

An approximate approach for a Stefan problem with periodic boundary condition

4.1 Introduction

Stefan problems (phase change problems) occur in many important areas of science, engineering and industry. These problems have wide applications in separation process, food technology, image development in electro photography, medical sciences, heat and mixture migration in solid grounds, etc. The history and some classical solutions to the Stefan problems are well covered by Crank (1987) and Hill (1987). These problems are interesting both because of diversity of their application and because of its nonlinearity nature which is associated with the moving interface (Carslaw and Jaeger (1959)). Due to presence of moving interface their exact solution are limited. Therefore, many approximate (Annamalai et al. (1986), Rizwan-Uddin (1998, 1999), Caldwell and Chan (2000), Rajeev (2014)) and numerical solutions (Asaithambi (1992, 1997) and Rajeev (2009)) have been used to solve such problems. However, Stefan problems with time-dependent boundary condition require some special techniques (Yao and Prusa (1989), Rizwan-Uddin (1998, 1999)). Savovic and Caldwell (2003) used finite difference method to solve one-dimensional Stefan problem with periodic boundary conditions. Ahmed (2006) discussed a new algorithm for a moving boundary problem subject to periodic boundary conditions. In 2009, Rajeev et al. (2009) used variational iteration method to solve a phase change problem with time dependent boundary condition and the result is obtained in term of Mittag-Leffler function.

The approximate analytical approach taken in this chapter is Adomian decomposition method. Adomian decomposition method was developed by Adomian (1988, 1994, 1998) and has been applied to solve various partial differential equations (Wazwaz (2000, 2007)). Grzymkowski and Słota (2004) presented the solution of one-phase inverse Stefan problem by Adomian decomposition method. Recently, Das and Rajeev (2010) also used and Adomian decomposition method to solve time-fractional diffusion equation with a moving boundary condition which is related to the diffusional release of a solute from a polymer matrix in which the initial loading is higher/lower than the solubility.

In this chapter, an approximate solution of one dimensional Stefan problem with periodic boundary conditions is obtained by using Adomian decomposition method. The explicit expressions of the temperature distribution and position of interface are obtained in series form and the numerical results for different particular cases are depicted through graphs. The obtained results are compared with the existing exact solution for a particular case. The dependence of moving interface on time for various Stefan number is also discussed.

4.2 Mathematical Model

In this section, a mathematical model (Rizwan-Uddin (1999)) of a melting solid is considered. The solid is at its phase change temperature initially. The governing equations of problem for the liquid region of a melting solid are as follow:

$$\frac{\partial^2 \bar{T}(\bar{x}, \bar{t})}{\partial \bar{x}^2} = \frac{1}{\alpha} \frac{\partial \bar{T}(\bar{x}, \bar{t})}{\partial \bar{t}}, \quad 0 < \bar{x} < \bar{s}(\bar{t}), \quad (4.2.1)$$

$$\bar{T}(0, \bar{t}) = \bar{T}_0(\bar{t}), \quad (4.2.2)$$

$$\bar{T}(\bar{s}(\bar{t}), \bar{t}) = T_m, \quad (4.2.3)$$

$$\bar{T}(\bar{x}, 0) = T_m, \quad (4.2.4)$$

The additional conditions on the moving interface are

$$\rho h \frac{d\bar{s}(\bar{t})}{d\bar{t}} = -k \frac{\partial \bar{T}(\bar{s}(\bar{t}), \bar{t})}{\partial \bar{x}} \quad (4.2.5)$$

$$\bar{s}(0) = 0 \quad (4.2.6)$$

where \bar{T} is the temperature distribution in liquid region, T_m is the phase change temperature, $\bar{T}_0(\bar{t})$ is the time dependent boundary conditions at the $\bar{x}=0$, α is the liquid diffusivity, ρ is the density of solid, h is the latent heat, k is the thermal conductivity, \bar{t} is the time and $\bar{s}(\bar{t})$ is the position of moving interface.

