An approximate approach for a Stefan problem governed with space–time fractional derivatives

5.1 Introduction

In recent years, fractional derivatives have been used in various mathematical models by many researchers due to its applicability in different fields of science and engineering (Chaves (1998), Benson et al. (2000), Aoki et al. (2008), Jiang et al.(2012)). It is well known that fractional derivative is a good tool for taking into account memory mechanism, particularly in some diffusive processes (Tomovskia et al. (2012)). Stefan problems (moving boundary problems) with fractional derivatives ((Li et al. (2007, 2008), Liu and Xu (2009), Vogel et al. (2012), Das et al. (2011)) are typical problems from point of view of mathematics because of its non-linear nature and presence of moving interface. Some exact solutions of Stefan problems can be seen in the papers of Liu and Xu (2009), Voller (2010) and Zhou et al. (2014). Many Stefan problems are available in the literature whose exact solution is not known. Therefore, several approximate analytical methods (Li et al. (2009), Rajeev and Kushwaha (2013), Das and Rajeev (2010), Grzymkowski and Słota (2006), Rajeev et al.(2013)) have been used to solve the Stefan problems with fractional derivatives. The approximate analytical method taken in this literature is optimal homotopy asymptotic method.

Optimal homotopy asymptotic method was developed by Marinca and Herisanu (2008) and it has been applied to solve a wide class of non-linear differential equations (Marinca and Herisanu (2010a, 2010b), Iqbal et al.(2010), Iqbal and Javed (2011), Hashmi et al.(2012)). Ghoreishi et al. (2012) presented the comparison between homotopy analysis method and optimal homotopy asymptotic method for nonlinear age-structured population models. In 2013, Dinarvand and Hosseini (2013) also used this technique to investigate the temperature distribution equation in a convective straight fin with temperature-dependent thermal conductivity and the convective–radiative cooling of a lumped system with variable specific heat.

In this chapter, a mathematical model for a Stefan problem (Zhou et al. (2014)) with space-time fractional derivative is presented. In this model, optimal homotopy asymptotic method is used to find the expression of temperature distribution in given domain and location of moving interface with the help of Taylor's series. The obtained results are compared with the existing exact solution for standard case and are in good agreement. An approachable analysis for fractional case is also discussed.

5.2 Mathematical Formulation

In this section, a mathematical model of one-dimensional Stefan problem with latent heat a power function of position (Zhou et al. (2014)) is considered. For this problem, a fractional model is presented. This model involves space-time fractional derivatives as given by Voller (2010). The governing equations are as follow:

$$D_t^{\beta} T = \nu \frac{\partial}{\partial x}\left(D_x^{\alpha} T\right), \quad 0 < x < s(t),\ t > 0, \tag{5.2.1}$$

$$k\, D_x^{\alpha} T\left(x = 0, t > 0\right) = -b t^{(n-1)/2}, \tag{5.2.2}$$

$$T\left(s(t), t\right) = 0, \qquad t > 0, \tag{5.2.3}$$

$$k\, D_x^{\alpha}\left(T(s(t), t)\right) = -\gamma\, (s(t))^n D_t^{\beta} s(t)\,, \tag{5.2.4}$$

where $T(x,t)$ is the temperature distribution, $s(t)$ is moving interface, k is thermal conductivity, ν is the thermal diffusion coefficient, b is a constant ($b > 0$ for melting, $b<0$ for freezing), $\gamma(s(t))^n$ is the variable latent heat per unit volume and n is an non negative integer.

5.3 Solution of the problem

First we write Eqs. (5.2.1- 5.2.4) in operator form as follows:

$$\nu L(T(x,t)) - N(T(x,t)) = 0, \tag{5.3.5}$$

$$B\left(T, \frac{\partial T}{\partial x}\right) = 0, \tag{5.3.6}$$

where $L\left(=\dfrac{\partial^{1+\alpha}}{\partial x^{1+\alpha}}\right)$ is a linear operator, $N\left(=\dfrac{\partial^{\beta}}{\partial t^{\beta}}\right)$ is nonlinear operator and B is boundary operator.

