline{t}(p) \leq \overline{t}(p)$ for all $p \in \mathcal{S}$.

In [20], Wu introduced two types of convexity for IVF, i.e., 'LU-convexity' and 'UC-convexity'. However, in this article, we only use LU-convexity for IVF and we read an LU-convex IVF as simply a convex IVF. The definition of a convex IVF is

Definition 8 (*Convex IVF* [20]). Let $\mathcal{S}$ be convex. An IVF $\mathbf{T} : \mathcal{S} \to I(\mathbb{R})$ is said to be convex on $\mathcal{S}$ if for any $p_1, p_2 \in \mathcal{S}$,

$$\mathbf{T}(\gamma_1 p_1 + \gamma_2 p_2) \preceq \gamma_1 \odot \mathbf{T}(p_1) \oplus \gamma_2 \odot \mathbf{T}(p_2)$$

for all $\gamma_1, \gamma_2 \in [0,\ 1]$ with $\gamma_1 + \gamma_2 = 1$.

It is notable that in Definition 8, we have used the notation '$\preceq$' instead of '$\preceq_{LC}$'. Because the ordering relation '$\preceq_{LC}$' provided in [20] is the same as the ordering relation '$\preceq$'.

Lemma 1 (See [20]). $\mathbf{T}$ *is convex if both of its boundary functions* $\underline{t}$ *and* $\overline{t}$ *are convex and vice-versa.*

Definition 9 (*gH-continuity* [4, 12]). An IVF $\mathbf{T}$ on $\mathcal{S}$ is said to be a gH-continuous at $\bar{p} \in \mathcal{S}$ if for any $d \in \mathbb{R}^n$ with $\bar{p} + d \in \mathcal{S}$,

$$\lim_{\|d\| \to 0} \left(\mathbf{T}(\bar{p} + d) \ominus_{gH} \mathbf{T}(\bar{p})\right) = \mathbf{0}.$$

Lemma 2 (See [34]). *An IVF* $\mathbf{T}$ *on* $\mathcal{S}$ *is gH-continuous if both of its boundary functions* $\underline{t}$ *and* $\overline{t}$ *are continuous and vice-versa.*

Theorem 1 *Let* S *be open. If an IVF* $\mathbf{T}$ *on* $\mathcal{S}$ *is convex, then it is gH-continuous on* $\mathcal{S}$.

Proof Let the IVF $\mathbf{T}$ be convex on $\mathcal{S}$. Due to Lemma 1, both the boundary functions $\underline{t}$ and $\overline{t}$ are convex on $\mathcal{S}$. Therefore, by the property of real-valued functions, $\underline{t}$ and $\overline{t}$ are continuous on $\mathcal{S}$. Hence, according to Lemma 2, $\mathbf{T}$ is gH-continuous on $\mathcal{S}$. □

Definition 10 (*gH-Lipschitz continuous IVF* [2]). An IVF $\mathbf{T}$ on $\mathcal{S}$ with the following property:

$$\|\mathbf{T}(p) \ominus_{gH} \mathbf{T}(q)\|_{I(\mathbb{R})} \leq K\|p - q\| \forall p, q \in \mathcal{S},$$

where $K > 0$, is known as gH-Lipschitz continuous on $\mathcal{S}$.

Definition 11 (*gH-derivative* [4, 14]). Let $\mathbf{T}$ be an IVF on $\mathcal{Y} \subseteq \mathbb{R}$. If the following limit:

$$\mathbf{T}'(\bar{p}) = \lim_{d \to 0} \frac{\mathbf{T}(\bar{p} + d) \ominus_{gH} \mathbf{T}(\bar{p})}{d}$$

exists at a point $\bar{p} \in \mathcal{Y}$ for all $d \in \mathbb{R}$ with $\bar{p} + d \in \mathcal{Y}$, then $\mathbf{T}'(\bar{p})$ is known as gH-derivative of $\mathbf{T}$ at $\bar{p}$.

Remark 3 (See [4, 35]). Let $\mathcal{Y} \subset \mathbb{R}$. If the derivatives of $\underline{t}$ and $\overline{t}$ at $\bar{p} \in \mathcal{Y}$ exist, then gH-derivative of the IVF $\mathbf{T} : \mathcal{Y} \to I(\mathbb{R})$ at $\bar{p}$ exists and

$$\mathbf{T}'(\bar{p}) = \left[\min\left\{\underline{t}'(\bar{p}), \bar{t}'(\bar{p})\right\}, \max\left\{\underline{t}'(\bar{p}), \bar{t}'(\bar{p})\right\}\right].$$

But, the converse is not true.

Definition 12 (*Partial gH-derivative* [11, 12]). Let $\mathbf{T}$ be an IVF on $\mathcal{S}$. We define a function $\mathbf{G}_i$ by

$$\mathbf{G}_i(p_i) = \mathbf{T}(\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{i-1}, p_i, \bar{p}_{i+1}, \cdots, \bar{p}_n),$$

where $\bar{p} = (\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_n)^t \in \mathcal{S}$. If $\mathbf{G}_i'$ exists at $\bar{p}_i$, then the i-th partial gH-derivative of $\mathbf{T}$ at $\bar{p}$, denoted $D_i\mathbf{T}(\bar{p})$, is defined by

$$D_i\mathbf{T}(\bar{p}) = \mathbf{G}_i'(\bar{p}_i) \forall i = 1, 2, \cdots, n.$$

Definition 13 (*gH-gradient* [11, 12]). The gH-gradient of an IVF $\mathbf{T}$ on $\mathcal{S}$ at a point $\bar{p} \in \mathcal{S}$, denoted $\nabla\mathbf{T}(\bar{p})$, is defined by

$$\nabla\mathbf{T}(\bar{p}) = (D_1\mathbf{T}(\bar{p}), D_2\mathbf{T}(\bar{p}), \cdots, D_n\mathbf{T}(\bar{p}))^t.$$

Definition 14 (*Directional gH-derivative* [15, 18]). Let $\mathbf{T}$ be an IVF on $\mathcal{S}$. Let $\bar{p} \in \mathcal{S}$ and $h \in \mathbb{R}^n$ such that $\bar{p} + \gamma h \in \mathcal{S}$ for any small γ. The directional gH-derivative of $\mathbf{T}$ at $\bar{p}$ in the direction h, denoted by $\mathbf{T}'(\bar{p})(h)$, is defined by

$$\lim_{\gamma \to 0+} \frac{1}{\gamma} \odot \left(\mathbf{T}(\bar{p} + \gamma h) \ominus_{gH} \mathbf{T}(\bar{p})\right), \text{provided the limit exists.}$$

Definition 15 (*Linear IVF* [2]). An IVF $\mathbf{L} : \mathcal{X} \to I(\mathbb{R})$ with the following properties:

(i) $\mathbf{L}(\gamma p) = \gamma \odot \mathbf{L}(p) \forall p \in \mathcal{S}$ and for all $\gamma \in \mathbb{R}$,
(ii) for all $p, q \in \mathcal{X}$, either

$$\mathbf{L}(p) \oplus \mathbf{L}(q) = \mathbf{L}(p + q)$$

or none of $\mathbf{L}(p) \oplus \mathbf{L}(q)$ and $\mathbf{L}(p + q)$ dominates the other,

is known as linear IVF.

Remark 4 (See [2]). The IVF $\mathbf{L} : \mathbb{R}^n \to I(\mathbb{R})$ that is defined by

$$\mathbf{L}(p) = p^t \odot \widehat{\mathbf{U}} = \bigoplus_{i=1}^n p_i \odot \mathbf{U}_i = \bigoplus_{i=1}^n p_i \odot [\underline{u}_i, \bar{u}_i]$$

is a linear IVF, where '$\bigoplus_{i=1}^n$' denotes successive addition of n number of intervals.

Remark 5 It is to mention that if the boundary functions $\underline{l}$ and $\bar{l}$ of an IVF $\mathbf{L} : \mathbb{R}^n \to I(\mathbb{R})$ are linear, then the IVF $\mathbf{L}$ must be linear. However, converse is not true.

For instance, consider the IVF $\mathbf{L}(p) = [-1, 1] \odot |p|$ on $\mathbb{R}$. For any $\gamma \in \mathbb{R}$,

$$\mathbf{L}(\gamma p) = [-1, 1] \odot |\gamma p| = \gamma \odot ([-1, 1] \odot |p|) = \gamma \odot \mathbf{L}(p).$$

Further, for all $p, q \in \mathbb{R}$,

$$\mathbf{L}(p + q) = [-1, 1] \odot |p + q| = [-|p + q|, |p + q|]$$

and

$$\mathbf{L}(p) + \mathbf{L}(q) = [-|p|, |p|] \oplus [-|q|, |q|] = [-|p| - |q|, |q| + |q|].$$

Since $-|p + q| \geq -|p| - |q|$ and $|p + q| \leq |q| + |q|$, therefore $\mathbf{L}(p + q)$ and $\mathbf{L}(p) + \mathbf{L}(q)$ are either equal or none of them dominates other. Hence, the IVF $\mathbf{L}$ is linear. However, the real-valued boundary functions $\underline{l}(p) = -|p|$ and $\bar{l}(p) = |p|$ are not linear.

In [15], the definition of gH-differentiability for IVFs is provided using the midpoint-radius representation $\left[\frac{\bar{t}+\underline{t}}{2}, \frac{\bar{t}-\underline{t}}{2}\right]$ of an IVF $\mathbf{T}$. However, as our main intention in this article is to illustrate all the things regarding IVF whether its lower boundary function $\underline{t}$ and upper boundary function $\bar{t}$ are readily available or not, we consider the Proposition 7 of [15] as the definition of gH-differentiability for IVFs, which is as follows.