Introducing the following non-dimensional variables (Crank (1987), Hill (1987)):

$$x = \frac{\bar{x}}{l}, \quad s(t) = \frac{\bar{s}(\bar{t})}{l}, \quad t = \frac{\alpha \bar{t}}{l^2}$$

$$T = \frac{(\bar{T} - T_m)}{\Delta T_{ref}}, \quad Ste = \frac{c \Delta T_{ref}}{h}.$$

where l is the characteristic length, ΔT_{ref} is reference temperature, Ste is Stefan number, and x , t , $s(t)$ and T are dimensionless variables.

The Eqs. (4.2.1) to (4.2.6) become

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{\partial T(x,t)}{\partial t}, \quad 0 < x < s(t), \quad (4.2.7)$$

$$T(0,t) = \frac{(\bar{T}_0 - \bar{T}_m)}{\Delta T_{ref}} \equiv g(t), \quad (4.2.8)$$

$$T(s(t),t) = 0, \quad (4.2.9)$$

$$T(x,0) = 0, \quad (4.2.10)$$

$$\frac{ds(t)}{dt} = -Ste \frac{\partial T(s(t),t)}{\partial x}, \quad (4.2.11)$$

$$s(0) = 0, \quad (4.2.12)$$

where $g(t) = 1 + \varepsilon \sin \omega t$, ε is the oscillation amplitude and ω is the oscillation frequency.

4.3 Solution of the problem by Adomian Decomposition Method

We first write the Eq. (4.2.7) in an operator form

$$L_{xx}T(x,t) = \frac{\partial T(x,t)}{\partial t}, \quad (4.3.13)$$

where $L_{xx} = \frac{\partial^2}{\partial x^2}$.

Assuming that the inverse operator L_{xx}^{-1} exists and

$$L_{xx}^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dx dx.$$

Applying the inverse operator L_{xx}^{-1} on both side of the equation (4.3.13) which gives

$$T(x,t) - T(0,t) = L_{xx}^{-1} \left(\frac{\partial T}{\partial t} \right). \quad (4.3.14)$$

Choosing the following initial approximations of $T(x,t)$ and $s(t)$ as given by Das and Rajeev (2010):

$$T_0 = (1 + \varepsilon \sin \omega t) \left(1 - \frac{x}{s_0} \right),$$

and $s_0 = a_0 t^{1/2}$,

where $a_0 = (2c(1 + \varepsilon \sin \omega t))^{1/2}$.

According to the Adomian decomposition method (Adomian (1988, 1994, 1998)), decomposing the unknown function $T(x,t)$ as follows:

$$T(x,t) = T_0 + T_1 + T_2 + \dots, \quad (4.3.15)$$

where the components T_0, T_1, T_2, \dots are defined as :

$$T_0 = T(0,t) = (1 + \varepsilon \sin \omega t) \left(1 - \frac{x}{s_0} \right),$$

$$T_1 = L_{xx}^{-1} \left(\frac{\partial T_0(x,t)}{\partial t} \right) = \frac{x^2}{2!} \varepsilon \omega \cos \omega t - \frac{x^3}{3!} \frac{t^{-1/2}}{a_0} \varepsilon \omega \cos \omega t + \frac{x^3}{3!} \frac{t^{-3/2}}{2a_0} (1 + \varepsilon \sin \omega t),$$

$$T_2 = L_{xx}^{-1} \left(\frac{\partial T_1(x,t)}{\partial t} \right) = -\frac{x^4}{4!} \varepsilon \omega^2 \sin \omega t + \frac{x^5}{5!} \frac{t^{-1/2}}{a_0} \varepsilon \omega^2 \sin \omega t - \frac{3x^5}{2^2 5!} \frac{t^{-5/2}}{a_0} (1 + \varepsilon \sin \omega t) \\ + \frac{x^5}{5!} \frac{t^{-3/2}}{a_0} \varepsilon \omega \cos \omega t,$$

⋮

and so on.

