An approximate solution to a moving boundary problem with space-time fractional derivatives in fluvio-deltaic sedimentation process

3.1 Introduction

An interesting moving boundary problem in the field of earth surface science involves the movement of the shoreline in a sedimentary ocean basin (a shoreline problem). The classical diffusion transport models (Swenson et al. (2000), Voller et al. (2004) and Capart et al. (2007)) provide a reliable means of modeling the sediment transport in fluvial depositional systems. The assumptions of the classical diffusion equations are thin-tailed periods of inactivity and thin-tailed transport distances for sediment particles. From the literature (Schumer et al. (2009), Ganti et al. (2009), Nathan Bradley et al. (2010), Foufoula-Georgiou et al. (2010)), the deviation from normal (Fickian) diffusion in sediment tracer dispersion is observed that violates the assumption of statistical convergence to a Gaussian. Therefore, the fractional diffusion equations are widely used for the investigation of the mechanism of anomalous diffusion in transport processes through complex and/or disordered systems including fractal media (Li et al. (2008), Liu et al. (2009)). It is well known that fractional derivative is a good tool for taking into account memory mechanism, particularly in some diffusive processes (Tomovskia et al. (2012)). Both space and time fractional operators correspond to the diffusion limit of continuous time random walk models with long-tailed waiting time and/or jump length distributions (Gorenflo et al. (1998), Tomovskia et al. (2012)). Li et al. (2007)) used Caputo derivative $\beta \in (0, 1]$ and Riesz-Feller derivative $\alpha \in (0, 2]$ operators for the first order time derivative and second order space derivative, respectively and presented an analytic solution to fractional form of a moving boundary problem in drug release devices in term of Fox H function. Voller (2010) presented fractional (non-integer) form of a Stefan problem using Caputo derivatives for both space and time, and discussed exact solution of the problem. Recently, some researchers (Vogel et al.(2012), Zhao et al.(2012a, 2012b), Jiang et al. (2012)) also discussed various mathematical models governed with different fractional derivatives for both the space and time.

The most commonly used definitions in mathematical models are the Riemann–Liouville and Caputo. Riemann-Liouville fractional derivative requires initial conditions to be expressed in terms of fractional integrals and their derivatives which have no obvious physical interpretation. So, Riemann-Liouville fractional derivative is not always the most convenient definition for real applications (Podlubny (1999)). However, Caputo fractional derivative requires the initial conditions (including the mixed boundary conditions) in the same form as that of ordinary differential equations with integer-order derivatives (Podlubny (1999)). These integer-order derivatives represent well-understood features of a physical situation and therefore, their values can be measured accurately. Another advantage is that the Caputo derivative of a constant is zero, whereas the Riemann-Liouville fractional derivative of a constant is not zero. Therefore, it is interesting and applicable to use Caputo fractional derivative in diffusion model of sediment transport on earth surface. It can be seen by Schumer et al. (2003) and Meerschaert et al.(2004) that a pure power-law, heavy-tailed probability density function for the periods of inactivity without any truncation leads to a time-fractional diffusion equation which describes the evolution of surface elevation in time. In depositional system, the deviation of fluvial profiles from classical/standard diffusion is reported by Voller and Paola (2010). They also presented a diffusive model governed with fractional derivatives to describe the steady-state fluvial profiles. After that Ganti et al. (2011) assumed that the periods of inactivity are heavy-tailed and presented a time-fractional diffusion model for the surface dynamics of depositional system. A discussion of the physical basis for anomalous diffusion in bed load transport is reported in the paper of Martin et al. (2012). Rajeev and Kushwaha (2013) also discussed a mathematical model with time-fractional derivative for a moving boundary problem which occurs in sedimentation process. These models motivate to discuss space-time fractional diffusion model in sedimentation process to study the physical effect in complex domain.

The diffusion equation with a moving boundary (moving boundary problem) is a special nonlinear problem which is difficult to get the exact solution (Crank (1987), Carslaw et al. (1987)). Hence, many approximate and numerical methods have been used to solve the moving boundary problems (Lin et al. (2005), Abdekhodaie (1996), Das and Rajeev (2010), Li et al. (2009), Rajeev (2014), Voller et al. (2006), Rajeev (2009)). The approximate analytical approach taken in this literature is Adomian decomposition method. Adomian decomposition method was developed by Adomian (1988,1994, 1998) and has been applied to solve a wide class of non-linear differential and partial differential equations (Wazwaz (2000, 2007)). Grzymkowski and Słota (2006) presented the solution of one-phase inverse Stefan problem by Adomian decomposition method. Das and Rajeev (2010) also used and Adomian decomposition method to solve time-fractional diffusion equation with a moving boundary condition which is related to the diffusional release of a solute from a polymer matrix in which the initial loading is higher/lower than the solubility.