Definition 16 (*gH-differentiability* [13, 15]). An IVF $\mathbf{T}$ on $\mathcal{S}$ is said to be gH-differentiable at $\bar{p} \in \mathcal{S}$ if there exists an IVF $\mathbf{L}_{\bar{p}}(d) = d^t \odot \widehat{\mathbf{U}}$, where $d \in \mathbb{R}^n$ and $\widehat{\mathbf{U}} \in I(\mathbb{R})^n$, an IVF $\mathbf{E}(\mathbf{T}(\bar{p}); d)$ and a $\lambda > 0$ such that

$$\left(\mathbf{T}(\bar{p} + d) \ominus_{gH} \mathbf{T}(\bar{p})\right) = \mathbf{L}_{\bar{p}}(d) \oplus \|d\| \odot \mathbf{E}(\mathbf{T}(\bar{p}); d)$$

for all d with $\|d\| < \lambda$, where $\mathbf{E}(\mathbf{T}(\bar{p}); d) \to \mathbf{0}$ as $\|d\| \to 0$.

Theorem 2 (See [15]). *Let an IVF* $\mathbf{T}$ *on* $\mathcal{S}$ *be gH-differentiable at* $\bar{p} \in \mathcal{S}$. *Then,* $\mathbf{T}$ *has directional gH-derivative at* $\bar{p}$ *for every direction* $d \in \mathbb{R}^n$ *and*

$$\mathbf{T}'(\bar{p})(d) = d^t \odot \nabla\mathbf{T}(\bar{p}) = \bigoplus_{i=1}^n d_i \odot D_i\mathbf{T}(\bar{p}) \forall d \in \mathbb{R}^n.$$

Theorem 3 (See [34]). *Let* $\mathcal{S}$ *be convex and an IVF* $\mathbf{T}$ *on* $\mathcal{S}$ *be gH-differentiable at* $p \in \mathcal{S}$. *Then*

$$(q - p)^t \odot \nabla\mathbf{T}(p) \preceq \mathbf{T}(q) \ominus_{gH} \mathbf{T}(p) \forall p, q \in \mathcal{S},$$

if $\mathbf{T}$ is convex on $\mathcal{S}$.

3. Subdifferentiability of IVFs

In this section we describe the concepts gH-subgradient and gH-subdifferential for convex IVFs and study their characteristics. In order to do this, we adopt the concept of subgradient for convex FVFs provided in [32].

Definition 17 (*gH-subgradient*). Let $\mathcal{S}$ be convex. An element $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \cdots, \mathbf{G}_n) \in I(\mathbb{R})^n$ is said to be a gH-subgradient of the convex IVF $\mathbf{T} : \mathcal{S} \to I(\mathbb{R})$ at $\bar{p} \in \mathcal{S}$ if

$$(p - \bar{p})^t \odot \widehat{\mathbf{G}} \preceq \mathbf{T}(p) \ominus_{gH} \mathbf{T}(\bar{p}) \forall p \in \mathcal{S}. \qquad (1)$$

Due to Remark 4, we can also define gH-subgradient as a

gH-continuous linear IVF. A gH-continuous linear IVF $\mathbf{L}_{\bar{p}} : \mathcal{X} \rightarrow I(\mathbb{R})$ is said to be gH-subgradient of $\mathbf{T}$ at $\bar{p} \in \mathcal{S}$ if

$$\mathbf{L}_{\bar{p}}(p - \bar{p}) \preceq \mathbf{T}(p) \ominus_{gH} \mathbf{T}(\bar{p}) \forall p \in \mathcal{S}, \tag{2}$$

where $\mathcal{X}$ is the smallest linear subspace of $\mathbb{R}^n$ containing $\mathcal{S}$.

Definition 18 (*gH-subdifferential*). The set $\partial\mathbf{T}(\bar{p})$ of all gH-subgradients of the convex IVF $\mathbf{T} : \mathcal{S} \subset \mathbb{R}^n \rightarrow I(\mathbb{R})$ at $\bar{p} \in \mathcal{S}$, where $\mathcal{S}$ is convex, is called gH-subdifferential of $\mathbf{T}$ at $\bar{p}$.

Throughout this article, we express an element of $\partial\mathbf{T}(\bar{p})$ either as as an element of $I(\mathbb{R})^n$ satisfying (1) or as an element of $\widehat{\mathcal{X}}$ satisfying (2).

Remark 6 In view of Theorem 3, it is to be noted that if $\mathbf{T}$ is gH-differentiable at $\bar{p} \in \mathcal{S}$, then $\nabla\mathbf{T}(\bar{p}) \in \partial\mathbf{T}(\bar{p})$.

Example 1 Let $\mathcal{Y} \subseteq \mathbb{R}$ be convex and an IVF $\mathbf{T} : \mathcal{Y} \rightarrow I(\mathbb{R})$ be defined by $\mathbf{T}(p) = |p| \odot \mathbf{B}$, where $\mathbf{0} \preceq \mathbf{B}$. If $\mathbf{G} \in I(\mathbb{R})$ is a gH-subgradient of $\mathbf{T}$ at $\bar{p} = 0$, then according to Definition 17, we have

$$(p - \bar{p})^t \odot \mathbf{G} \preceq \mathbf{T}(p) \ominus_{gH} \mathbf{T}(\bar{p}) \Longrightarrow \mathbf{G} \odot p \preceq \mathbf{B} \odot |p|.$$

Therefore, for $p \leq 0$, we have

$$\mathbf{G} \odot p \preceq (-1) \odot \mathbf{B} \odot p \Longrightarrow (-1) \odot \mathbf{B} \preceq \mathbf{G} \tag{3}$$

and for $p \geq 0$, we have

$$\mathbf{G} \odot p \preceq \mathbf{B} \odot p \Longrightarrow \mathbf{G} \preceq \mathbf{B}. \tag{4}$$

With the help of (3) and (4), we obtain

$$(-1) \odot \mathbf{B} \preceq \mathbf{G} \preceq \mathbf{B}.$$

Hence, $\partial\mathbf{T}(0) = \{\mathbf{G} : (-1) \odot \mathbf{B} \preceq \mathbf{G} \preceq \mathbf{B}\}$.

Considering $\mathbf{B} = [1, 3]$, the IVF $\mathbf{T}$ is delineated in figure 1 by the shaded region within dashed lines, and two possible subgradients $\mathbf{G}_1$, $\mathbf{G}_2 \in \partial\mathbf{T}(0)$ of $\mathbf{T}$ are delineated by black and dark gray regions, respectively.

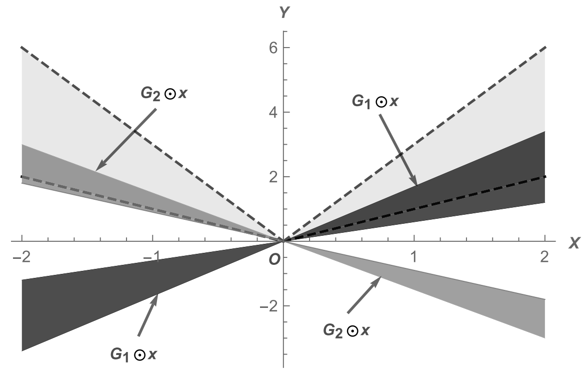

Figure 1. The IVF $\mathbf{T}$ of example 1 is depicted by the shaded region within dashed lines, and two possible subgradients $\mathbf{G}_1$ and $\mathbf{G}_2$ of $\mathbf{T}$ are illustrated by black and dark gray regions, respectively.

Example 2 Let $\mathcal{S}$ be convex and an IVF $\mathbf{T} : \mathcal{S} \rightarrow I(\mathbb{R})$ be defined by $\mathbf{T}(p) = |p_j| \odot \mathbf{U}$, where $j \in \{1, 2, \cdots, n\}$ and $\mathbf{0} \preceq \mathbf{U}$. If $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \cdots, \mathbf{G}_n) \in I(\mathbb{R})^n$ is a gH-subgradient of $\mathbf{T}$ at $\bar{p} = (\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_j \cdots, \bar{p}_n)$, where $\bar{p}_j = 0$, then due to Definition 17, we have

$$h^t \odot \widehat{\mathbf{G}} \preceq \mathbf{T}(\bar{p} + h) \ominus_{gH} \mathbf{T}(\bar{p}) \forall h \in \mathcal{S},$$

which implies

$$\bigoplus_{i=1}^{n} \mathbf{G}_i \odot h_i \preceq \mathbf{U} \odot |h_j| \forall h \in \mathcal{S}. \tag{5}$$

Let us choose $\widehat{\mathbf{G}} = \left(\mathbf{0}, \mathbf{0}, \cdots, \mathbf{0}, \mathbf{G}_j, \mathbf{0}, \cdots \mathbf{0}\right)$. From the relation (5) we obtain that

$$\mathbf{G}_j \odot h_j \preceq \mathbf{U} \odot |h_j|.$$

Thus, for all h with $h_j \leq 0$, we get

$$\mathbf{G}_j \odot h_j \preceq (-1) \odot \mathbf{U} \odot h_j \Longrightarrow (-1) \odot \mathbf{U} \preceq \mathbf{G}. \tag{6}$$

Hence, for all h with $h_j \geq 0$,

$$\mathbf{G}_j \odot h_j \preceq \mathbf{U} \odot h_j \Longrightarrow \mathbf{G}_j \preceq \mathbf{U}. \tag{7}$$

By (6) and (7), we have

$$(-1) \odot \mathbf{U} \preceq \mathbf{G}_j \preceq \mathbf{U}. \tag{8}$$

Therefore, $\widehat{\mathbf{G}} = \left(\mathbf{0}, \mathbf{0}, \cdots, \mathbf{0}, \mathbf{G}_j, \mathbf{0}, \cdots \mathbf{0}\right)$ that satisfies the condition (8) is a gH-subgradient of $\mathbf{T}$ at $\bar{p}$.

Remark 7 It is noteworthy that

(i) the author of [33] in Definition 2 has proposed the concept of subgradient for IVFs by considering L as linear real-valued function. However, in Definition 17 of the present article, we consider $\mathbf{L}_{\bar{p}}$ as linear IVF. That's why our concept of subgradient in terms of linear function is more general.