According to optimal homotopy asymptotic method (Grzymkowski and Słota (2006), Iqbal et al. (2010)), constructing an optimal as:

$$T(x,t,p):[0,s(t)]\times[0,1]\to R,$$

which satisfies

$$(1-p)[\nu L(T(x,t,p))] = H(p)[\nu L(T(x,t,p)) - N(T(x,t,p))], \tag{5.3.7}$$

$$B\left(T(x,t,p), \frac{\partial T(x,t,p)}{\partial x}\right) = 0, \tag{5.3.8}$$

where $p\in[0,1]$ is an embedding parameter, $T(x,t;p)$ is an unknown function, $H(p)$ is a nonzero auxiliary function for $p\neq 0$ and $H(0)=0$.

Obviously, if $p=0$,

$$T(x,t;0) = T_0(x,t), \tag{5.3.9}$$

and when $p=1$ then

$$T(x,t;1)=T(x,t). \tag{5.3.10}$$

Therefore, as p increase from 0 to1, the unknown function $T(x,t,p)$ varies from $T_0(x,t)$ to the solution $T(x,t)$.

Now, we choose the auxiliary function $H(p)$ in the following form:

$$H(p)=pc_1+p^2c_2+p^3c_3+\cdots, \tag{5.3.11}$$

where $c_{1,}\, c_{2,}\, c_3\cdots$ are constants to be determined later.

Considering the solution of (5.3.7) in the following series form:

$$T(x,t;p,c_i)=\sum_{n=0}^{\infty}T_n(x,t,c_i)p^n,\; i=0,1,2,\cdots l, \tag{5.3.12}$$

and

$$s(t)=\sum_{n=0}^{\infty}p^n s_n(t), \tag{5.3.13}$$

where $c_0=0$ and $T_0(x,t,0)=T_0(x,t)$.

As given by Ghoreishi et al. (2012), expanding $N(T(x,t;p,c_j))$ in the following series form:

$$N(T(x,t;p,c_j))=N_0(T_0)+\sum_{m\geq 1}N_m(T_0,T_1,T_2,\cdots T_m)p^m,\; j=1,2,\cdots. \tag{5.3.14}$$

Now, substituting Eqs. (5.3.12), (5.3.14) into Eq. (5.3.7) and equating the coefficients of like powers of p, the following problems are obtained:

$$p^0:\quad L(T_0(x,t))=0, \tag{5.3.15}$$

$$p^1:\quad \nu\, L(T_1(x,t))=-c_1N_0(T_0(x,t)), \tag{5.3.16}$$

$$p^2: \ \nu L(T_2(x,t)) - \nu L(T_1(x,t)) = c_1 \nu L(T_1(x,t)) - c_2 N_0(T_0(x,t)) - c_1 N_1(T_0(x,t), T_1(x,t)), \tag{5.3.17}$$

and the general equation for $T_k(x,t)$ are given as:

$$\nu L(T_k(x,t)) = \nu L(T_{k-1}(x,t)) - c_k N_0(T_0(x,t)) + \sum_{i=1}^{k-1} c_i [\nu L(T_{k-i}(x,t)) - N_{k-i}(T_0(x,t), T_1(x,t), \cdots T_{k-1}(x,t))], \tag{5.3.18}$$

where $k = 2,3,\cdots$.

Substituting (5.3.12), (5.3.13) in the Eq. (5.2.2) and Eq. (5.2.3), one can obtain

$$k D_x^{\alpha} \left(\sum_{n=0}^{\infty} T_n(x = 0, t, c_i) \right) p^n = -bt^{(n-1)/2} , \tag{5.3.19}$$

and

$$\sum_{n=0}^{\infty} T_n \left(\sum_{n=0}^{\infty} p^n s_n , t, c_i \right) p^n = 0, \tag{5.3.20}$$

where $i = 0,1, 2,\cdots l.$

For the comparisons of various powers of p, expending $T_i(x,t)$ in Taylor's series form (Li et al. (2009), Rajeev and Kushwaha (2013)) about a point (s_0, t) as:

$$T_l(x,t,c_i) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n T_l(s_0, t, c_i)}{\partial x^n} (x - s_0)^n , \tag{5.3.21}$$

where $l = 0,1,2,3\cdots$ and $i = 0,1,2,3\cdots,l$.