In this study, we consider the non-classical or non-Fickian, anomalous sediment transport in braided networks. Our attention in this chapter is to discuss a moving boundary problem governed by fractional space-time derivative in Caputo sense which arises during the movement of the shoreline in a sedimentary ocean basin. This model is a generalization of previous model (Rajeev and kushwaha (2013)). The physical purpose for adopting and investigating diffusion equations with fractional space-time derivative is to describe phenomena of anomalous (non- Fickian) sediment transport through complex and/or disordered systems including fractal media which occurs in sedimentation process. Adomian decomposition method is successfully applied to find an approximate solution of the proposed problem. The obtained results are compared with the existing exact solutions. Three particular cases, the standard diffusion, the time-fractional and the space fractional diffusions are also discussed.

3.2 The fluvio- deltaic sedimentation model

Fluvio-deltaic sedimentation problem involves the shoreline propagation in a sedimentary ocean basin due to a sediment line flux, tectonic subsidence of the earth's crust, and sea level change. The mathematical model of fluvio-deltaic sedimentation process is discussed in (Swenson et al. (2000), Voller et al. (2004), Capart et al. (2007)). In this chapter, we consider a fixed line flux, a constant ocean level ($z = 0$), no tectonic subsidence of the earth's crust, and a constant sloping basement $b < a$. A schematic cross section of such a basin indicating the variables is revealed in Fig. 3.1 (Voller et al. (2004)). Under this limit case, the dynamics of the sedimentation process become a moving boundary problem with variable latent heat (as given by Voller et al. (2004)) which is as follows:

$$\frac{\partial \eta}{\partial t} = \nu \frac{\partial^2 \eta}{\partial x^2}, \qquad 0 < x < s(t), \tag{3.2.1}$$

with initial and boundary conditions

$$\nu \left. \frac{\partial \eta}{\partial x} \right|_{x=0} = -q(t), \tag{3.2.2}$$

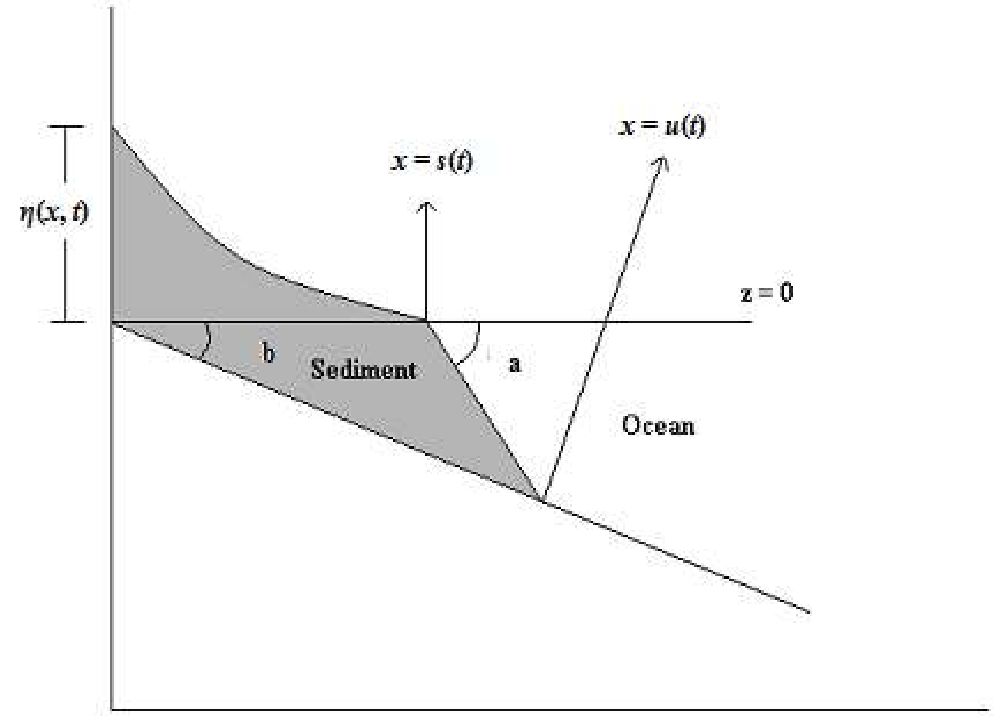

Fig. 3.1. A schematic cross section of a basin with no tectonic subsidence and sea level change.

and $\eta(s,t)=0$, (3.2.3)

where $\eta(x,t)$ is height of sediment above datum, ν is a diffusion coefficient, $q(t)$ is the time-dependent sediment line flux and $s(t)$ is the moving contact point (moving interface).