(ii) as IVFs are the special case of FVFs, one may think that we can adopt the concept of subgradient for FVFs of the article [36] as the concept of subgradient for IVFs. However, according to Definition 3.1 of [36], if we define the gH-subgradient $\widehat{\mathbf{G}}$ satisfying the condition

$$(p - \bar{p})^t \odot \widehat{\mathbf{G}} \oplus \mathbf{T}(\bar{p}) \preceq \mathbf{T}(p) \tag{9}$$

instead of satisfying the condition (1) in Definition 17, then Definition 17 will be quite restrictive even for a gH-differentiable IVF. The following example reveals that case.

Example 3 Let an IVF $\mathbf{T} : [0, 2.5] \to I(\mathbb{R})$ be defined by

$$\begin{aligned}\mathbf{T}(p) &= [1,1] \odot p^4 \oplus [0,1] \odot (p^2 - p^4 + 34) \oplus [1,6]\\ &= [p^4 + 1, p^2 + 40]\\ &= [\underline{t}(p), \overline{t}(p)]\end{aligned}$$

Clearly, $\underline{t}$ and $\overline{t}$ are differentiable at $\bar{p} = 1$. Hence, the *gH*-derivative $\mathbf{T}'(\bar{p})$ of $\mathbf{T}$ at $\bar{p} = 1$ exists due to Remark 3, and

$$\nabla\mathbf{T}(1) = \mathbf{T}'(1) = [2, 4].$$

Since

$$\nabla\mathbf{T}(1) \odot (2-1) = [2,4] \preceq [3,15] = \mathbf{T}(2) \ominus_{gH} \mathbf{T}(1)$$

but

$$\nabla\mathbf{T}(1) \odot (2-1) \oplus \mathbf{T}(1) = [4,45] \npreceq [17,44] = \mathbf{T}(2)$$

therefore, $\nabla\mathbf{T}(1) \in \partial\mathbf{T}(1)$ with respect to condition (1), not respect to the condition (9).

Remark 8 One may think that the study of *gH*-subgradients and *gH*-subdifferentials of an IVF $\mathbf{T}$ is equivalent to the study of subgradients and subdifferentials of its real-valued boundary functions $\underline{t}$ and $\overline{t}$ together. Practically, it is not true. The following two reasons clarify the fact.

(i) From the definition of subgradient (Definition 17), it is clear that a *gH*-subgradient of an IVF $\mathbf{T}$ is a linear IVF ($\mathbf{L}$ say); also, it is well known that the subgradients of the real-valued boundary functions $\underline{t}$ and $\overline{t}$ are linear. Therefore, one may think that the boundary functions of the subgradient $\mathbf{L}$ must be linear. However, Remark 5 reveals that real-valued boundary functions of a linear IVF are not necessarily linear. Hence, just by the properties of the boundary functions of $\mathbf{L}$, one cannot expect to capture the properties of *gH*-subgradient of the IVF $\mathbf{T}$.

(ii) Further, it is noteworthy that the subgradients $\underline{g}$ and $\overline{g}$ of the real-valued boundary functions $\underline{t}$ and $\overline{t}$ can be defined by

$$(p - \bar{p})^t \underline{g} + \underline{t}(\bar{p}) \le \underline{t}(p)$$

and

$$(p - \bar{p})^t \overline{g} + \overline{t}(\bar{p}) \le \overline{t}(p)$$

respectively. However, example 3 shows that we cannot define *gH*-subgradient of the IVF $\mathbf{T}$ by the relation (9).

So, it can be said that *gH*-subgradients and *gH*-subdifferentials of an IVF $\mathbf{T}$ are not the obvious extension of the subgradients and subdifferentials of the real-valued boundary functions $\underline{t}$ and $\overline{t}$ together.

Now we provide an example of *gH*-subdifferential as a collection of *gH*-continuous linear IVF through Theorem 4. To do so, we introduce the concept of a norm on the set $\widehat{\mathcal{X}}$ of all *gH*-continuous linear IVFs on a linear subspace $\mathcal{X}$ of $\mathbb{R}^n$.

Definition 19 (*Norm on* $\widehat{\mathcal{X}}$). A norm on the set $\widehat{\mathcal{X}}$ of all *gH*-continuous linear IVF $\mathbf{L}$ on $\mathcal{X}$ is defined by the function $\|\cdot\|_{\widehat{\mathcal{X}}} : \widehat{\mathcal{X}} \to \mathbb{R}_+$ such that

$$\|\mathbf{L}\|_{\widehat{\mathcal{X}}} = \sup_{p \neq 0} \frac{\|\mathbf{L}(p)\|_{I(\mathbb{R})}}{\|p\|}, \text{where } p \in \mathcal{X}.$$

To prove that the function $\|\cdot\|_{\widehat{\mathcal{X}}}$ satisfies all the properties of a norm please see Appendix II.

Lemma 3 Let $\mathbf{L} \in \widehat{\mathcal{X}}$ be such that

$$\mathbf{L}(p) \preceq \mathbf{B} \odot \|p\| \forall p \in \mathcal{X},$$

where $\mathbf{B} \in I(\mathbb{R})$. Then,

$$\|\mathbf{L}(p)\|_{I(\mathbb{R})} \le \|\mathbf{B}\|_{I(\mathbb{R})} \|p\| \forall p \in \mathcal{X}.$$

Proof Please see Appendix III. □

Theorem 4 *Let* $\mathbf{T} : \mathcal{X} \to I(\mathbb{R})$ *be a convex IVF, defined by*

$$\mathbf{T}(p) = \mathbf{B} \odot \|p\| \forall p \in \mathcal{X},$$

where $\mathbf{B} \in I(\mathbb{R}_+)$. Then,

$$\partial\mathbf{T}(0) = \left\{\mathbf{L}_0 \in \widehat{\mathcal{X}} \mid \|\mathbf{L}_0\|_{\widehat{\mathcal{X}}} \le \|\mathbf{B}\|_{I(\mathbb{R})}\right\}.$$

Proof Let $\mathbf{L}_0 \in \partial\mathbf{T}(0)$. Therefore, for all nonzero $p \in \mathcal{X}$,

$$\begin{aligned}&\mathbf{L}_0(p - 0) \preceq \mathbf{T}(p) \ominus_{gH} \mathbf{T}(0)\\ \Longrightarrow &\mathbf{L}_0(p) \preceq \mathbf{B} \odot \|p\|\\ \Longrightarrow &\|\mathbf{L}_0(p)\|_{I(\mathbb{R})} \le \|\mathbf{B}\|_{I(\mathbb{R})}\|p\|, \text{by Lemma 3}\\ \Longrightarrow &\frac{\|\mathbf{L}_0(p)\|_{I(\mathbb{R})}}{\|p\|} \le \|\mathbf{B}\|_{I(\mathbb{R})}\\ \Longrightarrow &\sup_{p\neq 0}\frac{\|\mathbf{L}_0(p)\|_{I(\mathbb{R})}}{\|p\|} \le \|\mathbf{B}\|_{I(\mathbb{R})}\\ \Longrightarrow &\|\mathbf{L}_0\|_{\widehat{\mathcal{X}}} \le \|\mathbf{B}\|_{I(\mathbb{R})}.\end{aligned}$$

Hence,

$$\partial\mathbf{T}(0) = \left\{\mathbf{L}_0 \in \widehat{\mathcal{X}} \mid \|\mathbf{L}_0\|_{\widehat{\mathcal{X}}} \le \|\mathbf{B}\|_{I(\mathbb{R})}\right\}.$$

□

Next, we show that a *gH*-differentiable convex IVF has only one *gH*-subgradient, which is the *gH*-gradient of the IVF. Thereafter, we show that on a real linear subspace if the *gH*-subgradients of a convex IVF at a point exists, then the directional *gH*-derivative of the IVF at that point in each direction is the maximum of all the products of *gH*-subgradients and the direction.

Lemma 4 *Let $\mathcal{S}$ be convex and* $\mathbf{T}$ *be a convex IVF on* $\mathcal{S}$. *Then, for an arbitrary* $\bar{p} \in \mathbb{R}^n$

$$\partial\mathbf{T}(\bar{p}) = \left\{\widehat{\mathbf{G}} \in I(\mathbb{R})^n \mid h^t \odot \widehat{\mathbf{G}} \preceq \mathbf{T}'(\bar{p})(h) \forall h \in \mathcal{S}\right\}.$$

Proof For an arbitrary $\widehat{\mathbf{G}} \in \partial\mathbf{T}(\bar{p})$, we have

$$(p - \bar{p})^t \odot \widehat{\mathbf{G}} \preceq \mathbf{T}(p) \ominus_{gH} \mathbf{T}(\bar{p}) \forall p \in \mathcal{S}.$$

Replacing p by $\bar{p} + \gamma h$, where $\gamma > 0$, we get

$$(\gamma h)^t \odot \widehat{\mathbf{G}} \preceq \mathbf{T}(\bar{p} + \gamma h) \ominus_{gH} \mathbf{T}(\bar{p}),$$

which implies

$$h^t \odot \widehat{\mathbf{G}} \preceq \lim_{\gamma \to 0+} \frac{1}{\gamma} \odot \left(\mathbf{T}(\bar{p} + \gamma h) \ominus_{gH} \mathbf{T}(\bar{p})\right)$$
$$\Longrightarrow h^t \odot \widehat{\mathbf{G}} \preceq \mathbf{T}'(\bar{p})(h).$$

□

Theorem 5 *Let an IVF* $\mathbf{T}$ *on* $\mathcal{S}$ *is gH-differentiable at* $\bar{p} \in \mathcal{S}$, *then*

$$\partial\mathbf{T}(\bar{p}) = \{\nabla\mathbf{T}(\bar{p})\}.$$

Proof Let $\mathbf{G} \in \partial\mathbf{T}(\bar{p})$. Since $\mathbf{T}$ is *gH*-differentiable at $\bar{p}$, in view of Theorem 2 and Lemma 4, we have

$$h^t \odot \widehat{\mathbf{G}} \preceq h^t \odot \nabla\mathbf{T}(\bar{p}) \forall h \in \mathbb{R}^n. \tag{10}$$