From Eqs. (5.3.20) and (5.3.21), we have

$$\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{p^l}{m!} \left(\sum_{n=1}^{\infty} p^n s_n \right)^m \frac{\partial^m}{\partial x^m} T_l(s_0, t, c_i) = 0 . \tag{5.3.22}$$

The interface condition (5.2.4) becomes

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{p^l}{m!}\left(\sum_{n=1}^{\infty}p^n s_n(t)\right)^m \frac{\partial^{m+\alpha}}{\partial x^{m+\alpha}}T_l(s_0,t,c_i)=-\gamma\left(\sum_{m=0}^{\infty}p^m(s_m(t))\right)^n D_t^{\beta}\left(\sum_{m=0}^{\infty}p^m(s_m(t))\right). \quad (5.3.23)$$

Considering Eq. (5.3.15) and comparing the coefficients of power of p^0 from Eqs. (5.3.19), (5.3.22) and (5.3.23), then the following system can be obtained:

$$\begin{cases}\frac{\partial}{\partial x}\left(D_x^{\alpha}T_0(x,t)\right)=0,\\ k\,D_x^{\alpha}\left(T_0(x=0,t)\right)=-bt^{(n-1)/2},\\ T_0(s_0,t)=0,\\ k\frac{\partial^{\alpha}T_0(s_0,t)}{\partial x^{\alpha}}=-\gamma\left(s_0(t)\right)^n D_t^{\beta}\left(s_0(t)\right).\end{cases} \quad (5.3.24)$$

Taking Eq. (5.3.16) and comparing the coefficients of power of p^1 from Eqs. (5.3.19), (5.3.22) and (5.3.23), we have

$$\begin{cases}\nu\frac{\partial}{\partial x}\left(D_x^{\alpha}T_1(x,t,c_1)\right)=c_1\,D_t^{\beta}\left(T_0(x,t)\right),\\ D_x^{\alpha}\left(T_0(0,t)\right)=0,\\ T_1(s_0,t,c_1)+s_1\frac{\partial T_0(s_0,t)}{\partial x}=0,\\ \frac{\partial^{\alpha}T_1(s_0,t,c_1)}{\partial x^{\alpha}}+s_1\frac{\partial^{1+\alpha}T_0(s_0,t)}{\partial x^{1+\alpha}}=-\frac{\gamma}{k}\left((s_0)^n D_t^{\beta}\left(s_1(t)\right)+n(s_0)^{n-1}s_1\,D_t^{\beta}\left(s_0\right)\right).\end{cases} \quad (5.3.25)$$

⋮

and so on.

It is calculated that the solutions of zeroth-order problem (given in (5.3.24)) are

$$T_0(x,t)=\frac{b}{k\,\Gamma(1+\alpha)}\left(s_0^{\alpha}-x^{\alpha}\right)t^{(n-1)/2} \quad (5.3.26)$$

and

$$s_0 = a_0\, t^{\theta}\,, \tag{5.3.27}$$

where

$$\theta = \frac{\beta + (n-1)/2}{n+1} \quad \text{and} \quad a_0 = \left(\frac{b\,\Gamma(1+\theta-\beta)}{\gamma\,\Gamma(1+\theta)}\right)^{\frac{1}{1+n}} .$$