Thus,

$$T(x,t) = T_0 + T_1 + T_2 + \dots \\ = (1 + \varepsilon \sin \omega t) \left(1 - \frac{x}{s_0} \right) + \frac{x^2}{2!} \varepsilon \omega \cos \omega t - \frac{x^3}{3!} \frac{t^{-1/2}}{a_0} \varepsilon \omega \cos \omega t \\ + \frac{x^3}{3!} \frac{t^{-3/2}}{2a_0} (1 + \varepsilon \sin \omega t) - \frac{x^4}{4!} \varepsilon \omega^2 \sin \omega t + \frac{x^5}{5!} \frac{t^{-1/2}}{a_0} \varepsilon \omega^2 \sin \omega t \\ - \frac{3x^5}{2^2 5!} \frac{t^{-5/2}}{a_0} (1 + \varepsilon \sin \omega t) + \frac{x^5}{5!} \frac{t^{-3/2}}{a_0} \varepsilon \omega \cos \omega t + \dots \quad (4.3.16)$$

Eq. (4.3.16) gives temperature distribution in liquid region.

Now, using Eq. (4.3.16) and writing the interface condition (4.2.11) in operator form as:

$$s(t) = \varphi - Ste L_t^{-1}(F(s)), \quad (4.3.17)$$

where initial approximation $\varphi = s_0 = a_0 t^{\frac{1}{2}}$,

and

$$\begin{aligned} F(s) &= \frac{\partial}{\partial x} T(s(t), t) \\ &= - \frac{(1 + \varepsilon \sin \omega t)}{s(t)} + s(t) \varepsilon \omega \cos \omega t - \frac{(s(t))^2 t^{\frac{-1}{2}} \varepsilon \omega \cos \omega t}{2! a_0} \\ &\quad + \frac{(s(t))^2 t^{\frac{-3}{2}} (1 + \varepsilon \sin \omega t)}{2! 2 a_0} - \frac{(s(t))^3 \varepsilon \omega^2 \sin \omega t}{3!} + \frac{(s(t))^4 t^{\frac{-1}{2}} \varepsilon \omega^2 \sin \omega t}{4! a_0} \\ &\quad - \frac{3(s(t))^4 t^{\frac{-5}{2}} (1 + \varepsilon \sin \omega t)}{4! 2^2 a_0} + \frac{(s(t))^4 t^{\frac{-3}{2}} \varepsilon \omega \cos \omega t}{4! a_0} - \dots \end{aligned}$$

Accordingly (Adomian, 1998), decomposing $s(t)$ as:

$$s(t) = \sum_{n=0}^{\infty} s_n. \quad (4.3.18)$$

Eqs. (4.3.17) and (4.3.18) give

$$\sum_{n=0}^{\infty} s_n = \varphi - Ste L_t^{-1} \left(\sum_{n=0}^{\infty} A_n \right), \quad (4.3.19)$$

where A_n are so-called Adomian polynomials for non-linear terms and defined as:

$$A_0 = F(s_0), \quad (4.3.20 \text{ a})$$

$$A_1 = s_1 F'(s_0), \quad (4.3.20 \text{ b})$$

$$A_2 = s_2 F'(s_0) + \frac{1}{2} s_1^2 F''(s_0), \quad (4.3.20 \text{ c})$$

⋮

and so on.

The components of $s_n(t)$, $n \geq 1$, can be completely determined as:

$$s_1 = -Ste L_t^{-1}(A_0), \quad (3.3.21)$$

where

$$A_0 = \left(-\frac{(1 + \varepsilon \sin \omega t)}{s_0} + s_0 \varepsilon \omega \cos \omega t - \frac{(s_0)^2 t^{\frac{-1}{2}} \varepsilon \omega \cos \omega t}{2! a_0} \right. \\ \left. + \frac{(s_0)^2 t^{\frac{-3}{2}} (1 + \varepsilon \sin \omega t)}{2! 2 a_0} - \frac{(s_0)^3 \varepsilon \omega^2 \sin \omega t}{3!} + \frac{(s_0)^4 t^{\frac{-1}{2}} \varepsilon \omega^2 \sin \omega t}{4! a_0} \right. \\ \left. - \frac{3(s_0)^4 t^{\frac{-5}{2}} (1 + \varepsilon \sin \omega t)}{2^2 4! a_0} + \frac{(s_0)^4 t^{\frac{-3}{2}} \varepsilon \omega \cos \omega t}{4! a_0} - \dots \right),$$