The additional conditions on the moving interface are

$$-\nu\frac{\partial\eta}{\partial x}\bigg|_{x=s(t)}=\gamma s\frac{ds}{dt}, \qquad (3.2.4)$$

and $s(0)=0$, (3.2.5)

where $a(u-s)=\dfrac{ab\,s}{a-b}=\gamma s$.

3.3 The fractional model

In order to describe phenomena of anomalous (non- Fickian) sediment transport through complex and/or disordered systems including fractal media, we consider above moving boundary problem with fractional space-time derivatives. Using Caputo fractional derivatives (as given by Voller (2010)), a space-time fractional form of the equations (3.2.1-3.2.5) can be described as follows:

$$D_t^{\beta}\eta(x,t)=\nu\frac{\partial}{\partial x}(D_x^{\alpha}\eta(x,t)), \quad (0<x<s(t),\ 0<\alpha,\beta\le 1), \qquad (3.3.6)$$

with the following posed conditions:

$$\nu D_x^{\alpha}\eta(0,t)=-q, \qquad (3.3.7)$$

$$\eta(s,t)=0. \qquad (3.3.8)$$

The additional conditions on the moving interface are

$$-\nu D_x^{\alpha}\eta(s(t),t) = \gamma s D_t^{\beta} s(t), \tag{3.3.9}$$

and $s(0) = 0$. (3.3.10)

where $a(u-s) = \dfrac{abs}{a-b} = \gamma s$ and q is prescribed sediment line flux that is considered as a constant.

3.4 Solution of the problem by Adomian decomposition method

We first write the equation (3.3.6) in operator form

$$L_{xx}\eta(x,t) = D_x^{1-\alpha}\left(\frac{1}{\nu}\frac{\partial^{\beta}\eta(x,t)}{\partial^{\beta} t}\right), \tag{3.4.11}$$

where $L_{xx} = \dfrac{\partial^2}{\partial x^2}$.

Assuming that the inverse operator L_{xx}^{-1} exists and

$$L_{xx}^{-1}(.) = \int_0^x\int_0^x (.)dx\,dx.$$

Applying the inverse operator L_{xx}^{-1} on the both side of equation (3.4.11)

$$\eta(x,t) - \eta(0,t) = L_{xx}^{-1}\left(D_x^{1-\alpha}\left(\frac{1}{\nu}\frac{\partial^{\beta}\eta(x,t)}{\partial^{\beta} t}\right)\right). \tag{3.4.12}$$

Choosing the following initial approximations of $\eta(x,t)$ and $s(t)$ as given by Das and Rajeev (2010):

$$\eta_0 = c(s_0^{\alpha} - x^{\alpha}),$$

where $c = \dfrac{q}{\nu\Gamma(1+\alpha)}$,

and $s_0 = a_0 t^{\beta/2}$,

where $a_0 = \left(\dfrac{c\,v}{\gamma} \dfrac{\Gamma(1-\beta/2)\Gamma(1+\alpha)}{\Gamma(1+\beta/2)} \right)^{1/2}$.

According to the Adomian decomposition method (Adomian (1988)), decomposing the unknown function $\eta(x,t)$ as follows:

$$\eta(x,t) = \eta_0 + \eta_1 + \eta_2 + \cdots. \qquad (3.4.13)$$

From the equations (3.4.12) and (3.4.13), the components $\eta_0, \eta_1, \eta_2 ......$ are recursively determined by:

$$\eta_0 = \eta(0,t) = c(s_0^\alpha - x^\alpha),$$

$$\eta_1 = L_{xx}^{-1}\left(D_x^{1-\alpha}\left(\frac{1}{v} \frac{\partial^\beta \eta_0(x,t)}{\partial^\beta t} \right) \right) = \frac{c\,a_0^\alpha}{v} \frac{\Gamma(1+\frac{\alpha\beta}{2})}{\Gamma(1+\frac{\alpha\beta}{2}-\beta)\Gamma(\alpha+2)} t^{\frac{\alpha\beta}{2}-\beta} x^{\alpha+1},$$