Replacing h by $-h$ in the last relation we get

$$(-h)^t \odot \widehat{\mathbf{G}} \preceq (-h)^t \odot \nabla\mathbf{T}(\bar{p}),$$

which implies

$$h^t \odot \nabla\mathbf{T}(\bar{p}) \preceq h^t \odot \widehat{\mathbf{G}} \forall h \in \mathbb{R}^n. \tag{11}$$

Thus, the relations (10) and (11) together yield

$$h^t \odot \nabla\mathbf{T}(\bar{p}) = h^t \odot \widehat{\mathbf{G}} \forall h \in \mathbb{R}^n. \tag{12}$$

For each $i \in \{1, 2, \cdots, n\}$, by choosing $h = e_i$, we have

$$D_i\mathbf{T}(\bar{p}) = \mathbf{G}_i.$$

Therefore,

$$\nabla\mathbf{T}(\bar{p}) = \widehat{\mathbf{G}}$$

and hence,

$$\partial\mathbf{T}(\bar{p}) = \{\nabla\mathbf{T}(\bar{p})\}.$$

□

Theorem 6 *Let an IVF* $\mathbf{T}$ *be a convex and gH-continuous IVF on* $\mathcal{X}$. *If gH-subdifferential* $\partial\mathbf{T}(\bar{p})$ *of* $\mathbf{T}$ *at* $\bar{p} \in \mathcal{X}$ *is nonempty, then*

$$\mathbf{T}'(\bar{p})(h) = \max\left\{h^t \odot \widehat{\mathbf{G}} \mid \widehat{\mathbf{G}} \in \partial\mathbf{T}(\bar{p})\right\} \forall h \in \mathcal{X}.$$

Proof Let *gH*-subdifferential $\partial\mathbf{T}(\bar{p})$ of $\mathbf{T}$ at $\bar{p} \in \mathcal{X}$ is nonempty. Since $\mathbf{T}$ is convex on $\mathcal{X}$, the directional *gH*-derivative of $\mathbf{T}$ at $\bar{p}$ in every direction $h \in \mathcal{X}$ exists due to Theorem 3.1 in [2]. By Lemma 3.1 in [2], we have

$$\frac{1}{\gamma} \odot \left(\mathbf{T}(\bar{p} + \gamma h) \ominus_{gH} \mathbf{T}(\bar{p})\right) \preceq \mathbf{T}(\bar{p} + h) \ominus_{gH} \mathbf{T}(\bar{p}),$$
$$\text{where } \gamma > 0$$
$$\Longrightarrow \mathbf{T}'(\bar{p})(h) \preceq \mathbf{T}(\bar{p} + h) \ominus_{gH} \mathbf{T}(\bar{p})$$

for all $h \in \mathcal{X}$. Hence, in view of Lemma 4, we obtain

$$\mathbf{T}'(\bar{p})(h) = \max\left\{\widehat{\mathbf{G}}^t \odot h \mid \widehat{\mathbf{G}} \in \partial\mathbf{T}(\bar{p})\right\} \forall h \in \mathcal{X}.$$

□

Next, we show that *gH*-subdifferentials of a convex IVF are bounded and closed. To do so, we use the mapping $\mathcal{W} : I(\mathbb{R})^n \to \mathbb{R}^n$ defined in [2] by

$$\begin{aligned} \mathcal{W}(\widehat{\mathbf{U}}) =& \mathcal{W}(\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_n) \\ =& (w\underline{u}_1 + w'\overline{u}_1, w\underline{u}_2 + w'\overline{u}_2, \cdots, w\underline{u}_n + w'\overline{u}_n)^t, \end{aligned} \tag{13}$$

where $w, w' \in [0, 1]$ with $w + w' = 1$.

Lemma 5 *For any* $\widehat{\mathbf{U}} \in I(\mathbb{R})^n$ *and* $d \in \mathbb{R}^n$,

$$d^t \odot \widehat{\mathbf{U}} \preceq [c, c] \Longrightarrow d^t\mathcal{W}\left(\widehat{\mathbf{U}}\right) \le 2c,$$

where the map $\mathcal{W}$ *is defined by* (13).

Proof Please see Appendix IV. □

Lemma 6 *For any* $\widehat{\mathbf{U}} \in I(\mathbb{R})^n$,

$$\|\mathcal{W}\left(\widehat{\mathbf{U}}\right)\| \text{is finite} \Longrightarrow \|\widehat{\mathbf{U}}\|_{I(\mathbb{R})^n} \text{is finite},$$

where the map $\mathcal{W}$ *is defined by* (13).

Proof Please see Appendix V. □

Theorem 7 *Let* $\mathcal{S}$ *be compact and convex and* $\mathbf{T}$ *be a convex IVF on* $\mathcal{S}$. *Then,* $\bigcup_{p \in \mathcal{S}} \partial\mathbf{T}(p)$ *is bounded.*

Proof We claim that the set $\bigcup_{p \in \mathcal{S}} \partial\mathbf{T}(p)$ is bounded. On contrary, there exists a sequence $\{p_k\}$ on $\mathcal{S}$ and an unbounded sequence $\{\widehat{\mathbf{G}}_k\}$, where $\widehat{\mathbf{G}}_k \in \partial\mathbf{T}(p_k)$, such that

$$0 < \|\widehat{\mathbf{G}}_k\| < \|\widehat{\mathbf{G}}_{k+1}\|, k \in \mathbb{N}.$$

Let us take $d_k = \frac{\mathcal{W}(\widehat{\mathbf{G}}_k)}{\|\mathcal{W}(\widehat{\mathbf{G}}_k)\|}$, where the mapping $\mathcal{W}$ is defined by (13). By Definition 17 we have

$$\begin{aligned} & d_k^t \odot \widehat{\mathbf{G}}_k \preceq \mathbf{T}(p_k + d_k) \ominus_{gH} \mathbf{T}(p_k) \\ & = \max\{\underline{t}(p_k + d_k) - \underline{t}(p_k), \overline{t}(p_k + d_k) - \overline{t}(p_k)\} \\ \Longrightarrow & d_k^t \odot \widehat{\mathbf{G}}_k \preceq [c, c], \text{where } \max\{\overline{t}(p) | p \in \mathcal{S}\} \le c \\ \Longrightarrow & d_k^t \mathcal{W}\left(\widehat{\mathbf{G}}_k\right) \le 2c, \text{by Lemma 3} \\ \Longrightarrow & \|\mathcal{W}\left(\widehat{\mathbf{G}}_k\right)\| \le 2c. \end{aligned}$$

Since $\mathbf{T}$ is convex on $\mathbb{R}^n$, in view of Theorem 1 and Lemma 2, $\underline{t}$ and $\overline{t}$ are continuous on $\mathbb{R}^n$. As $\{p_k\}$ and $\{d_k\}$ are bounded and the boundary functions $\underline{t}$ and $\overline{t}$ are continuous, by the property of real-valued function, c is finite. Thus, $\|\mathcal{W}\left(\widehat{\mathbf{G}}_k\right)\|$ is finite and hence, due to Lemma 6, $\|\widehat{\mathbf{G}}_k\|_{I(\mathbb{R}^n)}$ is finite. Therefore, the sequence $\{\widehat{\mathbf{G}}_k\}$ is bounded, which is a contradiction. Hence, the set $\bigcup_{p\in\mathcal{S}} \partial\mathbf{T}(p)$ is bounded.

Theorem 8 *Let $\mathcal{S}$ be convex and $\mathbf{T}$ be a convex IVF on $\mathcal{S}$. Then, for every $\bar{p} \in \mathcal{S}$, $\partial\mathbf{T}(\bar{p})$ is closed.*

Proof Let $\left\{\widehat{\mathbf{G}}_k\right\}$ be an arbitrary sequence in $\partial\mathbf{T}(p)$ which converges to $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$, where $\widehat{\mathbf{G}}_k = (\mathbf{G}_{k1}, \mathbf{G}_{k2}, \cdots, \mathbf{G}_{kn})$ and $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \cdots, \mathbf{G}_n)$.

Since, $\widehat{\mathbf{G}}_k \in \partial\mathbf{T}(\bar{p})$, for all $d \in \mathcal{S}$ we have

$$d^t \odot \widehat{\mathbf{G}}_k \preceq \mathbf{T}(\bar{p} + d) \ominus_{gH} \mathbf{T}(\bar{p}),$$

i.e.,

$$\bigoplus_{i=1}^n d_i \odot \mathbf{G}_{ki} \preceq \mathbf{T}(\bar{p} + d) \ominus_{gH} \mathbf{T}(\bar{p}). \tag{14}$$

In view of Remark 1, without loss of generality, let the first m components of d be nonnegative and the rest $n - m$ components be negative. Therefore, from (14), we get

$$\begin{aligned} & \bigoplus_{i=1}^m d_i \odot \mathbf{G}_{ki} \oplus \bigoplus_{j=m+1}^n d_j \odot \mathbf{G}_{kj} \preceq \mathbf{T}(\bar{p} + d) \ominus_{gH} \mathbf{T}(\bar{p}) \\ \Longrightarrow & \bigoplus_{i=1}^m \left[\underline{g_{ki}} d_i, \overline{g_{ki}} d_i\right] \oplus \bigoplus_{j=m+1}^n \left[\overline{g_{kj}} d_j, \underline{g_{kj}} d_j\right] \\ & \preceq \mathbf{T}(\bar{p} + d) \ominus_{gH} \mathbf{T}(\bar{p}) \\ \Longrightarrow & \left[\sum_{i=1}^m \underline{g_{ki}} d_i + \sum_{j=m+1}^n \overline{g_{kj}} d_j, \sum_{i=1}^m \overline{g_{ki}} d_i + \sum_{j=m+1}^n \underline{g_{kj}} d_j\right] \\ & \preceq \mathbf{T}(\bar{p} + d) \ominus_{gH} \mathbf{T}(\bar{p}). \end{aligned}$$