$$s_2 = -Ste L_t^{-1}(A_1), \quad (4.3.22)$$

where

$$A_1 = s_1 \frac{d}{d s_0} \left(-\frac{(1 + \varepsilon \sin \omega t)}{s_0} + s_0 \varepsilon \omega \cos \omega t - \frac{(s_0)^2 t^{\frac{-1}{2}} \varepsilon \omega \cos \omega t}{2! a_0} \right. \\ \left. + \frac{(s_0)^2 t^{\frac{-3}{2}} (1 + \varepsilon \sin \omega t)}{2! 2 a_0} - \frac{(s_0)^3 \varepsilon \omega^2 \sin \omega t}{3!} \right. \\ \left. + \frac{(s_0)^4 t^{\frac{-1}{2}} \varepsilon \omega^2 \sin \omega t}{4! a_0} - \frac{3(s_0)^4 t^{\frac{-5}{2}} (1 + \varepsilon \sin \omega t)}{2^2 4! a_0} + \frac{(s_0)^4 t^{\frac{-3}{2}} \varepsilon \omega \cos \omega t}{4! a_0} - \dots \right),$$

⋮

and so on.

The approximate analytical solution of $s(t)$ is given by

$$s(t) = s_0 + s_1 + s_2 + \dots \quad (4.3.23)$$

4.4 Numerical discussion

In this section, all the numerical results for temperature distribution $T(x,t)$ and interface position $s(t)$ are calculated by using MATHEMATICA software. When $\varepsilon \neq 0$, the exact solution of the problem is not known (Savovic and Caldwell (2003)). For accuracy, the obtained approximate solutions by Adomian decomposition method are compared with the existing exact solution at $\varepsilon = 0$. It can be seen that the exact solutions (Hill (1987)) of the problem defined by Eqs. (4.2.7-4.2.12) are

$$T(x,t) = 1 - \frac{\operatorname{erf}(x/2\sqrt{t})}{\operatorname{erf}\sqrt{\gamma/2}}, \quad (4.4.24)$$

$$s(t) = \sqrt{2\gamma t}, \quad (4.4.25)$$

where γ is a root of following transcendental equation:

$$e^{\gamma/2} \sqrt{\frac{\pi\gamma}{2}} \operatorname{erf}\left(\sqrt{\frac{\gamma}{2}}\right) = Ste. \quad (4.4.26)$$

Table 4.1 present absolute errors for the temperature distribution in liquid region at $Ste = 0.5$ and $Ste = 1.0$. It is clear from table that the proposed approximate solution is closed to the exact.

Figure 4.1 and Figure 4.2 show the dependence of temperature distribution $T(x,t)$ on x at the fixed value of $Ste = 1.0$, oscillation frequency $\omega = \pi/2$ and $t = 5$ for two oscillation amplitudes $\varepsilon = 0.5$ and $\varepsilon = 1.0$, respectively. It is

observed from figures (4.1- 4.2) that increase in the oscillation amplitude leads to a more prominent change in the temperature distribution in the domain $0 \leq x \leq 1$. Moreover, the temperature distribution is maximum at $x = 0$ and then continuously decreases up to the zero at the $x = 1$. This result confirms the result of Savovic and Caldwell (2003) and Rajeev et al. (2009).