$$\eta_2 = L_{xx}^{-1}\left(D_x^{1-\alpha}\left(\frac{1}{v} \frac{\partial^\beta \eta_1(x,t)}{\partial^\beta t} \right) \right) = \frac{c\,a_0^\alpha}{v^2} \frac{\Gamma(1+\frac{\alpha\beta}{2})}{\Gamma(1+\frac{\alpha\beta}{2}-2\beta)\Gamma(2\alpha+3)} t^{\frac{\alpha\beta}{2}-2\beta} x^{2\alpha+2},$$

$$\eta_3 = L_{xx}^{-1}\left(D_x^{1-\alpha}\left(\frac{1}{v} \frac{\partial^\beta \eta_2(x,t)}{\partial^\beta t} \right) \right) = \frac{c\,a_0^\alpha}{v^3} \frac{\Gamma(1+\frac{\alpha\beta}{2})}{\Gamma(1+\frac{\alpha\beta}{2}-3\beta)\Gamma(3\alpha+4)} t^{\frac{\alpha\beta}{2}-3\beta} x^{3\alpha+3},$$

$$\eta_4 = L_{xx}^{-1}\left(D_x^{1-\alpha}\left(\frac{1}{v} \frac{\partial^\beta \eta_3(x,t)}{\partial^\beta t} \right) \right) = \frac{c\,a_0^\alpha}{v^4} \frac{\Gamma(1+\frac{\alpha\beta}{2})}{\Gamma(1+\frac{\alpha\beta}{2}-4\beta)\Gamma(4\alpha+5)} t^{\frac{\alpha\beta}{2}-4\beta} x^{4\alpha+4},$$

$$\vdots$$

and so on.

Thus,

$$\eta(x,t) = \eta_0 + \eta_1 + \eta_2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{c(a_0)^\alpha}{v^n} \frac{\Gamma(1+\frac{\alpha\beta}{2})}{\Gamma(1+\frac{\alpha\beta}{2}-n\beta)\Gamma(n\alpha+n+1)} t^{\frac{\alpha\beta}{2}-n\beta} x^{n(\alpha+1)} - c\,x^\alpha, \qquad (3.4.14)$$

which gives height of the sediment above the datum.

Now, using (3.4.14) and writing the interface condition (3.3.9) in operator form

$$s(t) = \varphi - L_t^{-1}\left(F(s)\right), \qquad (2.4.15)$$

where $\varphi = s_0 = a_0 t^{\frac{\beta}{2}}$,

and

$$F(s) = -\frac{v}{\gamma s_0}\frac{\partial^\alpha}{\partial x^\alpha}\left(\eta(s(t),t)\right)$$

$$= -\frac{c\,v}{\gamma}\left(\frac{-\Gamma(1+\alpha)}{s(t)} + \frac{a_0^\alpha}{v}\frac{\Gamma(1+\frac{\alpha\beta}{2})}{\Gamma(1+\frac{\alpha\beta}{2}-\beta)}t^{\frac{\alpha\beta}{2}-\beta} + \frac{a_0^\alpha}{v^2}\frac{\left(s(t)\right)^{\alpha+1}\Gamma(1+\frac{\alpha\beta}{2})}{\Gamma(1+\frac{\alpha\beta}{2}-2\beta)\Gamma(\alpha+3)}t^{\frac{\alpha\beta}{2}-2\beta} \right.$$

$$\left. + \frac{a_0^\alpha}{v^3}\frac{\left(s(t)\right)^{2\alpha+2}\Gamma(1+\frac{\alpha\beta}{2})}{\Gamma(1+\frac{\alpha\beta}{2}-3\beta)\Gamma(2\alpha+4)}t^{\frac{\alpha\beta}{2}-3\beta} + \cdots \right).$$

Accordingly (Adomian (1994, 1998), decomposing $s(t)$ as:

$$s(t) = \sum_{n=0}^{\infty} s_n\,, \qquad (3.4.16)$$

Using (3.4.15) and (3.4.16), we have

$$\sum_{n=0}^{\infty} s_n = \varphi - \alpha\, L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \qquad (3.4.17)$$

where A_n are so-called Adomian polynomials for non-linear terms and defined as:

$$A_0 = F(s_0)\,,$$

$$A_1 = s_1\, F'(s_0)\,, \qquad (3.4.18)$$

$$A_2 = s_2\, F'(s_0) + \frac{1}{2} s_1^2\, F''(s_0)\,,$$

$\vdots$

and so on.