Substituting T_0 and s_0 into first-order problem (5.3.25) and using the above process, we can obtain the following expressions of $T_1(x,t,c_1)$:

$$T_1(x,t,c_1) = \frac{c_1\, b}{\nu\, k}\left(m_3\left(x^{1+\alpha} - s_0{}^{1+\alpha}\right) t^{\alpha\theta + \frac{n-1}{2} - \beta} + \frac{m_2}{\Gamma(2+2\alpha)}\left(s_0{}^{1+2\alpha} - x^{1+2\alpha}\right) t^{\frac{n-1}{2} - \beta} \right)$$
$$+ \frac{\alpha b}{k\,\Gamma(1+\alpha)}\, s_1 s_0{}^{1-\alpha}\, t^{\frac{n-1}{2}}\,, \tag{5.3.28}$$

where $m_1 = \dfrac{\Gamma(1+\frac{n-1}{2}+\alpha\theta)}{\Gamma(1+\frac{n-1}{2}+\alpha\theta-\beta)}$,

$$m_2 = \frac{\Gamma(1+\frac{n-1}{2})}{\Gamma(1+\frac{n-1}{2}-\beta)},$$

and $\quad m_3 = \dfrac{m_1\, a_0{}^{\alpha}}{\Gamma(1+\alpha)\Gamma(2+\alpha)}$.

The expression of $s_1(t)$ can be calculated as

$$s_1(t) = a_1\, t^{\phi} \tag{5.3.29}$$

where

$$a_1 = \frac{-c_1\, b\, a_0^{(1+\alpha-n)} \left(m_1 - \frac{m_2}{(1+\alpha)} \right)}{\nu\, \gamma\, \Gamma(1+\alpha) \left(\frac{\Gamma(1+\phi)}{\Gamma(1+\phi-\beta)} + n \frac{\Gamma(1+\theta)}{\Gamma(1+\theta-\beta)} \right)},$$

and

$$\phi = (1+\alpha-n)\theta + (n-1)/2 .$$

The approximate solution of temperature distribution is determined as:

$$T(x,t) = T_0(x,t) + T_1(x,t,c_1) + T_2(x,t,c_1,c_2) + \cdots . \tag{5.3.30}$$

and an approximate solution of $s(t)$ is given as:

$$s(t) = s_0(t) + s_1(t) + s_2(t) + \cdots . \tag{5.3.31}$$

In order to get the constants involved in the Eq. (5.3.30) of $T(x,t)$, Least Square Method is used (Ghoreishi et al.(2012)). For this purpose, we define the residual as:

$$R(x,t;c_1,c_2,\cdots c_l) = \nu\, L\left(\tilde{T}(x,t,c_1,c_2,\cdots c_l)\right) - N\left(\tilde{T}(x,t,c_1,c_2,\cdots c_l)\right). \tag{5.3.32}$$

where $\tilde{T}(x,t,c_1,c_2,\cdots c_l)$ is an approximate solution of $T(x,t)$ which can be found from Eq. (5.3.30).

Clearly, if $R(x,t;c_i)=0$ then $T(x,t;c_i)$ will be exact solution. Generally, optimal homotopy asymptotic method gives an approximate solution. Therefore, $R(x,t;c_i)\neq 0$ in such a case, but we can minimize the functional

$$J(c_i) = \int_0^t \int_0^{s(t)} R^2(x,t;c_i)\, dx dt , \tag{5.3.33}$$

where R is the residual. The constants $c_i\ (i = 1, 2 \cdots, l)$ can be optimally obtained from the following conditions:

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = \cdots = \frac{\partial J}{\partial c_l} = 0 . \qquad (5.3.34)$$

5.4 Numerical results and discussion

In this section, numerical results for interface position $s(t)$ are obtained and the results are presented through tables and figures. Tables (5.1-5.2) represent comparisons between exact and approximate value of positions of phase front $s(t)$ at particular times t for $\alpha = \beta = 1.0$ (standard motion). For standard problem, it is clear from the tables that our approximate results are sufficient accurate with the exact solution given by Zhou et al. (2014).

Figure 5.1 and Figure 5.2 represent the dependence of trajectory of the movement of phase front $s(t)$ on the thermal diffusion coefficient ν for n = 0 at $\alpha = 0.25,\ \beta = 0.75$ and $\alpha = 0.5,\ \beta = 1.0$, respectively. Figure 5.3 and Figure 5.4 also depict the dependence of path of phase front $s(t)$ on the thermal diffusion coefficient ν for n = 1 at $\alpha = 0.25,\ \beta = 0.75$ and $\alpha = 0.5,\ \beta = 1.0$, respectively. The figures (5.1-5.4) portray that movement of interface increases with the increase in the value of thermal diffusion coefficient for fractional cases (non-classical or non-Fickian) which is similar to the case of standard motion (Zhou et al. (2014)).