Figure 4.3 and Figure 4.4 describe the growth of the moving interface for different values of Stefan number ($Ste = 0.1, 0.5, 1.0$) at the fixed value of oscillation frequencies $\omega = \pi/4$ for two oscillation amplitudes $\varepsilon = 0.2$ and $\varepsilon = 0.9$, respectively. It is seen from both the figures that the interface covers more distance as the values of Stefan number increases at a particular time. It means movement of interface and melting process becomes fast with the increment in Stefan number. This fact shows that the result is well consistent with the statement of Rizwan-Uddin (1999) and Ahmed (2006).

4.5 Conclusion

The Adomian decomposition method is successfully applied to find explicit expressions of temperature distribution in liquid region and interface position of a Stefan problem with periodic boundary condition. From present study, it is observed that melting process and velocity of interface become faster as the value of Stefan number increases. It is seen that Adomian decomposition method is a straight forward, efficient and sufficiently accurate method for finding the solution of Stefan problems of science and engineering. The authors believe that the procedure as described in the present study will be beneficial to researcher working in this field.

<i>Ste</i>	<i>x</i>	Exact Value T_E	Approximate Value T_A	Absolute error $ T_E - T_A $
0.5	0.1	0.948413	0.955286	0.00687309
	0.2	0.896877	0.910617	0.0137399
	0.3	0.845444	0.866037	0.0205929
	0.4	0.794165	0.82159	0.0274254
	0.5	0.74309	0.777321	0.0342314
	0.6	0.692271	0.733273	0.0410021
	0.7	0.641755	0.689488	0.0477333
	0.8	0.591592	0.646009	0.0544167
	0.9	0.541829	0.602875	0.0610464
	1.0	0.492511	0.560128	0.0676172
1.0	0.1	0.959276	0.968382	0.00910649
	0.2	0.918592	0.936797	0.018205
	0.3	0.877989	0.905274	0.0272848
	0.4	0.837509	0.873845	0.0363364
	0.5	0.797189	0.842542	0.0453535
	0.6	0.757071	0.811396	0.0543246
	0.7	0.717193	0.780435	0.0632419
	0.8	0.677593	0.74969	0.0720974
	0.9	0.638309	0.71919	0.0808815
	1.0	0.599376	0.688739	0.0893635