Therefore, we get

$$\begin{aligned} & \sum_{i=1}^m \underline{g_{ki}} d_i + \sum_{j=m+1}^n \overline{g_{kj}} d_j \le \min\{\underline{t}(\bar{p} + h) \\ & -\underline{t}(\bar{p}), \overline{t}(\bar{p} + h) - \overline{t}(\bar{p})\} \end{aligned} \tag{15}$$

$$\begin{aligned} & \sum_{i=1}^m \overline{g_{ki}} d_i + \sum_{j=m+1}^n \underline{g_{kj}} \\ & d_j \le \max\{\underline{t}(\bar{p} + h) - \underline{t}(\bar{p}), \overline{t}(\bar{p} + h) - \overline{t}(\bar{p})\} \end{aligned} \tag{16}$$

Since the sequence $\left\{\widehat{\mathbf{G}}_k\right\}$ converges to $\widehat{\mathbf{G}}$, in view of Remark 2, the sequences $\left\{\underline{g_{ki}}\right\}$ and $\{\overline{g_{ki}}\}$ converge to $\underline{g_i}$ and $\overline{g_i}$, respectively, for all i. Thus, by (15) and (16), we have

$$\begin{aligned} \left(\sum_{i=1}^m \underline{g_{ki}} d_i + \sum_{j=m+1}^n \overline{g_{kj}} d_j\right) \to \left(\sum_{i=1}^m \underline{g_i} d_i + \sum_{j=m+1}^n \overline{g_j} d_j\right) \\ \le \min\Big\{\underline{t}(\bar{p} + h) - \underline{t}(\bar{p}), \\ \overline{t}(\bar{p} + h) - \overline{t}(\bar{p})\Big\} \end{aligned}$$

and

$$\begin{aligned} \left(\sum_{i=1}^m \overline{g_{ki}} d_i + \sum_{j=m+1}^n \underline{g_{kj}} d_j\right) \to \left(\sum_{i=1}^m \overline{g_i} d_i + \sum_{j=m+1}^n \underline{g_j} d_j\right) \\ \le \max\Big\{\underline{t}(\bar{p} + h) - \underline{t}(\bar{p}), \\ \overline{t}(\bar{p} + h) - \overline{t}(\bar{p})\Big\}. \end{aligned}$$

Hence,

$$\begin{aligned} & \left[\sum_{i=1}^m \underline{g_i} d_i + \sum_{j=m+1}^n \overline{g_j} d_j, \sum_{i=1}^m \overline{g_i} d_i + \sum_{j=m+1}^n \underline{g_j} d_j\right] \\ & \preceq \mathbf{T}(\bar{p} + d) \ominus_{gH} \mathbf{T}(\bar{p}) \\ \Longrightarrow & \bigoplus_{i=1}^m \left[\underline{g_i} d_i, \overline{g_i} d_i\right] \oplus \bigoplus_{j=m+1}^n \left[\overline{g_j} d_j, \underline{g_j} d_j\right] \\ & \preceq \mathbf{T}(\bar{p} + d) \ominus_{gH} \mathbf{T}(\bar{p}) \\ \Longrightarrow & \bigoplus_{i=1}^m d_i \odot \mathbf{G}_i \oplus \bigoplus_{j=m+1}^n d_j \odot \mathbf{G}_j \preceq \mathbf{T}(\bar{p} + d) \ominus_{gH} \mathbf{T}(\bar{p}) \\ \Longrightarrow & d^t \odot \widehat{\mathbf{G}} \preceq \mathbf{T}(\bar{p} + d) \ominus_{gH} \mathbf{T}(\bar{p}) \end{aligned}$$

for all $d \in \mathcal{S}$. Therefore, $\widehat{\mathbf{G}} \in \partial\mathbf{T}(\bar{p})$ and hence, $\partial\mathbf{T}(\bar{p})$ is closed. □

In the following theorem, we prove that if a convex IVF has gH-subgradients in all over its domain, then the IVF is gH-Lipschitz continuous on its domain.

Lemma 7 *For any $p \in \mathbb{R}^n$ and $\widehat{\mathbf{U}} = (\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_n) \in I(\mathbb{R})^n$,*

$$p^t \odot \widehat{\mathbf{U}} \preceq \|p\| \odot \left[\|\widehat{\mathbf{U}}\|_{I(\mathbb{R})^n}, \|\widehat{\mathbf{U}}\|_{I(\mathbb{R})^n}\right].$$

Proof Please see Appendix VI. □

Lemma 8 *Let $\mathbf{T}$ be an IVF on $\mathcal{S}$ such that*

$$\mathbf{T}(p) \ominus_{gH} \mathbf{T}(q) \preceq \mathbf{B} \odot \|p - q\| \forall p, q \in \mathcal{S},$$

where $\mathbf{B} = [\underline{b}, \overline{b}] = [b, b]$. *Then,*

$$\|\mathbf{T}(p) \ominus_{gH} \mathbf{T}(q)\|_{I(\mathbb{R})} \leq b\|p - q\| \forall p, q \in \mathcal{S}.$$

Proof Please see Appendix VII. □

Theorem 9 *Let* $\mathcal{S}$ *be compact and convex, and* $\mathbf{T}$ *be a convex IVF on* $\mathcal{S}$ *such that* $\mathbf{T}$ *has gH-subgradient at every* $p \in \mathcal{S}$. *Then,* $\mathbf{T}$ *is gH-Lipschitz continuous on* $\mathcal{S}$.

Proof Since $\mathbf{T}$ has *gH*-subgradient at every $p \in \mathcal{S}$, there exists a $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$ such that

$$\begin{aligned}
&(q-p)^t \odot \widehat{\mathbf{G}} \preceq \mathbf{T}(q) \ominus_{gH} \mathbf{T}(p) \\
\Longrightarrow &(-1) \odot \left((p-q)^t \odot \widehat{\mathbf{G}}\right) \preceq \mathbf{T}(q) \ominus_{gH} \mathbf{T}(p) \\
\Longrightarrow &\mathbf{T}(p) \ominus_{gH} \mathbf{T}(q) \preceq (p-q)^t \odot \widehat{\mathbf{G}} \\
\Longrightarrow &\mathbf{T}(p) \ominus_{gH} \mathbf{T}(q) \preceq \|p-q\| \odot \left[\|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n}, \|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n}\right], \\
&\text{by Lemma 3} \\
\Longrightarrow &\|\mathbf{T}(p) \ominus_{gH} \mathbf{T}(q)\|_{I(\mathbb{R})} \leq \|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} \|p-q\|, \text{by Lemma 3.}
\end{aligned}$$

Considering $K = \sup_{\widehat{\mathbf{G}} \in \bigcup_{p \in \mathcal{S}} \partial \mathbf{T}(p)} \|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n}$, we have

$$\|\mathbf{T}(p) \ominus_{gH} \mathbf{T}(q)\|_{I(\mathbb{R})} \leq K\|p - q\| \forall p, q \in \mathcal{S}.$$

Hence, $\mathbf{T}$ is *gH*-Lipschitz continuous on $\mathcal{S}$. □

Now we show another two important characteristics of *gH*-subdifferential of a convex IVF.

Theorem 10 (Chain rule). *Let* $\mathcal{S}$ *be convex and an IVF* $\mathbf{T}$ *be defined by*

$$\mathbf{T}(p) = \mathbf{H}(Ap) \forall p \in \mathcal{S},$$

where $\mathbf{H} : \mathbb{R}^m \to I(\mathbb{R})$ *is a convex IVF and* A *is a* $m \times n$ *matrix with real entries. Then,*

$$\partial \mathbf{T}(p) = \{A^t \odot \widehat{\mathbf{G}}_m \mid \widehat{\mathbf{G}}_m \in \partial \mathbf{H}(Ap)\},$$

where $\widehat{\mathbf{G}}_m \in I(\mathbb{R})^m$ *and* $p \in \mathcal{S}$.

Proof By the definition of *gH*-subdifferentiability of $\mathbf{H}$ at $A(p)$, for any $p \in \mathcal{S}$, we have a $\widehat{\mathbf{G}}_m \in I(\mathbb{R})^m$ such that

$$(Ay - Ap)^t \odot \widehat{\mathbf{G}}_m \preceq \mathbf{H}(Ay) \ominus_{gH} \mathbf{H}(Ap) \forall q \in \mathcal{S},$$

which implies

$$\begin{aligned}
&(A(q-p))^t \odot \widehat{\mathbf{G}}_m \preceq \mathbf{H}(Ay) \ominus_{gH} \mathbf{H}(Ap) \\
\Longrightarrow &(q-p)^t \odot (A^t \odot \widehat{\mathbf{G}}_m) \preceq \mathbf{H}(Ay) \ominus_{gH} \mathbf{H}(Ap) \\
\Longrightarrow &(q-p)^t \odot (A^t \odot \widehat{\mathbf{G}}_m) \preceq \mathbf{T}(q) \ominus_{gH} \mathbf{T}(p).
\end{aligned}$$

Since $(A^t \odot \widehat{\mathbf{G}}_m) \in I(\mathbb{R})^n$, by Definition 17,

$$\partial \mathbf{T}(p) = \{A^t \odot \widehat{\mathbf{G}}_m \mid \widehat{\mathbf{G}}_m \in \partial \mathbf{H}(Ap)\},$$

where $\widehat{\mathbf{G}}_m \in I(\mathbb{R})^m$ and $p \in \mathcal{S}$. □

Remark 9 (*gH-subdifferential of a sum*). Let $\mathcal{S}$ be a convex set and an IVF $\mathbf{T}$ be defined by

$$\mathbf{T}(p) = \bigoplus_{i=1}^m \mathbf{T}_i(p) \forall p \in \mathcal{S},$$

where each $\mathbf{T}_i : \mathcal{S} \to I(\mathbb{R})$ is a convex IVF on $\mathcal{S}$. We write

$$\mathbf{T}(p) = \mathbf{H}(Ap) \forall p \in \mathcal{S},$$

where A is a matrix, defined by $Ap = (p, p, \cdots, p)^t$ for all $p \in \mathcal{S}$ and $\mathbf{H} : \mathbb{R}^{mn} \to I(\mathbb{R})$ is an IVF, defined by

$$\mathbf{H}(q) = \mathbf{H}(q_1, q_2, \cdots, q_m) = \bigoplus_{i=1}^m \mathbf{T}_i(q_i) \forall q \in \mathbb{R}^{mn}.$$

Thus, by Theorem 10, we have

$$\partial \mathbf{T}(p) = \bigoplus_{i=1}^m \partial \mathbf{T}_i(p) \forall p \in \mathcal{S}.$$

4. Convex IOP and its optimality conditions

Here we explore the relation of efficient solutions (ESs) to the following IOP:

$$\min_{p \in \mathcal{S}} \mathbf{T}(p), \tag{17}$$

where $\mathcal{S}$ is convex and $\mathbf{T}$ is a convex IVF on $\mathcal{S}$, with the *gH*-subgradients of $\mathbf{T}$. The IOP with convex IVFs is known as convex IOP.

