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Inductive algebras for the affine group of a finite field

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Abstract
Each irreducible representation of the affine group of a finite field has a unique maximal inductive algebra, and it is self-adjoint.

Keywords Inductive algebra · Induced representation · Affine group · Finite field

Mathematics Subject Classification 20C15

1 Introduction

Let G be a separable locally compact group and π an irreducible unitary representation of G on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded operators on $\mathcal{H}$. An *inductive algebra* is a weakly closed abelian sub-algebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ that is normalized by $\pi(G)$, i.e., $\pi(g)\mathcal{A}\pi(g)^{-1} = \mathcal{A}$ for each $g \in G$. If we wish to emphasize the dependence on π, we will use the term π-inductive algebra. A *maximal inductive algebra* is a maximal element of the set of inductive algebras, partially ordered by inclusion.

The identification of inductive algebras can shed light on the possible realizations of $\mathcal{H}$ as a space of sections of a homogeneous vector bundle (see e.g. [7–10]). For self-adjoint maximal inductive algebras there is a precise result known as Mackey's Imprimitivity Theorem, as explained in the introduction to [7]. Inductive algebras have also found applications in operator theory (see e.g. [3, 4]).

In [6], it was shown that finite dimensional inductive algebras for a connected group are trivial. However, there are interesting inductive algebras for finite groups (see e.g. [5]). In this note, we classify the maximal inductive algebras for the representations of the affine group (the "$ax + b$" group) of a finite field.

In Sect. 2, we recall the structure of the affine group of a finite field, and set up the notation. In Sect. 3, we recall its representation theory, and formulate our main result. The main result is proved in Sect. 4.

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2 The affine group of a finite field

Let k be a finite field of order $q = p^n$, where p is prime. Let $k^\times$ denote the multiplicative group of non-zero elements of k. Recall that the affine group of k is the group G of affine automorphisms of k. Thus an element g of G is a map $g : k \to k$ of the form $g(x) = ax + b$ where $a \in k^\times$ and $b \in k$, and the group law is composition. The group G may be identified with the group of matrices

$$\left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \middle| \; a \in k^\times, \; b \in k \right\}.$$

Let $\iota : k \to G$, $p : G \to k^\times$ and $s : k^\times \to G$ be defined by

$$\iota(b) = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad p\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right) = a, \text{ and } \quad s(a) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}.$$

Then ι, p and s are homomorphisms, $\iota(k) \lhd G$ and

$$0 \longrightarrow k \xrightarrow{\;\iota\;} G \underset{s}{\overset{p}{\rightleftarrows}} k^\times \longrightarrow 1$$

is an exact sequence with splitting s. Thus G is a semidirect product $k^\times \ltimes k$. We note for future reference that $s(a')\iota(b)s(a')^{-1} = \iota(a'b)$.

3 The representations and their inductive algebras

The irreducible unitary representations of G may be constructed using the *Mackey machine* (see [1,Sect. 3.9]). There are $q - 1$ characters (one-dimensional representations), and one $(q - 1)$-dimensional representation (up to unitary equivalence).

Obviously, the characters have only the trivial inductive algebra $\mathbb{C}$, which is self-adjoint.

The $(q - 1)$-dimensional representation is

$$\pi = \mathrm{Ind}_{\iota(k)}^{G} \chi,$$

where $\chi : k \to \mathbb{C}^\times$ is a non-trivial homomorphism (i.e. $\chi \not\equiv 1$). Let $\mathcal{H}$ denote the Hilbert space of all complex-valued functions on $k^\times$ equipped with the inner product

$$\langle F_1, F_2 = \sum_{a' \in k^\times} F_1(a')\overline{F_2(a')}.$$

The representation π may be realized on $\mathcal{H}$ by

$$(\pi(g)F)(a') = \chi(a'b)F(a'a), \qquad g = \iota(b)s(a).$$

For each $\varphi \in \mathcal{H}$, let $m_\varphi : \mathcal{H} \to \mathcal{H}$ be defined by $m_\varphi(F) = \varphi F$. Let

$$\mathcal{B} = \{m_\varphi \mid \varphi \in \mathcal{H}\}.$$

Then $\mathcal{B}$ is a maximal-abelian subalgebra of $\mathcal{B}(\mathcal{H})$, and $\mathcal{B}$ is π-inductive. Therefore $\mathcal{B}$ is a maximal π-inductive algebra. Moreover, it is self-adjoint. Our main result is the following theorem.