Figures (5.5-5.6) show variation of path of $s(t)$ for different value of b for non-classical or non-Fickian case. From these figures it is clear that the movement of phase front increases with the increase in the value of constant b i.e., the melting (or freezing) process becomes fast as the value of constant b increases.

b	**T**	**Exact value of S(t)**	**Approximate value of S(t) by OHAM**	**Absolute Error**
0.1	**0**	**0.0**	**0.0**	**0.0**
	5	**0.4428497413051738**	**0.4449844728214594**	**0.0021347315162856**
	10	**0.6262841102471934**	**0.6293030765095499**	**0.003018966262356377**
	15	**0.7668507025490686**	**0.77077357155060199**	**0.0036974634467252074**
	20	**0.8856994826103476**	**0.8899689456429188**	**0.0042694630325712**
	25	**0.990242125376565**	**0.9950155301606909**	**0.0047733404784125801**
0.25	**0**	**0.0**	**0.0**	**0.0**
	1	**0.4728215497051472**	**0.48477006891316846**	**0.0118791394265374**
	2	**0.6686706481752838**	**0.6854702882616138**	**0.016799640086330142**
	3	**08189509470027683**	**0.8395262200397257**	**0.02057527303695739**
	4	**0.9456430994102945**	**0.9694013782633693**	**0.0237582788530748**
	5	**1.0030059722629254**	**1.028205432392426**	**0.02519946012949521**
0.5	**0**	**0.0**	**0.0**	**0.0**
	0.25	**0.4193648240191325**	**0.4439987507075641**	**0.024633926688431618**
	0.5	**0.5930714217100636**	**0.627909054927348**	**0.03483763321728454**
	0.75	**0.7263611821083185**	**0.769028394722609**	**0.0426672126142905**
	1.0	**0.838729648038265**	**0.8879975014151282**	**0.049267853376863235**
	1.3	**0.9562989329952792**	**1.012472928662966**	**0.05617399566768704**

Table 5.1. Comparison between exact and approximate solution of $s(t)$ at $n = 0$

b	***t***	**Exact value of *S*(*t*)**	**Approximate value of *S*(*t*) by OHAM**	**Absolute error**
0.1	**0**	**0.0**	**0.0**	**0.0**
	1	**0.427525268155265**	**0.43321518187742797**	**0.005689913722162987**
	2	**0.60461203248237**	**0.6126587856369857**	**0.008046753154615675**
	3	**0.7404954859644275**	**0.7503507056218971**	**0.009855219657469638**
	4	**0.85505053631053**	**0.8664303637548559**	**0.011379827444325974**
	5	**0.9559755616939987**	**0.968698595562864**	**0.01272303386886529**
0.25	**0**	**0.0**	**0.0**	**0.0**
	0.5	**0.45275560721474817**	**0.471999375385378206**	**0.019243768139033897**
	1.0	**0.6402931201635628**	**0.6675079180569478**	**0.02721479789338499**
	1.5	**0.7841957151076419**	**0.8175268992535237**	**0.03333118414588188**
	2.0	**0.9055112144294963**	**0.9439987507075641**	**0.03848753627806779**
	2.5	**1.0123923149263712**	**1.0554226886284956**	**0.04303037370212442**
0.5	**0.0**	**0.0**	**0.0**	**0.0**
	0.25	**0.4222512866672956**	**0.45363176182240944**	**0.03138047515515383**
	0.5	**0.5971534963343791**	**0.6415321898924531**	**0.04437869355807393**
	0.75	**0.7313606820690868**	**0.7857132594033969**	**0.05435257733439995**
	1.0	**0.8445025733345912**	**0.9072635236448189**	**0.06276095031022766**
	1.25	**0.9441825805748236**	**1.0143514561879015**	**0.07016887561307779**