The following definition reveals the concept of an ES to the IOP (17).

Definition 20 (*ES* [2]). Let $\bar{p} \in \mathcal{S}$. If $\mathbf{T}(p) \nprec \mathbf{T}(\bar{p}) \forall p (\neq \bar{p}) \in \mathcal{S}$, then $\bar{p}$ is known as an ES to the IOP (17).

Remark 10 Let $\mathcal{S}$ be a convex set and $\mathbf{T} : \mathcal{S} \to I(\mathbb{R})$ be a convex IVF. If $\widehat{\mathbf{0}} \in \partial \mathbf{T}(\bar{p})$ for some $\bar{p} \in \mathcal{S}$, where $\widehat{\mathbf{0}} = (\mathbf{0}, \mathbf{0}, \cdots, \mathbf{0})$, then $\bar{p}$ is an ES to the IOP (17). The reason is follows. If $\widehat{\mathbf{0}} \in \partial \mathbf{T}(\bar{p})$, then for all $p \in \mathcal{S}$,

$$\begin{aligned}
(p - \bar{p})^t \odot \widehat{\mathbf{0}} \preceq \mathbf{T}(p) \ominus_{gH} \mathbf{T}(\bar{p}) &\Longrightarrow \mathbf{0} \preceq \mathbf{T}(p) \ominus_{gH} \mathbf{T}(\bar{p}) \\
&\Longrightarrow \mathbf{T}(\bar{p}) \preceq \mathbf{T}(p) \\
&\Longrightarrow \mathbf{T}(p) \nprec \mathbf{T}(\bar{p}).
\end{aligned}$$

Hence, $\bar{p}$ is an ES to the IOP (17).

An important point to note for the result in this remark is that the result is applicable for any general convex IVF, irrespective of *gH*-differentiability. Thus, this result is not identical to Theorem 3 in [37], which is applicable only for *gH*-differentiable IVFs.

The following two examples reveal that Remark 10 is true.

Example 4 Consider the following IOP:

$$\min_{p\in\mathcal{S}=[-2,6]} \mathbf{T}(p) = \begin{cases} [-2,5]\ominus_{gH}[-1,0]\odot|p-2|, & \text{for } 1\leqslant p\leqslant 3 \\ [-2,3]\oplus[1,2]\odot|p-2|, & \text{otherwise.} \end{cases} \tag{18}$$

Choosing $\bar{p}=2$ we have

$$\mathbf{T}(p)\ominus_{gH}\mathbf{T}(\bar{p}) = \begin{cases} [0,|p-2|], & \text{for } 1\leqslant p\leqslant 3 \\ [2|p-2|-2,|p-2|], & \text{for } 0\leqslant p\leqslant 1 \text{ and } 3\leqslant p\leqslant 4 \\ [|p-2|,2|p-2|-2], & \text{otherwise.} \end{cases}$$

Thus,

$$\mathbf{0} = (p-\bar{p})^t\odot\mathbf{0}\preceq\mathbf{T}(p)\ominus_{gH}\mathbf{T}(\bar{p})\forall p\in\mathcal{S}.$$

Therefore, $\mathbf{0}\in\partial\mathbf{T}(\bar{p})$.

The graph of the IVF **T** is presented by the gray shaded region in figure 2. From figure 2, it is to be observed that there does not exist any $p(\neq\bar{p})\in\mathcal{S}$ such that $\mathbf{T}(p)\prec\mathbf{T}(\bar{p})=[-2,5]$. Hence, $\bar{p}=2$ is the ES to the IOP (18).

Remark 11 As in example 4, the result has been tested at $p=2$, which is an ES of the problem (18), readers may wonder how to determine the point of interest in general cases. For a detailed answer to this aspect, we refer to the article [38].

Example 5 Consider the following IOP:

$$\min_{p\in\mathcal{S}=[-6,6]\times[-6,6]} \mathbf{T}(p) = [2,7]\odot|p_1|\oplus[1,3]\odot|p_2|\oplus[6,15]. \tag{19}$$

Taking $\bar{p}=(0,0)$ we have

$$\mathbf{T}(p)\ominus_{gH}\mathbf{T}(\bar{p}) = [2,7]\odot|p_1|\oplus[1,3]\odot|p_2|$$

Therefore,

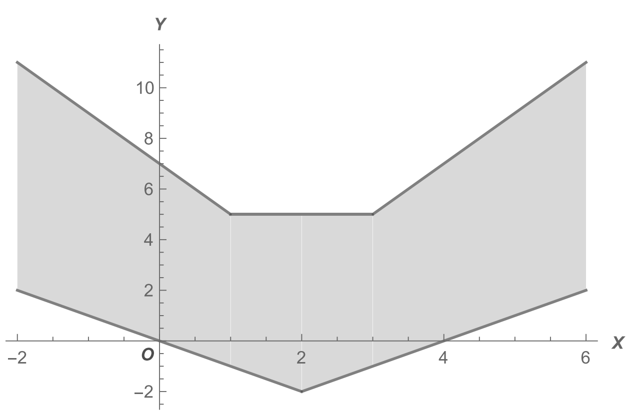

Figure 2. IVF **T** of the IOP (18) is depicted by gray region.

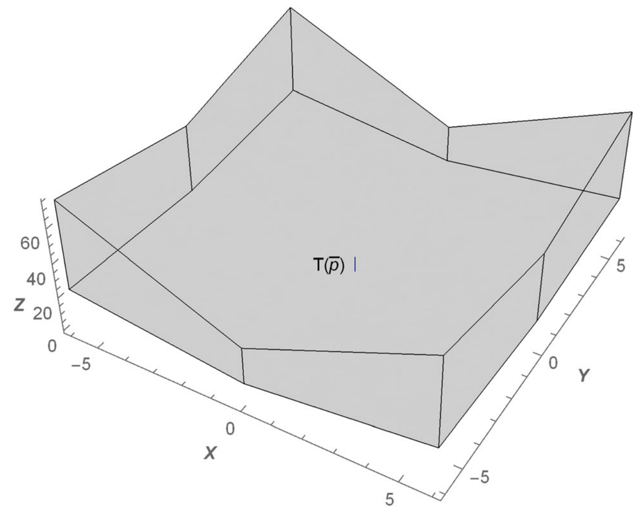

Figure 3. IVF **T** of the IOP (19) is illustrated by gray region and the value of $\mathbf{T}(\bar{p})$ is represented by blue line.

$$\mathbf{0} = (p-\bar{p})^t\odot\widehat{\mathbf{0}}\preceq\mathbf{T}(p)\ominus_{gH}\mathbf{T}(\bar{p})\forall p\in\mathcal{S}.$$

So, $\widehat{\mathbf{0}}\in\partial\mathbf{T}(\bar{p})$.

The graph of the IVF **T** is shown by the gray region in figure 3 and the interval $\mathbf{T}(\bar{p})$ is represented by the vertical line near $\mathbf{T}(\bar{p})$. From figure 3, it is to observe that there does not exist any $p(\neq\bar{p})\in\mathcal{S}$ such that $\mathbf{T}(p)\prec\mathbf{T}(\bar{p})=[6,15]$. Hence, $\bar{p}=(0,0)$ is an ES to the IOP (19).

The converse of Remark 10 is not true, which will be proved by the next example.

Example 6 Consider the following IOP:

$$\min_{p\in\mathcal{S}} \mathbf{T}(p) = [1,2]\odot p^2\ominus[0,2]\odot(p+1)\oplus[4,6], \tag{20}$$

where $\mathcal{S}=[-1,2]$.

Since $\underline{t}(p)=p^2-2p+2$ and $\overline{t}(p)=2p^2+6$ are convex on $\mathcal{S}$, the IVF **T** is convex on $\mathcal{S}$ by Lemma 1. Further, as $\underline{t}$ and $\overline{t}$ are differentiable in $\mathcal{S}$, the IVF **T** is *gH*-differentiable in $\mathcal{S}$ by Remark 3. Hence,

$$\partial\mathbf{T}(p) = \{\nabla\mathbf{T}(p)\} = \{[2,4]\odot p\ominus[0,2]\}\forall p\in\mathcal{S}.$$

The graph of the IVF **T** is presented by the gray shaded region in figure 4. From figure 4, It is clear that for any $\bar{p}\in[0,1]$, there does not exist any $p(\neq\bar{p})\in\mathcal{S}$ such that $\mathbf{T}(p)\prec\mathbf{T}(\bar{p})$. Therefore, each $\bar{p}\in[0,1]$ is an ES to the IOP (20). The region of the ESs of the IOP (20) is illustrated by bold black line on the *x*-axis in figure 4. However, for each $p\in[0,1]$,

$$\nabla\mathbf{T}(p) = [2p-2,4p]\neq\mathbf{0}$$

and hence, $\mathbf{0}\notin\partial\mathbf{T}(p)$.