The components of $s_n(t),\ n \geq 1,$ can be completely determined as:

$$s_1 = L_t^{-1}(A_0), \tag{3.4.19}$$

where,

$$A_0 = -\frac{c\,v}{\gamma}\left(\frac{-\Gamma(1+\alpha)}{s_0} + \frac{a_0^{\alpha}}{v} \frac{\Gamma(1+\frac{\alpha\,\beta}{2})}{\Gamma(1+\frac{\alpha\,\beta}{2}-\beta)} t^{\frac{\alpha\,\beta}{2}-\beta} + \frac{a_0^{\alpha}}{v^2} \frac{(s_0)^{\alpha+1}\,\Gamma(1+\frac{\alpha\,\beta}{2})}{\Gamma(1+\frac{\alpha\,\beta}{2}-2\beta)\Gamma(\alpha+3)} t^{\frac{\alpha\,\beta}{2}-2\beta} \right.$$
$$\left. + \frac{a_0^{\alpha}}{v^3} \frac{(s_0)^{2\alpha+2}\,\Gamma(1+\frac{\alpha\,\beta}{2})}{\Gamma(1+\frac{\alpha\,\beta}{2}-3\beta)\Gamma(2\alpha+4)} t^{\frac{\alpha\,\beta}{2}-3\beta} + \cdots \right),$$

$$s_2 = L_t^{-1}(A_1), \tag{3.4.20}$$

where,

$$A_1 = s_1 \frac{d}{d\,s_0}\left[-\frac{c\,v}{\gamma}\left(\frac{-\Gamma(1+\alpha)}{s_0} + \frac{a_0^{\alpha}}{v} \frac{\Gamma(1+\frac{\alpha\,\beta}{2})}{\Gamma(1+\frac{\alpha\,\beta}{2}-\beta)} t^{\frac{\alpha\,\beta}{2}-\beta} + \frac{a_0^{\alpha}}{v^2} \frac{(s_0)^{\alpha+1}\,\Gamma(1+\frac{\alpha\,\beta}{2})}{\Gamma(1+\frac{\alpha\,\beta}{2}-2\beta)\Gamma(\alpha+3)} t^{\frac{\alpha\,\beta}{2}-2\beta} \right.\right.$$
$$\left.\left. + \frac{a_0^{\alpha}}{v^3} \frac{(s_0)^{2\alpha+2}\,\Gamma(1+\frac{\alpha\,\beta}{2})}{\Gamma(1+\frac{\alpha\,\beta}{2}-3\beta)\Gamma(2\alpha+4)} t^{\frac{\alpha\,\beta}{2}-3\beta} + \cdots \right)\right],$$

$\vdots$

and so on.

Therefore, approximate analytical solution of $s(t)$ is given by:

$$s(t) = s_0 + s_1 + s_2 + \cdots, \tag{3.4.21}$$

which give height of the sediment above the datum and the shoreline position at a particular time.

3.5. Numerical comparison and discussion

In this section, numerical results for height of sediment $\eta(x,t)$ and shoreline positions $s(t)$ are calculated using MATHEMATICA software and depicted through figures. The solution of the problem is discussed in detail by considering three particular cases:

Case1. When $\alpha = 1$, $\beta = 1$, the equations (3.3.6-3.3.10) reduce to the equations (3.2.1-3.2.5) which is standard moving boundary problem. In order to show the accuracy of the proposed approximate solution, we compare it with the existing exact solution for integer order which is given by Voller et al.(2004). Fig. 3.2 and Fig. 3.3 represent the dependence of height of sediment $\eta(x,t)$ on space x for standard moving boundary problem ($\alpha = 1, \beta = 1$) at the fixed value of diffusion coefficient (v = 2.0), sediment line flux ($q = 0.5$) and time t = 3.0 for $\gamma = 10$ and $\gamma = 15$, respectively. Fig. 3.4 and Fig. 3.5 depict the dependence of the shoreline position on time at the fixed value of diffusion coefficient (v = 2.0) and sediment line flux ($q = 0.5$) for $\gamma = 10$ and $\gamma = 15$, respectively. It can be seen from figures (3.2-3.5) that the proposed approximate solution is close to the exact solution. Moreover, it is clear from figures (3.4-3.5) that the movement of shoreline position decreases as the value of γ increases. In this case, the sedimentation process becomes slow and the sediments will be deposited towards the land side which causes the increase of the thickness of earlier sediments. As a consequence of this there will be least shifting of the contact point towards the land side and sedimentation process will be slower.