Table 4.1. Absolute error for temperature at $\varepsilon = 0$ and $t = 5.0$

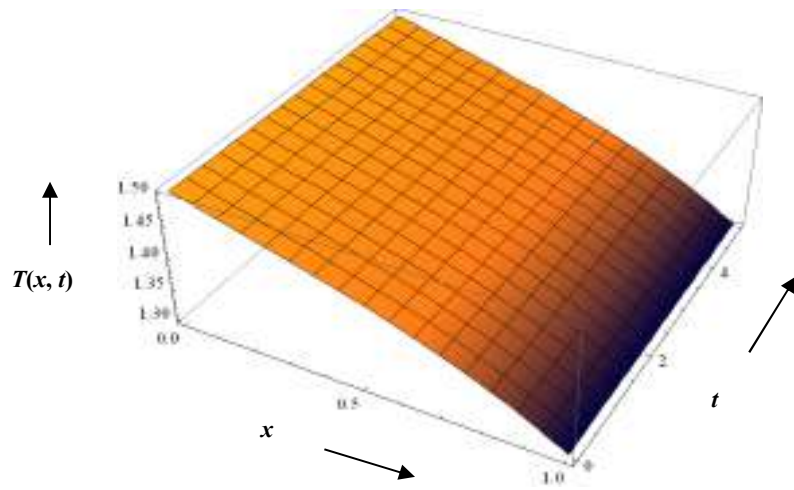


Fig.4.1. Plot of $T(x,t)$ vs. x for $\varepsilon = 0.5$ and $\omega = \pi/2$

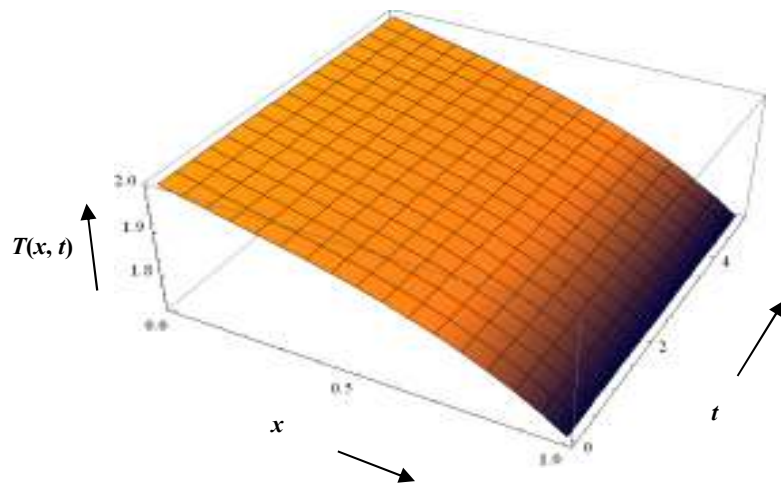


Fig.4.2. Plot of $T(x,t)$ vs. x for $\varepsilon = 1.0$ and $\omega = \pi/2$

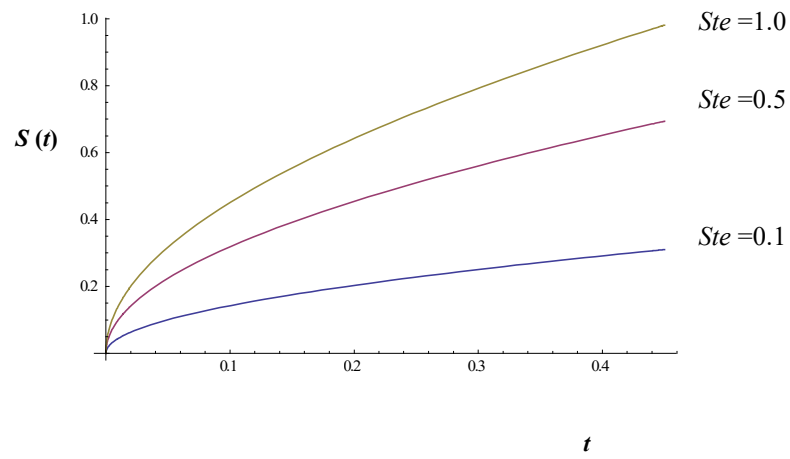


Fig.4.3. Plot of $s(t)$ vs. t for $\varepsilon = 0.2$ and $\omega = \pi/4$

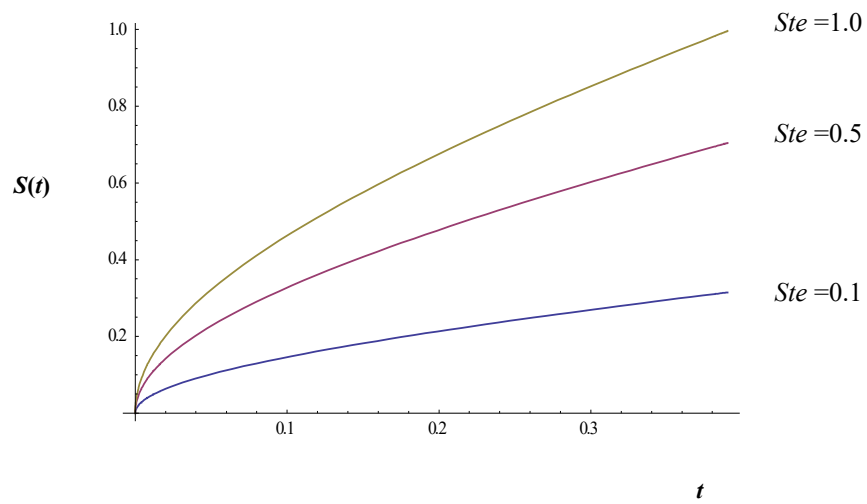


Fig.4.4. Plot of $s(t)$ vs. t for $\varepsilon = 0.9$ and $\omega = \pi/4$