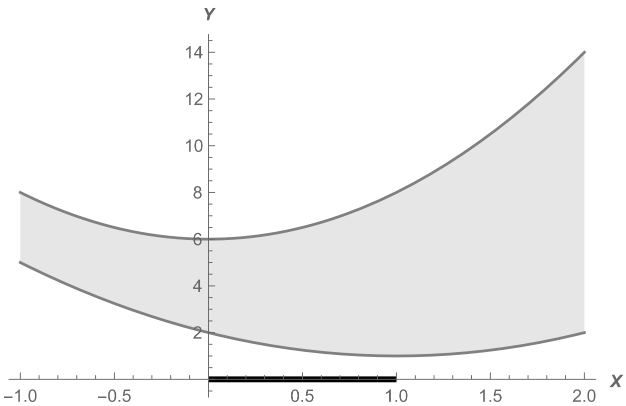

Figure 4. IVF **T** and ESs of the IOP (20) are depicted by gray shaded region and bold black line on x-axis, respectively.

Theorem 11 (Optimality condition). *Let $\mathcal{S}$ be convex and $\mathbf{T} : \mathcal{S} \to I(\mathbb{R})$ be a convex IVF. If there exists a $\widehat{\mathbf{G}} \in \partial \mathbf{T}(\bar{p})$ for some $\bar{p} \in \mathcal{S}$, such that*

$$(p - \bar{p})^t \odot \widehat{\mathbf{G}} \not\prec \mathbf{0} \forall p \in \mathcal{S}, \tag{21}$$

then $\bar{p}$ is an ES to the IOP (17).

Proof Let there exists a $\widehat{\mathbf{G}} \in \partial \mathbf{T}(\bar{p})$ for which the relation (21) is true. Then, by definition 17 of gH-subgradient and the relation (21), we obtain

$$\begin{aligned} &\mathbf{T}(p) \ominus_{gH} \mathbf{T}(\bar{p}) \not\prec \mathbf{0} \\ \Longrightarrow &\mathbf{T}(p) \not\prec \mathbf{T}(\bar{p}) \end{aligned}$$

for all $p \in \mathcal{S}$. Hence, $\bar{p}$ is an ES to the IOP (17). □

Remark 12 The converse of theorem 11 is not true. For example, consider the IOP (20) of example 6. We have seen that each point $\bar{p} \in [0, 1]$ is an ES to the IOP (20). However, at $\bar{p} = 0$,

$$(p - \bar{p})^t \odot \widehat{\mathbf{G}} = (p - \bar{p})^t \odot \nabla \mathbf{T}(\bar{p}) = [-2, 0] \odot p \prec \mathbf{0}$$

for all $p \in (0, 2] \subset \mathcal{S}$.

5. Conclusion and future directions

The concepts of gH-subgradients and gH-subdifferentials of convex IVFs with their several important characteristics have been provided in this article. It has been shown that the gH-subdifferential of a convex IVF is closed (Theorem 7) and convex (Theorem 8); the gH-subdifferential of a gH-differentiable convex IVF contains only gH-gradient (Theorem 5). It has been observed that on a real linear subspace if the gH-subgradients of a convex IVF at a point exists, then the directional gH-derivative of the IVF at that point in each direction is the maximum of all the products of gH-subgradients and the direction (Theorem 6). Also, it has been shown that a convex IVF is gH-Lipschitz continuous if it has gH-subgradient at each point in its domain (Theorem 9). The chain rule of a convex IVF (Theorem 10) and the gH-subgradient of the sum of finite numbers of convex IVFs (Remark 9) have been depicted. Furthermore, the relations between ESs of an IOP with gH-subgradient of its objective function have been illustrated (Remark 10 and theorem 11).

Although in this article, we have studied various properties of gH-subgradients and gH-subdifferentials of convex IVFs, we could not make any conclusion about the nonemptiness of gH-subdifferentials. In the future, we shall try to make a conclusion about the nonemptiness of gH-subdifferentials. Also, based on the proposed research, future research can be performed in the following directions.

- The concept of subdifferential of the dual problem of a constrained convex IOP can be illustrated.
- A gH-subgradient technique to obtain the whole solution set of a nonsmooth convex IOP can be derived.
- The derived results can be applied to solve *lasso problem* with interval-valued data.
- The notions of quasidifferentiability for IVFs without the help of its parametric representation can be illustrated.
- As IVFs are the special case of FVFs and IOPs are the special case of fuzzy optimization problems, similar results can be extended for FVFs and nonsmooth fuzzy optimization problems.

Appendix I. Proof of norm on $I(\mathbb{R})^n$

Proof

$$\|\widehat{\mathbf{U}}\|_{I(\mathbb{R})^n} = \sqrt{\sum_{i=1}^{n} \|\mathbf{U}_i\|^2_{I(\mathbb{R})}},$$

(i) For any element $\widehat{\mathbf{U}} = (\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_n) \in I(\mathbb{R})^n$, we have

$$\|\widehat{\mathbf{U}}\|_{I(\mathbb{R})^n} = \sqrt{\sum_{i=1}^{n} \|\mathbf{U}_i\|^2_{I(\mathbb{R})}} \geq 0, \text{since} \|\mathbf{U}_i\|_{I(\mathbb{R})} \geq 0 \forall i$$

and

$$\begin{aligned} \|\widehat{\mathbf{U}}\|_{I(\mathbb{R})^n} = 0 &\iff \|\mathbf{U}_i\|_{I(\mathbb{R})} = 0 \forall i \\ &\iff \mathbf{U}_i = \mathbf{0} \forall i \\ &\iff \widehat{\mathbf{U}} = \widehat{\mathbf{0}} = (\mathbf{0}, \mathbf{0}, \cdots, \mathbf{0}). \end{aligned}$$

(ii) For any $\gamma \in \mathbb{R}$ and an element $\widehat{\mathbf{U}} \in I(\mathbb{R})^n$, we obtain

$$\begin{aligned}\|\gamma \odot \widehat{\mathbf{U}}\|_{I(\mathbb{R})^n} &= \sqrt{\sum_{i=1}^{n} \|\gamma \odot \mathbf{U}_i\|^2_{I(\mathbb{R})}}\\ &= |\gamma| \sqrt{\sum_{i=1}^{n} \|\mathbf{U}_i\|^2_{I(\mathbb{R})}}\\ &= |\gamma| \|\widehat{\mathbf{U}}\|_{I(\mathbb{R})^n}.\end{aligned}$$

(iii) For any two elements $\widehat{\mathbf{U}}, \widehat{\mathbf{V}} \in I(\mathbb{R})^n$, we have

$$\|\widehat{\mathbf{U}} \oplus \widehat{\mathbf{V}}\|_{I(\mathbb{R})^n} = \sqrt{\sum_{i=1}^{n} \|\mathbf{U}_i \oplus \mathbf{V}_i\|^2_{I(\mathbb{R})}}.$$

Without loss of generality, due to Definition 3, let

$$\|\mathbf{U}_i \oplus \mathbf{V}_i\|_{I(\mathbb{R})} = \begin{cases} |\underline{u}_i + \underline{v}_i| & \text{for } i = 1, 2, \cdots, m(\leq n) \\ |\overline{u}_i + \overline{v}_i| & \text{for } i = m+1, p+2, \cdots, n. \end{cases}$$

Therefore,

$$\begin{aligned}&\sqrt{\sum_{i=1}^{n} \|\mathbf{U}_i \oplus \mathbf{V}_i\|^2_{I(\mathbb{R})}}\\ &= \sqrt{\sum_{j=1}^{m} |\underline{u}_j + \underline{v}_j|^2 + \sum_{k=m+1}^{n} |\overline{u}_k + \overline{v}_k|^2}\\ &\leq \sqrt{\sum_{j=1}^{m} |\underline{u}_j|^2 + \sum_{k=m+1}^{n} |\overline{u}_k|^2}\\ &\quad + \sqrt{\sum_{j=1}^{m} |\underline{v}_j|^2 + \sum_{k=m+1}^{n} |\overline{v}_k|^2}\\ &\quad \text{by Minkowski inequality}\\ &\leq \sqrt{\sum_{i=1}^{n} \|\mathbf{U}_i\|^2} + \sqrt{\sum_{i=1}^{n} \|\mathbf{V}_i\|^2} \text{due to Definition 2.1.}\end{aligned}$$

Thus,

$$\|\widehat{\mathbf{U}} \oplus \widehat{\mathbf{V}}\|_{I(\mathbb{R})^n} \leq \sqrt{\sum_{i=1}^{n} \|\mathbf{U}_i\|^2} + \sqrt{\sum_{i=1}^{n} \|\mathbf{V}_i\|^2}$$

Hence, the function $\|\cdot\|_{I(\mathbb{R})^n}$ is a norm on $I(\mathbb{R})^n$. □

Appendix II. Proof of norm on $\widehat{\mathbf{Y}}$

Proof

(i) Since $\|\mathbf{L}(p)\|_{I(\mathbb{R})} \geq 0$ and $\|p\| > 0$,

$$\|\mathbf{L}\|_{\widehat{\mathcal{X}}} = \sup_{p \neq 0} \frac{\|\mathbf{L}(p)\|_{I(\mathbb{R})}}{\|p\|} \geq 0 \forall p \in \mathcal{X},$$

and

$$\begin{aligned}\|\mathbf{L}\|_{\widehat{\mathcal{X}}} = 0 &\Longleftrightarrow \sup_{p \neq 0} \frac{\|\mathbf{L}(p)\|_{I(\mathbb{R})}}{\|p\|} = 0\\ &\Longleftrightarrow \|\mathbf{L}(p)\|_{I(\mathbb{R})} = 0 \forall p \in \mathcal{X}\\ &\Longleftrightarrow \mathbf{L}(p) = \mathbf{0} \forall p \in \mathcal{X}\\ &\Longleftrightarrow \mathbf{L} \text{is the interval-valued zero mapping;}\end{aligned}$$

by an interval-valued zero mapping we mean an IVF which maps each element of its domain to $\mathbf{0} = [0, 0]$.