Table 5.2. Comparison between exact and approximate solution of $s(t)$ at *n* = 1

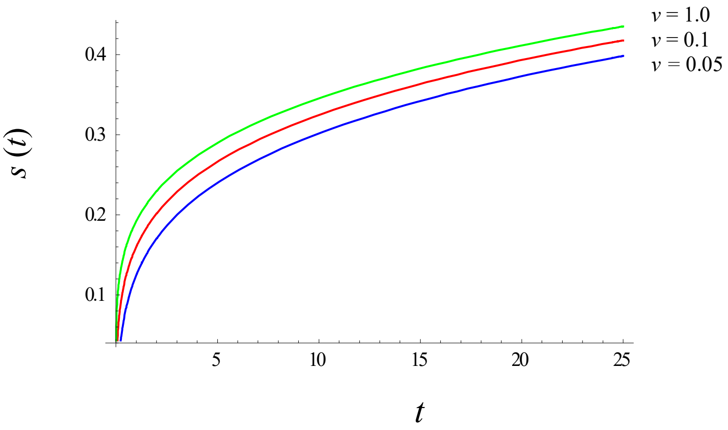

Fig.5.1. Plot of $s(t)$ **vs.** t **at** $\alpha = 0.25,\ \beta = 0.75$ and $\boldsymbol{n = 0}$

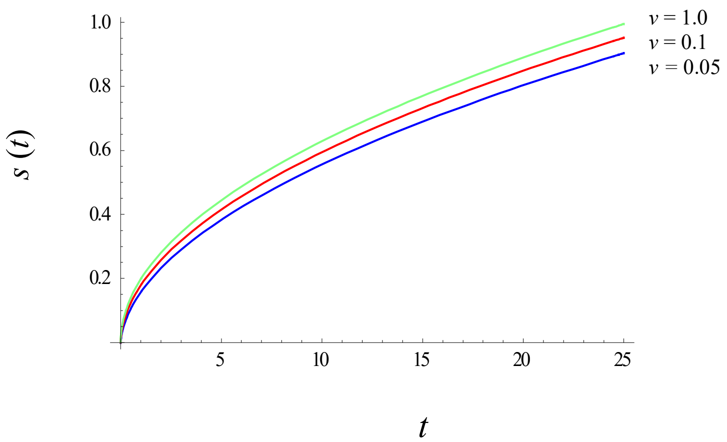

Fig. 5.2. Plot of $s(t)$ **vs.** t **at** $\alpha = 0.5,\ \beta = 1.0$ and $\boldsymbol{n = 0}$

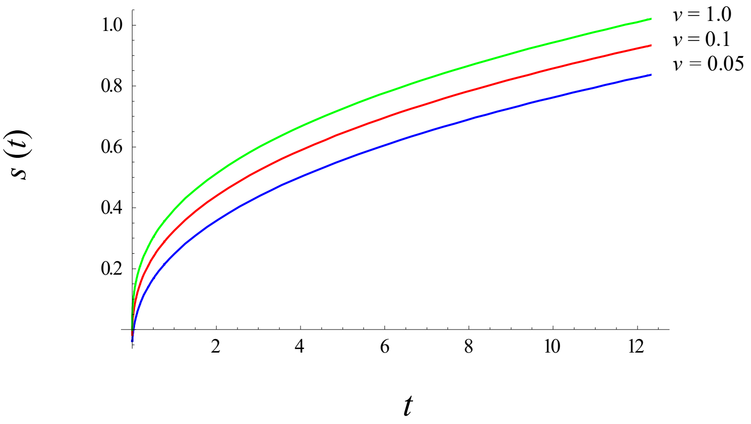

Fig. 5.3. Plot of $s(t)$ **vs.** t **at** $\alpha = 0.25$ $\beta = 0.75$ $\boldsymbol{n} = \mathbf{1}$