Case 2. When $\alpha = 1,\ 0 < \beta < 1$, the equations (3.3.6-3.3.10) degenerate into a moving boundary problem governed with time-fractional derivative as discussed in the second chapter. Fig. 3.6 and Fig. 3.7 explain the dependence of shoreline position on time for different Brownian motion $\beta = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, and also for the standard motion $\beta = 1.0$ at the fixed value of v = 2.0, $\gamma = 15$ and $\alpha = 1$. It is observed from figures (3.6-3.7) that the rate of increase of $s(t)$ decreases with the increase of β which confirms the exponential decay of regular Brownian motion. This result is in good agreement with the result of Das and Rajeev (2010).

Case 3. When $0 < \alpha < 1,\ \beta = 1,$ the proposed problem becomes a moving boundary problem with space-fractional derivatives. Fig. 3.8 and Fig. 3.9 show the plot of shoreline position $s(t)$ on time for different values of $\alpha = \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, 1$ at the fixed value of v = 2.0, $\gamma = 15$ and $\beta = 1$.

It can be seen from the Figures (3.6-3.9) that if the sediment line flux q increases($q = 0.5, 1.5$), the movement of the contact point (shoreline position) increases towards sea side with formation of inclined strata along the off-shore sediment wedge. This conclusion show the fact that the models are well consistent with truth. Figures (3.6-3.9) also show that trajectory of the movement of contact point deviates more from standard motion for the case of time fractional than space fractional case during sedimentation process.