(ii) Let $\mathbf{L} \in \widehat{\mathcal{X}}$ and $\gamma \in \mathbb{R}$. Then,

$$\|(\gamma \odot \mathbf{L})\|_{\widehat{\mathcal{X}}} = \sup_{p \neq 0} \frac{\|(\gamma \mathbf{L})(p)\|_{I(\mathbb{R})}}{\|p\|} = |\gamma| \|\mathbf{L}\|_{\widehat{\mathcal{X}}}.$$

(iii) Let $\mathbf{L}_1, \mathbf{L}_2 \in \widehat{\mathcal{X}}$. Then,

$$\begin{aligned}\|\mathbf{L}_1 \oplus \mathbf{L}_2\|_{\widehat{\mathcal{X}}} &= \sup_{p \neq 0} \frac{\|\mathbf{L}_1(p) \oplus \mathbf{L}_2(p)\|_{I(\mathbb{R})}}{\|p\|}\\ &\leq \sup_{p \neq 0} \frac{\|\mathbf{L}_1(p)\|_{I(\mathbb{R})} + \|\mathbf{L}_2(p)\|_{I(\mathbb{R})}}{\|p\|}\\ &= \|\mathbf{L}_1\|_{\widehat{\mathcal{X}}} + \|\mathbf{L}_2\|_{\widehat{\mathcal{X}}}.\end{aligned}$$

□

Appendix III. Proof of Lemma 3

Proof For all $p \in \mathcal{X}$, we have

$$\mathbf{L}(p) \preceq \mathbf{B} \odot \|p\|, \tag{22}$$

i.e.,

$$[\underline{l}(p), \overline{l}(p)] \preceq [\underline{b}\|p\|, \overline{b}\|p\|].$$

Therefore,

$$\underline{l}(p) \leq \underline{b}\|p\| \text{ and } \overline{l}(p) \leq \overline{b}\|p\|. \tag{23}$$

Replacing p by $-p$ in the relation (22), we get

$$\begin{aligned}&\mathbf{L}(-p) \preceq \mathbf{B} \odot \|p\|\\ &\Longrightarrow (-1) \odot \mathbf{L}(p) \preceq \mathbf{B} \odot \|p\|\\ &\Longrightarrow (-1) \odot \mathbf{B} \odot \|p\| \preceq \mathbf{L}(p)\\ &\Longrightarrow [-\overline{b}\|p\|, -\underline{b}\|p\|] \preceq [\underline{l}(p), \overline{l}(p)],\end{aligned}$$

which implies

$$\underline{l}(p) \geq -\overline{b}\|p\| \text{ and } \overline{l}(p) \geq -\underline{b}\|p\|. \tag{24}$$

By the inequalities (23) and (24) we obtain

$$\begin{aligned} & -\overline{b}\|p\| \leq \underline{l}(p) \leq \underline{b}\|p\| \text{ and } -\underline{b}\|p\| \leq \overline{l}(p) \leq \overline{b}\|p\| \\ \Longrightarrow & |\underline{l}(p)| \leq \max\{|\underline{b}|\|p\|, |\overline{b}|\|p\|\} \\ & \text{and} |\overline{l}(p)| \leq \max\{|\underline{b}|\|p\|, |\overline{b}|\|p\|\} \\ \Longrightarrow & \max\{|\underline{l}(p)|, |\overline{l}(p)|\} \leq \max\{|\underline{b}|\|p\|, |\overline{b}|\|p\|\} \\ \Longrightarrow & \|\mathbf{L}(p)\|_{I(\mathbb{R})} \leq \|\mathbf{B}\|_{I(\mathbb{R})}\|p\| \end{aligned}$$

for all $p \in \mathcal{X}$. □

Appendix IV. Proof of Lemma 5

Proof In view of Remark 1, without loss of generality, let the first m components of d be nonnegative and the rest $n-m$ components be negative. Therefore, $d^t \odot \widehat{\mathbf{U}}$ can be written as

$$\begin{aligned} d^t \odot \widehat{\mathbf{U}} &= \bigoplus_{i=1}^{n} d_i \odot \mathbf{U}_i \\ &= \bigoplus_{i=1}^{m} d_i \odot \mathbf{U}_i \oplus \bigoplus_{j=m+1}^{n-p} d_j \odot \mathbf{U}_j \\ &= \bigoplus_{i=1}^{m} [\underline{u}_i d_i, \overline{u}_i d_i] \oplus \bigoplus_{j=m+1}^{n} [\overline{u}_j d_j, \underline{u}_j d_j] \\ &= \left[\sum_{i=1}^{m} \underline{u}_i d_i + \sum_{j=m+1}^{n} \overline{u}_j d_j, \sum_{i=1}^{m} \overline{u}_i d_i + \sum_{j=m+1}^{n} \underline{u}_j d_j\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} & d^t \odot \widehat{\mathbf{U}} \preceq [c, c] \\ \Longrightarrow & \sum_{i=1}^{m} \underline{u}_i d_i + \sum_{j=m+1}^{n} \overline{u}_j d_j \leq \sum_{i=1}^{m} \overline{u}_i d_i + \sum_{j=m+1}^{n} \underline{u}_j d_j \leq c \\ \Longrightarrow & w \sum_{i=1}^{n} \underline{u}_i d_i + w' \sum_{i=1}^{n} \overline{u}_i d_i \leq 2c \\ \Longrightarrow & \sum_{i=1}^{n} (w\underline{u}_i + w'\overline{u}_i) d_i \leq 2c \\ \Longrightarrow & d^t \mathcal{W}\left(\widehat{\mathbf{U}}\right) \leq 2c. \end{aligned}$$

□

Appendix V. Proof of Lemma 6

Proof Let

$$\begin{aligned} & \|\mathcal{W}\left(\widehat{\mathbf{U}}\right)\| \\ = & \sqrt{(w\underline{u}_1 + w'\overline{u}_1)^2 + (w\underline{u}_2 + w'\overline{u}_2)^2 + \cdots + (w\underline{u}_n + w'\overline{u}_n)^2}. \end{aligned}$$

be finite. Therefore, all $\underline{u}_i$'s and $\overline{u}_i$'s are finite. Hence,

$$\|\widehat{\mathbf{U}}\|_{I(\mathbb{R})^n} = \sqrt{\sum_{i=1}^{n} \|\mathbf{U}_i\|_{I(\mathbb{R})}^2} = \sqrt{\sum_{i=1}^{n} \max\{|\underline{u}_i|, |\overline{u}_i|\}^2}$$

is finite. □

Appendix VI. Proof of Lemma 7

Proof Let $p^t \odot \widehat{\mathbf{U}} = \mathbf{V}$. According to Definition 3, we have

$$p^t \odot \widehat{\mathbf{U}} = \mathbf{V} \preceq \left[\|\mathbf{V}\|_{I(\mathbb{R})}, \|\mathbf{V}\|_{I(\mathbb{R})}\right],$$

which implies

$$p^t \odot \widehat{\mathbf{U}} \preceq \|p\| \odot \left[\|\widehat{\mathbf{U}}\|_{I(\mathbb{R})^n}, \|\widehat{\mathbf{U}}\|_{I(\mathbb{R})^n}\right]$$

because

$$\begin{aligned} \|\mathbf{V}\|_{I(\mathbb{R})} &= \|p_1 \odot \mathbf{U}_1 \oplus p_2 \odot \mathbf{U}_2 \oplus \cdots \oplus p_n \odot \mathbf{U}_n\|_{I(\mathbb{R})} \\ &\leq \|p_1 \odot \mathbf{U}_1\|_{I(\mathbb{R})} + \|p_2 \odot \mathbf{U}_2\|_{I(\mathbb{R})} + \cdots \\ &\quad + \|p_n \odot \mathbf{U}_n\|_{I(\mathbb{R})} \\ &= |p_1|\|\mathbf{U}_1\|_{I(\mathbb{R})} + |p_2|\|\mathbf{U}_2\|_{I(\mathbb{R})} + \cdots + |p_n|\|\mathbf{U}_n\|_{I(\mathbb{R})} \\ &\leq \|p\|\left(\|\mathbf{U}_1\|_{I(\mathbb{R})} + \|\mathbf{U}_2\|_{I(\mathbb{R})} + \cdots + \|\mathbf{U}_n\|_{I(\mathbb{R})}\right) \\ &= \|p\|\|\widehat{\mathbf{U}}\|_{I(\mathbb{R})^n}. \end{aligned}$$

□

Appendix VII. Proof of Lemma 8

Proof Since $\mathbf{T}(p) \ominus_{gH} \mathbf{T}(q) \preceq \mathbf{B} \odot \|p-q\|$, for all $p, q \in \mathcal{S}$, we have

$$\underline{t}(p) - \underline{t}(q) \leq b\|p-q\| \text{ and } \overline{t}(p) - \overline{t}(q) \leq b\|p-q\|. \quad (25)$$

Interchanging p and q in the inequalities (25), we obtain

$$\underline{t}(p) - \underline{t}(q) \leq b\|p-q\| \text{ and } \overline{t}(p) - \overline{t}(q) \leq b\|p-q\| \forall p, q \in \mathcal{S}. \quad (26)$$

With the help of the inequalities (25) and (26), we get

$$\begin{aligned} & |\underline{t}(p) - \underline{t}(q)| \leq b\|p-q\| \text{ and } \\ & |\overline{t}(p) - \overline{t}(q)| \leq b\|p-q\| \forall p, q \in \mathcal{S}, \end{aligned}$$

which implies

$$\|\mathbf{T}(p) \ominus_{gH} \mathbf{T}(q)\|_{I(\mathbb{R})} \leq b\|p-q\| \forall p, q \in \mathcal{S}.$$

□

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