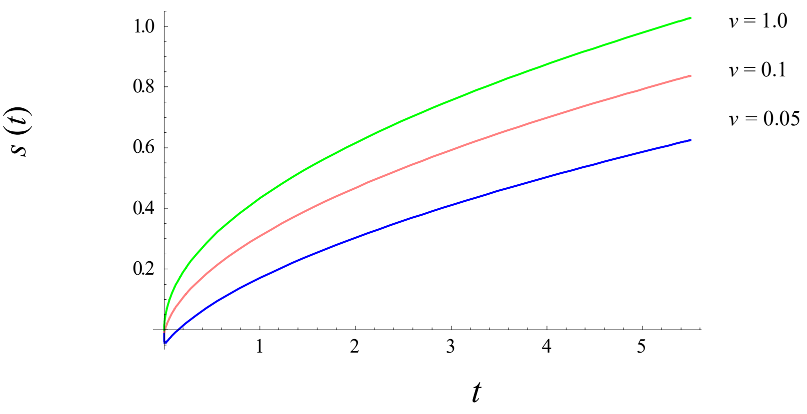

Fig. 5.4. Plot of $s(t)$ **vs.** t **at** $\alpha = 0.5$, $\beta = 1.0$ and $\boldsymbol{n} = \mathbf{1}$

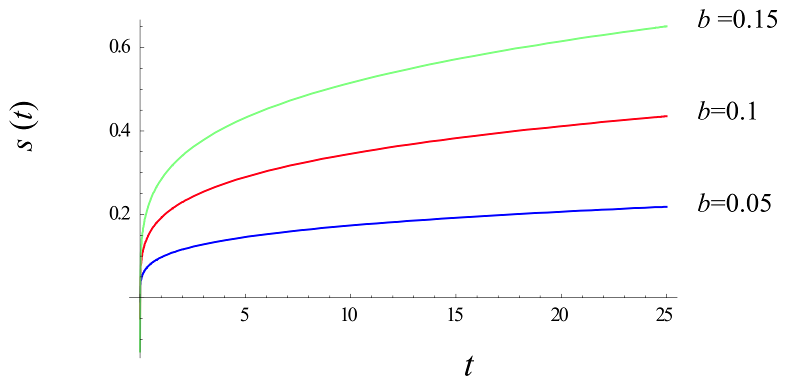

Fig.5. 5. Plot of $s(t)$ **vs.** t **at** $\alpha = 0.25$, $\beta = 0.75$ and $\boldsymbol{n = 0}$

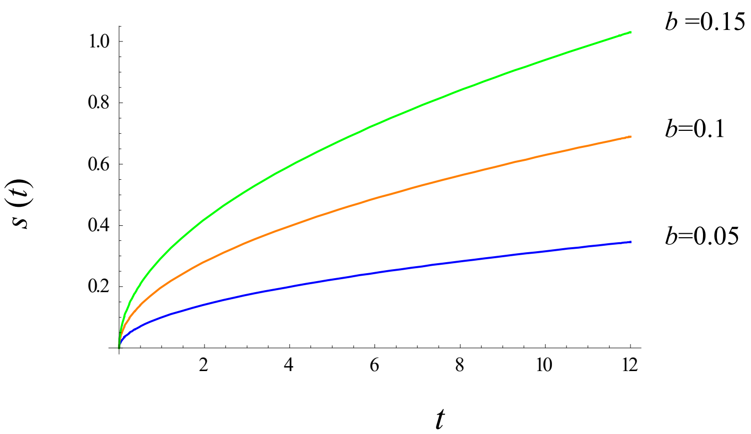

Fig. 5.6. Plot of $s(t)$ **vs.** t **at** $\alpha = 0.5$, $\beta = 1.0$ and $\boldsymbol{n = 0}$

5.5. Conclusion

In this work, a mathematical model that contains space- time fractional derivatives and time dependent surface heat flux is considered. An approximate solution of the problem is obtained by optimal homotopy asymptotic method. It is observed that movement of interface increases with the increase in the value of thermal diffusion coefficient v as well as constant b for non-classical or non-Fickian case. Moreover, it can be seen that proposed technique is sufficiently accurate and efficient for solving Stefan problems. It is also observed that it is convenient to control and adjust the convergence of the series solution through the control parameters c_i in optimal homotopy asymptotic method.