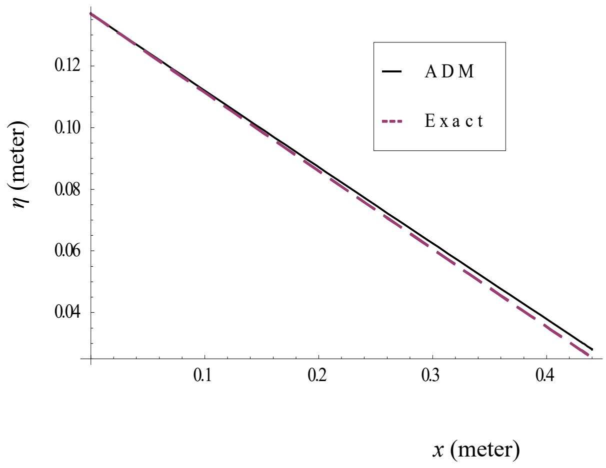

Fig.3.2. Plot of $\eta(x,t)$ vs. x for q =0.5, v=2.0 and

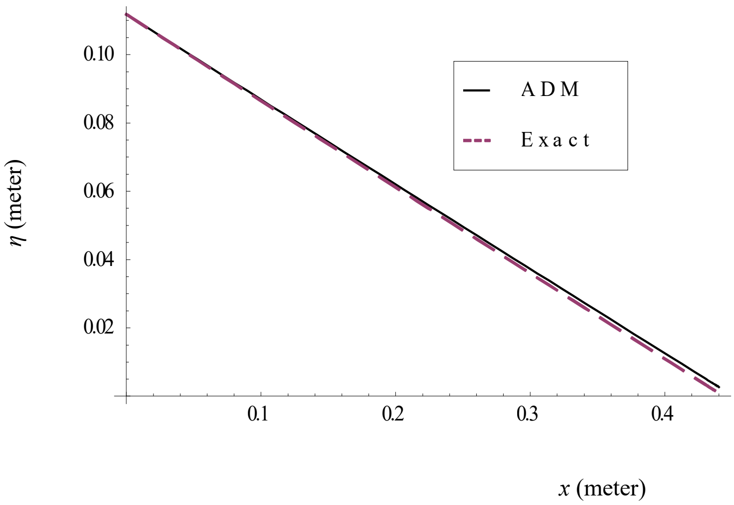

Fig.3.3. Plot of $\eta(x,t)$ vs. x for q =0.5, v=2.0 and

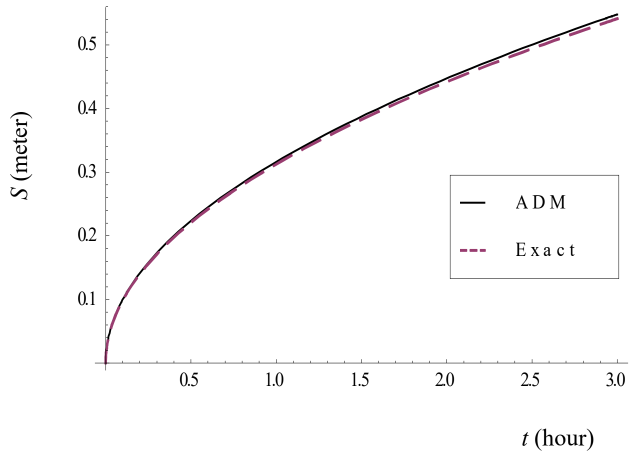

Fig.3.4. Plot of $s(t)$ vs. t for q =0.5, v=2.0 and $\gamma = 10$

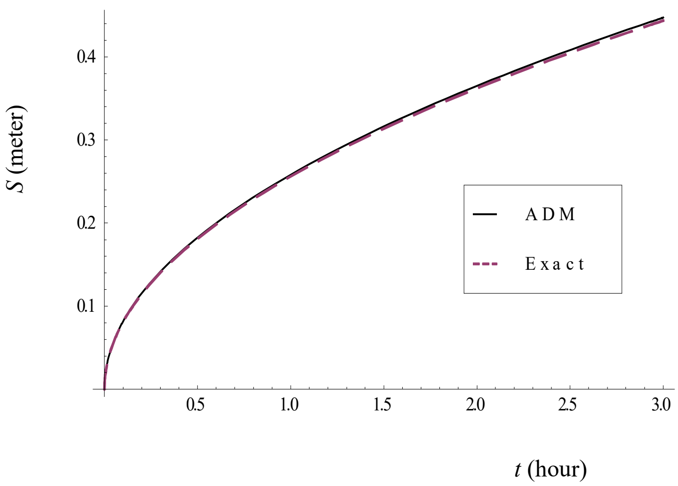

Fig.3.5. Plot of $s(t)$ vs. t for q =0.5, v=2.0 and $\gamma = 15$

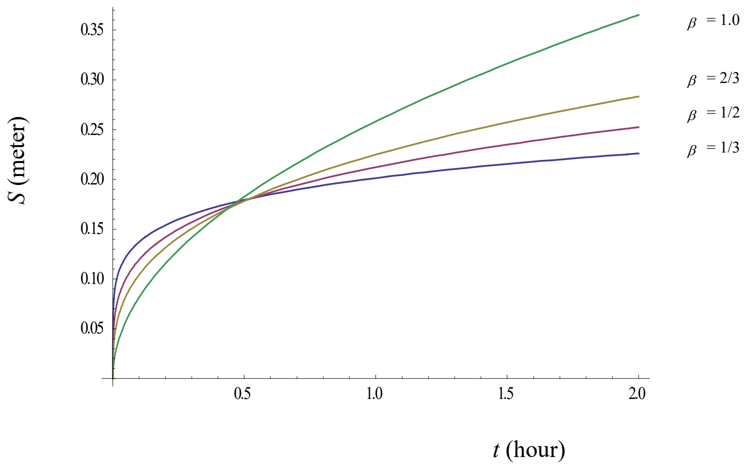

Fig.3.6. Plot of $s(t)$ vs. t for q = 0.5 , v =2 and $\gamma = 15$

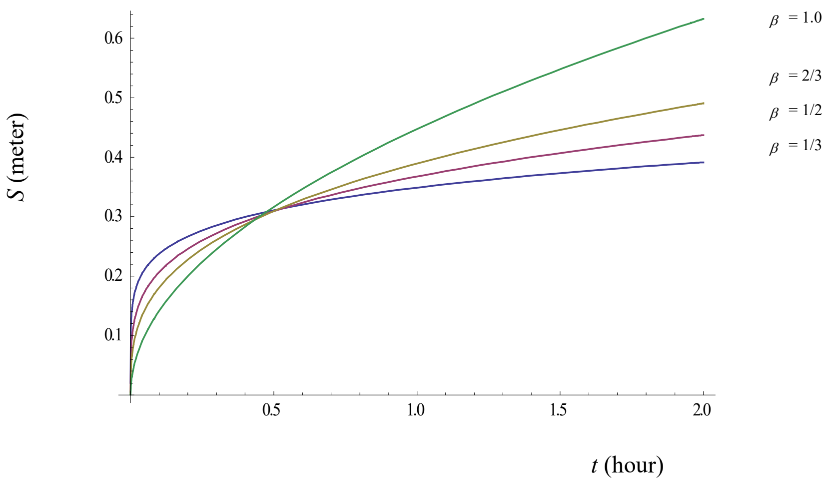

Fig.3.7. Plot of $s(t)$ vs. t for q = 1.5 v =2 and $\gamma = 15$

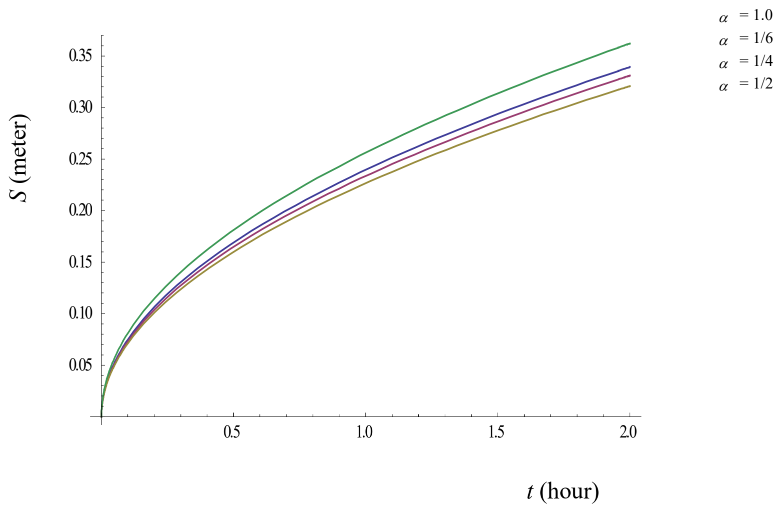

Fig.3.8. Plot of $s(t)$ vs. t for q = 0.5 , v=2 and $\gamma = 15$

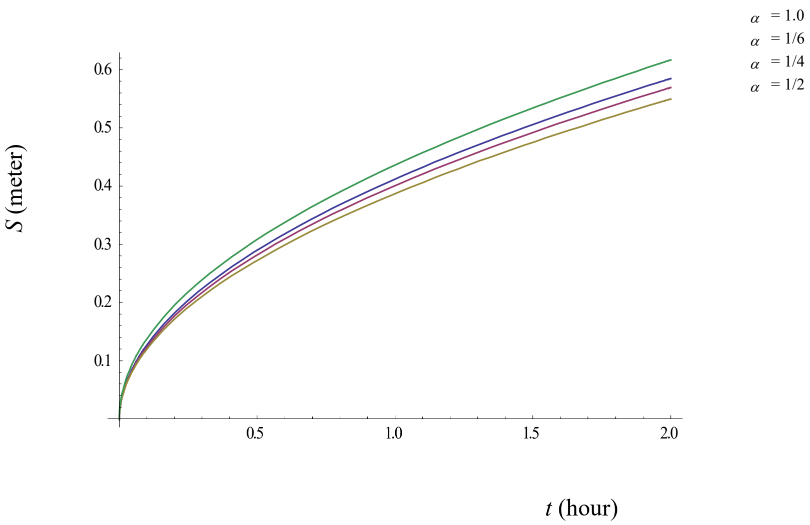

Fig.3.9. Plot of $s(t)$ vs. t for q =1.5 , v=2 and $\gamma = 15$

3.6. Conclusion

In this work, we discussed a mathematical model governed by space-time fractional derivative in Caputo sense for a moving boundary problem which occurs in fluvio-deltaic sedimentation process on earth surface. The solution of the proposed problem is obtained by Adomian decomposition method. It is found that sedimentation process becomes slow as the value of γ increases and sedimentation process becomes fast as the sediment line flux increases for standard as well as fractional Brownian motion. It is observed that time fractional is more pronounced than space fractional during sedimentation process. Moreover, it is seen that Adomian decomposition method is a powerful and accurate method for finding the solution of moving boundary problem. It is straight forward and avoids the hectic work of calculations. The author believes that the procedure as described in the present study will considerably benefit to engineers and scientists working in this field.