Theorem 1 *$\mathcal{B}$ is the only maximal π-inductive algebra.*

4 The proof

Let $\mathcal{A}$ be a maximal π-inductive algebra.

Lemma 2 *There are no non-zero nilpotent elements in* $\mathcal{A}$.

Proof Let $\mathcal{N}$ denote the set of nilpotent elements in $\mathcal{A}$ (the nilradical of $\mathcal{A}$). Let

$$\mathcal{K} = \{F \in \mathcal{H} \mid T(F) = 0, \quad \forall T \in \mathcal{N}\}.$$

By (a trivial case of) Engel's theorem [2,Sect. 3.3], $\mathcal{K} \neq 0$. Observe that $\mathcal{N}$ is normalized by $\pi(G)$, so $\mathcal{K}$ is $\pi(G)$-invariant. However, since π is irreducible, it follows that $\mathcal{K} = \mathcal{H}$, whence $\mathcal{N} = 0$. □

Corollary 3 $\dim(\mathcal{A}) \leq q - 1$.

Proof By Lemma 2, the Jordan–Chevalley decomposition [2,§4.2], and the fact that $\mathcal{A}$ is abelian, it follows that there is a (not necessarily orthonormal) basis for $\mathcal{H}$ in which each element of $\mathcal{A}$ is diagonal. Since $\dim(\mathcal{H}) = q - 1$, the result follows. □

For $b' \in k$, define $\kappa(b') : \mathcal{A} \to \mathcal{A}$ by $\kappa(b')T = \pi(\iota(b'))T\pi(\iota(b'))^{-1}$. Then κ is a representation of the finite abelian group k on the vector space $\mathcal{A}$, which decomposes as

$$\mathcal{A} = \bigoplus_{b \in k} \mathcal{A}_b$$

where

$$\mathcal{A}_b = \{T \in \mathcal{A} \mid \kappa(b')T = \chi(bb')T, \quad \forall b' \in k\}.$$

Here we are using the fact that every character of k is of the form χ_b where $\chi_b(b') = \chi(bb')$ for all $b' \in k$.

Observe that

(1) if $T \in \mathcal{A}_b$ and $T' \in \mathcal{A}_{b'}$, then $TT' \in \mathcal{A}_{b+b'}$, and
(2) for each $a \in k^\times$, the map $T \mapsto \pi(s(a))^{-1}T\pi(s(a))$ is a linear isomorphism $\mathcal{A}_b \to \mathcal{A}_{ab}$.

Lemma 4 $\mathcal{A}_1 = 0$.

Proof Suppose not. Then there exists a non-zero element $T \in \mathcal{A}_1$. By the first observation above, $T^p \in \mathcal{A}_0$. Since T is not nilpotent (see Lemma 2), it follows that $\dim(\mathcal{A}_0) \geq 1$. For $b \neq 0$, we have $\dim(\mathcal{A}_b) = \dim(\mathcal{A}_1) \geq 1$, by the second observation. Therefore

$$\dim(\mathcal{A}) = \sum_{b \in k} \dim(\mathcal{A}_b) \geq |k| = q,$$

contradicting Corollary 3. □

Lemma 5 $\mathcal{A}_0 \subseteq \mathcal{B}$.

Proof Observe that $\mathcal{H}$ is spanned by the set $\{\chi_{b'}|_{k^\times} \mid b' \in k\}$, hence $\mathcal{B}$ is spanned by the set $\{m_{(\chi_{b'}|_{k^\times})} \mid b' \in k\}$. If $T \in \mathcal{A}_0$, then T commutes with $m_{(\chi_{b'}|_{k^\times})}$ for each $b' \in k$, whence T commutes with $\mathcal{B}$. Since $\mathcal{B}$ is maximal-abelian, it follows that $T \in \mathcal{B}$. □

By Lemma 4, and the second observation above, it follows that $\mathcal{A}_b = 0$ for all $b \in k^\times$, whence $\mathcal{A} = \mathcal{A}_0 \subseteq \mathcal{B}$. Since $\mathcal{A}$ is maximal, it follows that $\mathcal{A} = \mathcal{B}$.

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